ELECTRODYNAMICS

Electrodynamics

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PREFACE

This book is designed as an advanced text in the theoretical aspects of electrodynamics. As there are many excellent texts available in which electrostatics and magnetostatics are treated in great detail, these special subjects have been omitted entirely except insofar as they may be incidental to the study of the general electromagnetic field. The relativistic attitude is adopted from the beginning, and Maxwell's field equations are deduced on the basis of the emission theory. In the authors' opinion, this is the only logical approach to electromagnetic theory. No attempt has been made to include quantum electrodynamics or any of the special results of the quantum theory.

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CONVERSION TABLE

Table of Equivalents in the Heaviside-Lorentz Units (h.l.u.) used in this book, Electromagnetic Units (e.m.u.) designated by the subscript m, and Electrostatic Units (e.s.u.) designated by the subscript s.

	h.l.u.	e.m.u.	e.s.u.
Charge Density..................	ρ	$c\sqrt{4\pi}\ \rho_m$	$\sqrt{4\pi}\ \rho_s$
Current Density.................	j	$c\sqrt{4\pi}\ j_m$	$\sqrt{4\pi}\ j_s$
Electric Intensity...............	E	$\frac{1}{c\sqrt{4\pi}} E_m$	$\frac{1}{\sqrt{4\pi}} E_s$
Magnetic Force..................	F	$\frac{1}{\sqrt{4\pi}} F_m$	$\frac{1}{c\sqrt{4\pi}} F_s$
Polarization.....................	P	$c\sqrt{4\pi}\, P_m$	$\sqrt{4\pi}\ P_s$
Intensity of Magnetization.......	I	$\sqrt{4\pi}\ I_m$	$c\sqrt{4\pi}\ I_s$
Electric Displacement...........	D	$\frac{c}{\sqrt{4\pi}} D_m$	$\frac{1}{\sqrt{4\pi}} D_s$
Magnetic Induction.............	B	$\frac{1}{\sqrt{4\pi}} B_m$	$\frac{c}{\sqrt{4\pi}} B_s$
Conductivity....................	σ	$4\pi c^2\sigma_m$	$4\pi\sigma_s$
Resistance......................	R	$\frac{1}{4\pi c^2} R_m$	$\frac{1}{4\pi} R_s$
Capacity........................	C	$4\pi c^2\ C_m$	$4\pi C_s$
Self-Inductance.................	L	$\frac{1}{4\pi c^2} L_m$	$\frac{1}{4\pi} L_s$

CHAPTER 1

THREE-DIMENSIONAL VECTOR ANALYSIS

1. Introduction. — Elementary physical quantities may be divided into two groups, *scalars* and *vectors*. A scalar quantity has magnitude alone. Thus volume, density, and mass are scalars. A vector on the other hand is a quantity which has both magnitude and direction, such as displacement, velocity, acceleration, force. A vector is represented by a straight line drawn in the direction of the vector, the sense being indicated by an arrowhead. In addition, the length of the line is made proportional to the magnitude of the vector, so that the graphical representation is complete. A vector $\mathbf{P}$ is illustrated in Fig. 1, the end A being called its *origin* and the end B its *terminus*. In printing, a vector is commonly denoted by a letter (or letters) in **black face** type. When it is desired to indicate the magnitude only of a vector, the latter is placed between vertical bars, or in the case of a vector symbolized by a single letter, that letter may be printed in italics. For example the magnitude of the vector $\mathbf{P} \times \mathbf{Q}$ is $|\mathbf{P} \times \mathbf{Q}|$ and that of $\frac{d\mathbf{P}}{dt}$ is $\left|\frac{d\mathbf{P}}{dt}\right|$, while the magnitude of $\mathbf{P}$ alone, is printed either as $|\mathbf{P}|$ or as P. The use of italics is simpler, of course, but it is usually feasible only in the case of single letters.

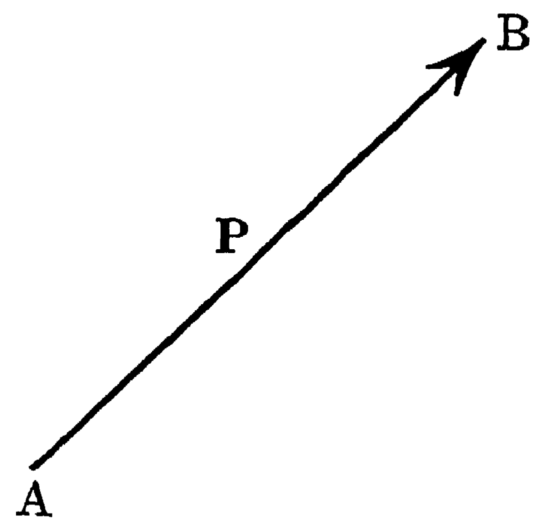

Fig. 1.

Taking the negative of a vector reverses its sense, that is $-\mathbf{P}$ has the same magnitude as $\mathbf{P}$ but it points in the opposite direction.

Multiplication of a vector by a positive scalar such as m increases the magnitude of the vector by the factor m without affecting its direction. Multiplication by the negative scalar $-m$ increases the magnitude by the factor m as before, but reverses the direction of the vector. Evidently $(-m)\mathbf{P} = m(-\mathbf{P}) = -(m\mathbf{P})$.

A *unit vector* is one whose magnitude is unity. It is often convenient to express a vector as the product of its magnitude and a unit vector having the same direction. Thus if $\mathbf{p}_1$ is a unit vector having the direction of $\mathbf{P}$, $\mathbf{P} = P\mathbf{p}_1$, from which we have at once $m\mathbf{P} = mP\mathbf{p}_1$ and $-m\mathbf{P} = -mP\mathbf{p}_1 = mP(-\mathbf{p}_1)$. The three unit vectors, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, parallel to the right-handed rectangular axes X, Y, Z respectively, are particularly useful.

In scalar algebra there are certain fundamental laws of combination, namely,

(I) the commutative law of addition,

$$a + b = b + a,$$

(II) the associative law of addition,

$$(a + b) + c = a + (b + c),$$

(III) the commutative law of multiplication,

$$ab = ba,$$

(IV) the associative law of multiplication,

$$(ab)c = a(bc),$$

(V) the distributive law of multiplication,

$$(a + b)c = ac + bc.$$

Not all of the above laws have general validity in vector analysis. We shall investigate their applicability in succeeding articles dealing with vector sums and products.

2. Addition and Subtraction of Vectors. — The addition of two vectors $\mathbf{P}$ and $\mathbf{Q}$ is carried out graphically by placing the origin of $\mathbf{Q}$ at the terminus of $\mathbf{P}$ (Fig. 2) and constructing the closing vector $\mathbf{R}$. This rule is based on the physical significance of the vectors. If $\mathbf{P}$ and $\mathbf{Q}$ are displacements, for example, the single displacement equivalent to $\mathbf{P}$ followed by $\mathbf{Q}$ obviously is $\mathbf{R}$. Analytically, we write

$$\mathbf{R} = \mathbf{P} + \mathbf{Q}, \tag{2–1}$$

where $\mathbf{R}$ is the *sum* or *resultant* of $\mathbf{P}$ and $\mathbf{Q}$. Since reversing the order of addition leads to the same resultant, the commutative law of addition holds. We note that the magnitude of $\mathbf{R}$ is given by

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha. \tag{2–2}$$

To find the sum **R** of several vectors **S**, **T**, **U**, obtain first the sum of **S** and **T** and then add **U**, (Fig. 3). The manner in which the vectors are grouped is immaterial. For example

$$(\mathbf{S} + \mathbf{T}) + \mathbf{U} = \mathbf{P} + \mathbf{U} = \mathbf{R}$$

and

$$\mathbf{S} + (\mathbf{T} + \mathbf{U}) = \mathbf{S} + \mathbf{Q} = \mathbf{R}.$$

Hence the associative law of addition is valid for vectors.

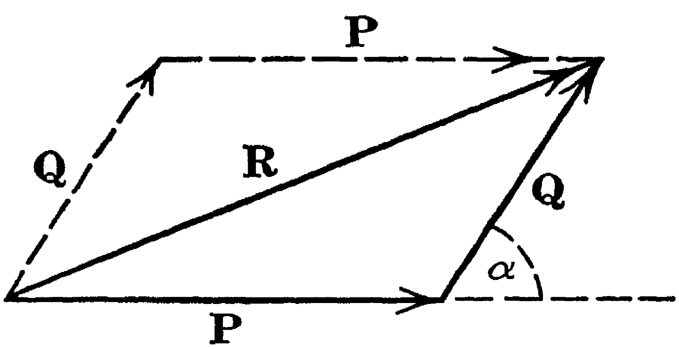

Fig. 2.

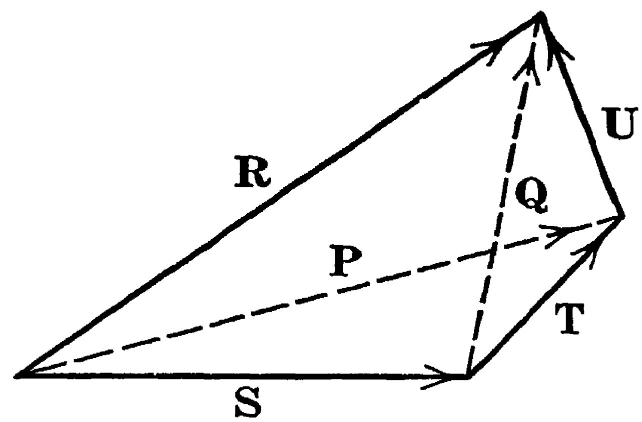

Fig. 3.

Subtraction of **Q** from **P** is accomplished by adding $-\mathbf{Q}$ to **P**.

The *components* of a vector are any vectors whose sum is the given vector. The components most commonly used are the three *rectangular components* parallel to the X, Y, Z axes, respectively. If P_x, P_y, P_z are the projections of **P** on the axes, its rectangular components are $\mathbf{i}P_x$, $\mathbf{j}P_y$, $\mathbf{k}P_z$. However, for brevity, it is customary to refer to P_x, P_y, P_z as the components since the subscripts indicate the appropriate directions. The vector **P** is completely determined by its components, for its magnitude is given by

$$P^2 = P_x^2 + P_y^2 + P_z^2 \tag{2-3}$$

and its direction cosines relative to XYZ by

$$l = \frac{P_x}{P}, \quad m = \frac{P_y}{P}, \quad n = \frac{P_z}{P}. \tag{2-4}$$

Inasmuch as equal vectors have equal magnitudes and the same directions, it follows that their rectangular components are equal each to each. Consider the relation $\mathbf{P} + \mathbf{Q} = \mathbf{R}$. Expressed in terms of rectangular components this is

$$\mathbf{i}(P_x + Q_x) + \mathbf{j}(P_y + Q_y) + \mathbf{k}(P_z + Q_z) = \mathbf{i}R_x + \mathbf{j}R_y + \mathbf{k}R_z$$

since addition is commutative and associative. As the left-hand side and the right-hand side of the equation represent equal vectors,

$$P_x + Q_x = R_x, \quad P_y + Q_y = R_y, \quad P_z + Q_z = R_z, \qquad (2\text{–}5)$$

showing that the sums of the components equal the corresponding components of the sum. This result holds, of course, for any number of vectors. Note that any vector equation expresses, fundamentally, three scalar equations.

3. Vector Representation of Surfaces. — Consider a plane surface of any shape such as ABC (Fig. 4). As it has a magnitude equal to its area and a direction specified by its normal, we may represent it by a vector $\boldsymbol{\sigma}$. It is necessary, however, to adopt a convention to determine the positive side of the surface, that is, to establish the positive sense of the normal. If ABC is part of a closed surface we draw the normal outward and there is no ambiguity. If, however, ABC is not part of a closed surface we can only make the positive sense of the normal depend on the positive sense of describing the periphery. The relationship chosen is such that when the positive sense of describing the periphery constitutes a clockwise motion to an observer the positive direction of the normal is away from him, that is, in the direction of advance of a right-handed screw rotated in the sense in which the periphery is described.

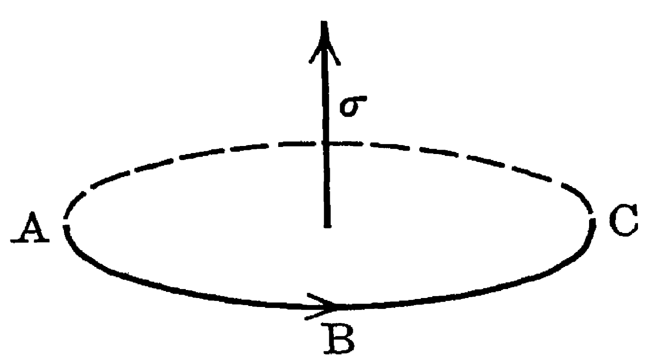

FIG. 4.

If a surface is not plane it may be divided into elements small enough so that each may be treated as plane. The vector representing the entire surface is then the sum of these elementary vectors.

Surfaces are equal vectorially if they are represented by equal vectors. Evidently there are an infinite number of surfaces corresponding to any given representative vector.

Let us next investigate the geometrical significance of the rectangular components of a vector representing a surface. Consider a plane surface ABC (Fig. 5) whose representative vector $\boldsymbol{\sigma}$ makes an angle γ with the Z axis. Let $A'B'C'$ be the projection of ABC on the XY plane. If we divide ABC into strips whose edges are parallel to a plane containing the Z axis and $\boldsymbol{\sigma}$, their projections on the XY plane

have the same width but are reduced in length by the factor $\cos\gamma$. Clearly then

$$A'B'C' = ABC\cos\gamma.$$

But

$$\sigma_z = \sigma\cos\gamma = ABC\cos\gamma.$$

Therefore σ_z is the projection of ABC on the XY plane or, of course, on any plane perpendicular to the Z axis. Note that the projection is negative when $\cos\gamma$ is negative. Similarly, σ_x and σ_y are the projections of ABC on planes perpendicular to the X and Y axes, respectively. These results apply to a curved surface as well as to a

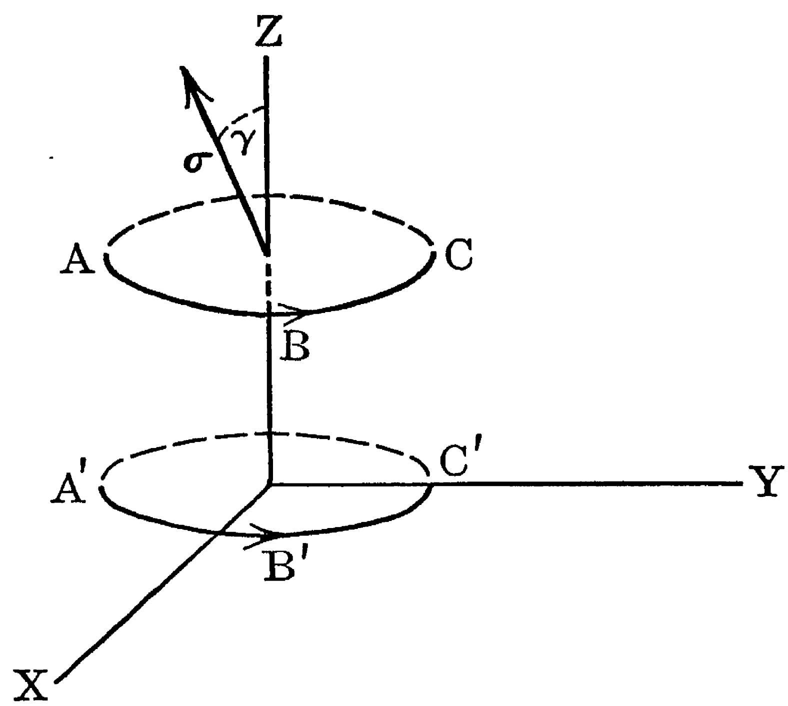

FIG. 5.

plane one, since the curved surface may be divided into a number of elementary surfaces which are effectively plane.

In the case of a closed surface the projection on any plane is zero. Therefore $\sigma_x = \sigma_y = \sigma_z = 0$ and $\boldsymbol{\sigma} = 0$.

We observe that, in general, the area of a curved surface cannot be deduced from the rectangular components of its representative vector. However, the size of a plane surface or of a surface element $d\boldsymbol{\sigma} = \boldsymbol{i}d\sigma_x + \boldsymbol{j}d\sigma_y + \boldsymbol{k}d\sigma_z$ is completely determined by its components.

4. Vector Product of Two Vectors. — The *vector* or *cross product* $\mathbf{P}\times\mathbf{Q}$ of two vectors $\mathbf{P}$ and $\mathbf{Q}$ (Fig. 6) is, by definition, a vector of

magnitude $PQ \sin \alpha$ in the direction of the normal to the plane determined by **P** and **Q**. Its sense is that of advance of a right-handed screw rotated from the first vector to the second through the angle α between them, this angle being defined as the smaller angle between their positive directions. It is clear from the figure that the commutative law of multiplication does not hold, but instead an anti-commutative law expressed by

$$\mathbf{Q} \times \mathbf{P} = -\mathbf{P} \times \mathbf{Q}, \quad \mathbf{P} \times \mathbf{P} = 0. \tag{4-1}$$

The vector product is defined as above because it represents a quantity which often appears in physical relations. Geometrically, the magnitude $|\mathbf{P} \times \mathbf{Q}| = PQ \sin \alpha$ gives the area of the parallelogram of which **P** and **Q** are the edges. We can make use of this fact

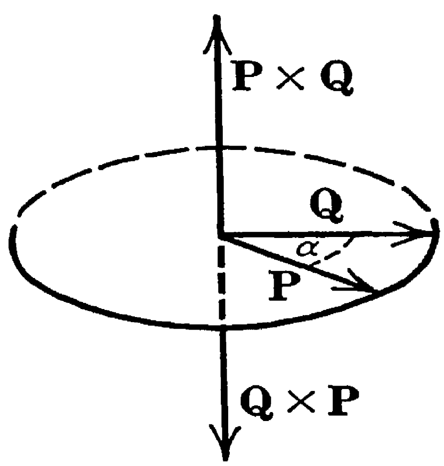

FIG. 6.

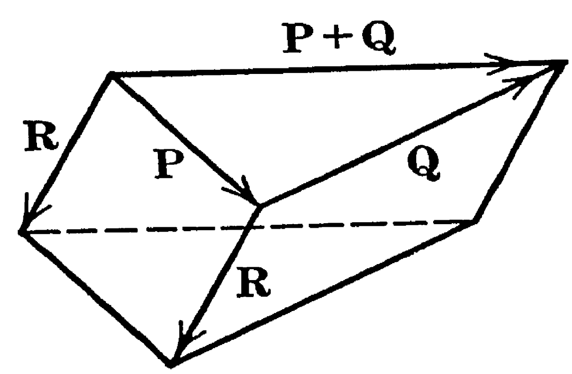

FIG. 7.

to show that the distributive law of multiplication is valid for vector products. Thus, consider the prism (Fig. 7) whose edges are **P**, **Q**, **P** + **Q** and **R**. We know that the vector representing the total surface of the prism is zero. Hence, expressing this surface vector as the sum of the surface vectors of the faces of the prism, all with positive normals drawn outward,

$$\tfrac{1}{2}(\mathbf{P} \times \mathbf{Q}) + \tfrac{1}{2}(\mathbf{Q} \times \mathbf{P}) + \mathbf{R} \times \mathbf{Q} + (\mathbf{P} + \mathbf{Q}) \times \mathbf{R} + \mathbf{R} \times \mathbf{P} = 0.$$

This reduces at once to

$$(\mathbf{P} + \mathbf{Q}) \times \mathbf{R} = \mathbf{P} \times \mathbf{R} + \mathbf{Q} \times \mathbf{R}, \tag{4-2}$$

which is the desired distributive law.

If we form cross products of the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ we see that

$$\left.\begin{aligned} &\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}, \\ &\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0. \end{aligned}\right\} \tag{4-3}$$

Making use of (4–3) we are able to expand the vector product $\mathbf{P} \times \mathbf{Q}$ in terms of the rectangular components of the two vectors. Since the distributive law holds

$$\begin{aligned}\mathbf{P} \times \mathbf{Q} &= (iP_x + jP_y + kP_z) \times (iQ_x + jQ_y + kQ_z) \\ &= i \times iP_xQ_x + i \times jP_xQ_y + i \times kP_xQ_z \\ &\quad + j \times iP_yQ_x + j \times jP_yQ_y + j \times kP_yQ_z \\ &\quad + k \times iP_zQ_x + k \times jP_zQ_y + k \times kP_zQ_z.\end{aligned}$$

This reduces to

$$\mathbf{P} \times \mathbf{Q} = i(P_yQ_z - P_zQ_y) + j(P_zQ_x - P_xQ_z) + k(P_xQ_y - P_yQ_x). \quad (4\text{–}4)$$

Expressed as a determinant this becomes

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} i & j & k \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}, \quad (4\text{–}5)$$

the cofactors of i, j, k being the corresponding components of $\mathbf{P} \times \mathbf{Q}$.

Problem 4a. Using vector methods compute the area of a triangle whose vertices have the rectangular coordinates (0, 0, 0), (5, 1, 3), (4, 5, 4) respectively. *Ans.* 12.52.

Problem 4b. If **P, Q, R** represent the sides of a triangle, as in Fig. 2, and θ_P, θ_Q, θ_R are the opposite angles, respectively, use the vector product to establish the relation $P : Q : R = \sin\theta_P : \sin\theta_Q : \sin\theta_R$.

Problem 4c. Suppose that the direction cosines of the rectangular axes $X'Y'Z'$ relative to the rectangular set XYZ are l_{11}, l_{12}, l_{13}; l_{21}, l_{22}, l_{23}; l_{31}, l_{32}, l_{33} respectively. Show by means of (4–4) that $l_{11} = l_{22}l_{33} - l_{23}l_{32}$, $l_{12} = l_{23}l_{31} - l_{21}l_{33}$, $l_{13} = l_{21}l_{32} - l_{22}l_{31}$, etc.

5. **Scalar Product of Two Vectors.** — The *scalar* or *dot product* of two vectors **P** and **Q**, written $\mathbf{P} \cdot \mathbf{Q}$, is a scalar quantity of magnitude $PQ \cos\alpha$, where α is the angle between **P** and **Q**, as in Fig. 6. As in the case of the vector product the scalar product represents a quantity of frequent occurrence in physical relations.

From the definition of the scalar product it appears that

$$\mathbf{Q} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{Q}, \quad \mathbf{P} \cdot \mathbf{P} = P^2, \quad (5\text{–}1)$$

so that the commutative law of multiplication applies. To show that the distributive also holds, consider the vectors **P**, **Q**, **P** + **Q**, and **R** shown in Fig. 8. Since the scalar product has the magnitude of either vector multiplied by the projection of the other upon it,

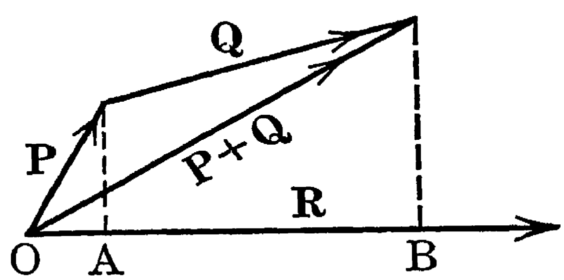

FIG. 8.

$$\mathbf{P}\cdot\mathbf{R} + \mathbf{Q}\cdot\mathbf{R} = \overline{OA}\,R + \overline{AB}\,R = \overline{OB}\,R,$$

$$(\mathbf{P} + \mathbf{Q})\cdot\mathbf{R} = \overline{OB}\,R.$$

Thus

$$(\mathbf{P} + \mathbf{Q})\cdot\mathbf{R} = \mathbf{P}\cdot\mathbf{R} + \mathbf{Q}\cdot\mathbf{R}, \tag{5-2}$$

which is the required relation.

The unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ yield the scalar products

$$\left.\begin{aligned} \boldsymbol{i}\cdot\boldsymbol{i} &= \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = 1, \\ \boldsymbol{i}\cdot\boldsymbol{j} &= \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = 0. \end{aligned}\right\} \tag{5-3}$$

With the aid of (5–2) and (5–3) we may expand $\mathbf{P}\cdot\mathbf{Q}$ in terms of rectangular components. Thus

$$\begin{aligned} \mathbf{P}\cdot\mathbf{Q} &= (\boldsymbol{i}P_x + \boldsymbol{j}P_y + \boldsymbol{k}P_z)\cdot(\boldsymbol{i}Q_x + \boldsymbol{j}Q_y + \boldsymbol{k}Q_z) \\ &= \boldsymbol{i}\cdot\boldsymbol{i}P_xQ_x + \boldsymbol{i}\cdot\boldsymbol{j}P_xQ_y + \boldsymbol{i}\cdot\boldsymbol{k}P_xQ_z \\ &\quad + \boldsymbol{j}\cdot\boldsymbol{i}P_yQ_x + \boldsymbol{j}\cdot\boldsymbol{j}P_yQ_y + \boldsymbol{j}\cdot\boldsymbol{k}P_yQ_z \\ &\quad + \boldsymbol{k}\cdot\boldsymbol{i}P_zQ_x + \boldsymbol{k}\cdot\boldsymbol{j}P_zQ_y + \boldsymbol{k}\cdot\boldsymbol{k}P_zQ_z, \end{aligned}$$

which gives

$$\mathbf{P}\cdot\mathbf{Q} = P_xQ_x + P_yQ_y + P_zQ_z. \tag{5-4}$$

Problem 5a. Using the scalar product deduce the relation (2–2).

Problem 5b. If θ is the angle between two vectors whose direction cosines are $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ respectively, show by means of (5–4) that $\cos\theta = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2$

6. Triple Scalar Product. — This product is the scalar $(\mathbf{P} \times \mathbf{Q})\cdot\mathbf{R}$. As the vector product $\mathbf{P} \times \mathbf{Q}$ is necessarily formed before the scalar product is taken, we may omit the parentheses without ambiguity and write simply $\mathbf{P} \times \mathbf{Q}\cdot\mathbf{R}$. From (4–1) and (5–1) it is clear as regards commutation of the vectors, that

$$\mathbf{P} \times \mathbf{Q}\cdot\mathbf{R} = -\,\mathbf{Q} \times \mathbf{P}\cdot\mathbf{R} = \mathbf{R}\cdot\mathbf{P} \times \mathbf{Q} = -\,\mathbf{R}\cdot\mathbf{Q} \times \mathbf{P}. \tag{6-1}$$

Let us investigate the result of interchanging the *cross* and the *dot* in the triple scalar product. If we construct a parallelopiped (Fig. 9) of which **P, Q, R** are the edges, it appears at once that, since $\mathbf{P} \times \mathbf{Q}$ gives the area of the base, $\mathbf{P} \times \mathbf{Q} \cdot \mathbf{R}$ is equal to the volume of the parallelopiped. But evidently $\mathbf{Q} \times \mathbf{R} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{Q} \times \mathbf{R}$ also is equal to the volume. Hence

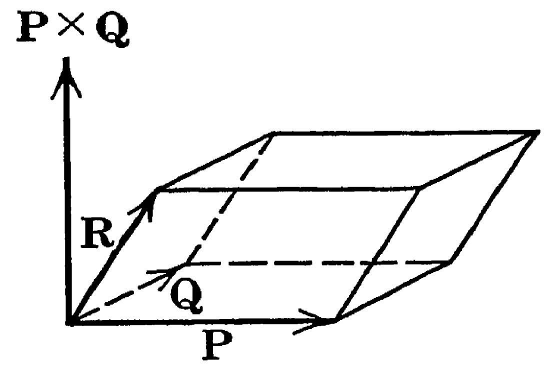

FIG. 9.

$$\mathbf{P} \times \mathbf{Q} \cdot \mathbf{R} = \mathbf{P} \cdot \mathbf{Q} \times \mathbf{R}, \qquad (6\text{–}2)$$

which shows that if the order of the vectors in a triple scalar product is not altered the *cross* and the *dot* may be interchanged without effect. This procedure is often useful in evaluating these products.

Finally, to express $\mathbf{P} \times \mathbf{Q} \cdot \mathbf{R}$ in terms of the rectangular components of the vectors, we have

$$\mathbf{P} \times \mathbf{Q} = \boldsymbol{i}(P_yQ_z - P_zQ_y) + \boldsymbol{j}(P_zQ_x - P_xQ_z) + \boldsymbol{k}(P_xQ_y - P_yQ_x),$$

$$\mathbf{R} = \boldsymbol{i}R_x + \boldsymbol{j}R_y + \boldsymbol{k}R_z.$$

Using (5–3) it follows that

$$\begin{aligned}\mathbf{P} \times \mathbf{Q} \cdot \mathbf{R} &= (P_yQ_z - P_zQ_y)R_x \\ &\qquad + (P_zQ_x - P_xQ_z)R_y + (P_xQ_y - P_yQ_x)R_z \\ &= \begin{vmatrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{vmatrix}. \end{aligned} \qquad (6\text{–}3)$$

Note that the cofactors of the components of any one of the three vectors in this determinant are the corresponding components of the vector product of the other two.

Problem 6a. Show that if three vectors are coplanar, their triple scalar product is zero.

Problem 6b. A force **F** acting at a distance **r** from the origin produces a torque about an axis through the origin. If the direction of the axis is given by the unit vector $\boldsymbol{\alpha}_1$, prove that the magnitude of the torque is $\mathbf{r} \times \mathbf{F} \cdot \boldsymbol{\alpha}_1$.

7. Triple Vector Product.— This product is the vector $(\mathbf{P}\times\mathbf{Q})\times\mathbf{R}$. The parentheses indicating that the vector product $\mathbf{P}\times\mathbf{Q}$ is to be formed first, cannot be omitted as in the case of the triple scalar product (6–1), since $(\mathbf{P}\times\mathbf{Q})\times\mathbf{R}$ and $\mathbf{P}\times(\mathbf{Q}\times\mathbf{R})$ are quite different. In other words, the associative law of multiplication does not hold. We observe from (4–1), however, that commutation of the vectors leads to the relations

$$(\mathbf{P}\times\mathbf{Q})\times\mathbf{R} = -(\mathbf{Q}\times\mathbf{P})\times\mathbf{R} = -\mathbf{R}\times(\mathbf{P}\times\mathbf{Q})$$
$$= \mathbf{R}\times(\mathbf{Q}\times\mathbf{P}). \qquad (7\text{–}1)$$

Since $\mathbf{P}\times\mathbf{Q}$ is perpendicular to the plane of $\mathbf{P}$ and $\mathbf{Q}$, while $(\mathbf{P}\times\mathbf{Q})\times\mathbf{R}$ is perpendicular to the plane of $\mathbf{P}\times\mathbf{Q}$ and $\mathbf{R}$, it is evident that the triple vector product must lie in the plane of $\mathbf{P}$ and $\mathbf{Q}$, as shown in Fig. 10. In order to expand $(\mathbf{P}\times\mathbf{Q})\times\mathbf{R}$ in terms of $\mathbf{P}$ and $\mathbf{Q}$ let us set

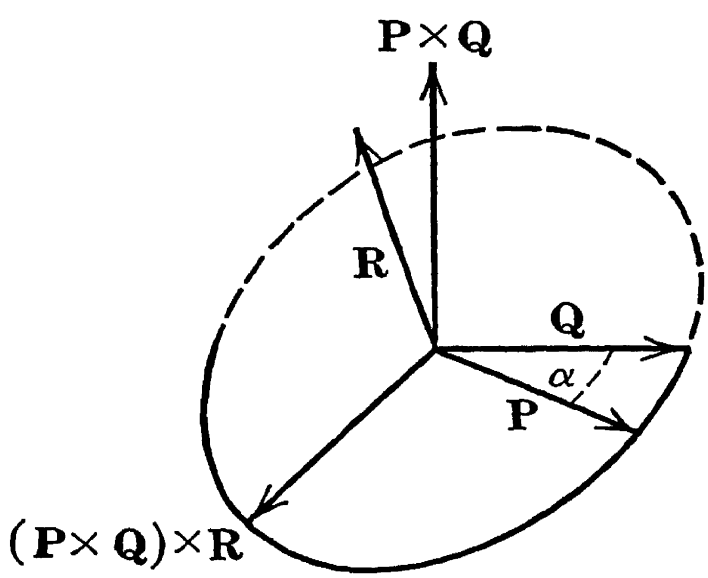

FIG. 10.

$$\mathbf{R} = a\mathbf{P} + b\mathbf{Q} + c(\mathbf{P}\times\mathbf{Q}), \qquad (7\text{–}2)$$

where the scalars a, b, c may be determined by finding the components of $\mathbf{R}$ parallel and perpendicular to $\mathbf{P}\times\mathbf{Q}$. Substituting (7–2) in the triple vector product, we have

$$(\mathbf{P}\times\mathbf{Q})\times\mathbf{R} = a(\mathbf{P}\times\mathbf{Q})\times\mathbf{P} + b(\mathbf{P}\times\mathbf{Q})\times\mathbf{Q}. \qquad (7\text{–}3)$$

It remains, then, to evaluate $(\mathbf{P}\times\mathbf{Q})\times\mathbf{P}$ and $(\mathbf{P}\times\mathbf{Q})\times\mathbf{Q}$.

Consider first $(\mathbf{P}\times\mathbf{Q})\times\mathbf{P}$. Since it, too, lies in the plane of $\mathbf{P}$ and $\mathbf{Q}$ we may write

$$(\mathbf{P}\times\mathbf{Q})\times\mathbf{P} = m\mathbf{Q} + n\mathbf{P}. \qquad (7\text{–}4)$$

Now, taking the vector product of $\mathbf{P}$ and (7–4),

$$m\mathbf{P}\times\mathbf{Q} = \mathbf{P}\times\{(\mathbf{P}\times\mathbf{Q})\times\mathbf{P}\},$$

and, equating magnitudes of the two sides of the equation,

$$mPQ\sin\alpha = P^3Q\sin\alpha,$$

giving $m = P^2$. On the other hand, from (7–4),

$$m\mathbf{P}\cdot\mathbf{Q} + nP^2 = \mathbf{P}\cdot\{(\mathbf{P}\times\mathbf{Q})\times\mathbf{P}\} = 0,$$

giving $n = -\mathbf{P}\cdot\mathbf{Q}$. Consequently

$$(\mathbf{P}\times\mathbf{Q})\times\mathbf{P} = P^2\mathbf{Q} - \mathbf{P}\cdot\mathbf{Q}\mathbf{P}, \tag{7–5}$$

and similarly

$$(\mathbf{P}\times\mathbf{Q})\times\mathbf{Q} = \mathbf{P}\cdot\mathbf{Q}\mathbf{Q} - Q^2\mathbf{P}. \tag{7–6}$$

Combining these results with (7–3),

$$(\mathbf{P}\times\mathbf{Q})\times\mathbf{R} = (aP^2 + b\mathbf{P}\cdot\mathbf{Q})\mathbf{Q} - (a\mathbf{P}\cdot\mathbf{Q} + bQ^2)\mathbf{P}. \tag{7–7}$$

Returning to (7–2) we see that

$$aP^2 + b\mathbf{P}\cdot\mathbf{Q} = \mathbf{R}\cdot\mathbf{P},$$

$$a\mathbf{P}\cdot\mathbf{Q} + bQ^2 = \mathbf{R}\cdot\mathbf{Q},$$

so that (7–7) yields the final result

$$(\mathbf{P}\times\mathbf{Q})\times\mathbf{R} = \mathbf{R}\cdot\mathbf{P}\mathbf{Q} - \mathbf{R}\cdot\mathbf{Q}\mathbf{P}. \tag{7–8}$$

Using (7–1) this may be written also as

$$\mathbf{R}\times(\mathbf{P}\times\mathbf{Q}) = \mathbf{R}\cdot\mathbf{Q}\mathbf{P} - \mathbf{R}\cdot\mathbf{P}\mathbf{Q}. \tag{7–9}$$

As an application of the triple vector product consider the vector expression $(\mathbf{r}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{c})$. Evidently we may treat $(\mathbf{b}\times\mathbf{c})$ as a single vector and expand according to (7–8) or we may regard $(\mathbf{r}\times\mathbf{a})$ as a single vector and expand by (7–9). Thus

$$(\mathbf{r}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\times\mathbf{c}\cdot\mathbf{r}\,\mathbf{a} - \mathbf{b}\times\mathbf{c}\cdot\mathbf{a}\,\mathbf{r}$$
$$= \mathbf{r}\times\mathbf{a}\cdot\mathbf{c}\,\mathbf{b} - \mathbf{r}\times\mathbf{a}\cdot\mathbf{b}\,\mathbf{c}.$$

Combining these, we find with the aid of (6–1) and (6–2),

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}\,\mathbf{r} = \mathbf{r}\cdot\mathbf{b}\times\mathbf{c}\,\mathbf{a} + \mathbf{r}\cdot\mathbf{c}\times\mathbf{a}\,\mathbf{b} + \mathbf{r}\cdot\mathbf{a}\times\mathbf{b}\,\mathbf{c}.$$

Provided the scalar $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ does not vanish, that is, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are not coplanar, the preceding equation may be put in the form

$$\mathbf{r} = \mathbf{r}\cdot\mathbf{a}'\mathbf{a} + \mathbf{r}\cdot\mathbf{b}'\mathbf{b} + \mathbf{r}\cdot\mathbf{c}'\mathbf{c}, \tag{7–10}$$

where

$$\mathbf{a}' \equiv \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}},\quad \mathbf{b}' \equiv \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}},\quad \mathbf{c}' \equiv \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}. \tag{7–11}$$

Again we may start with $(\mathbf{r} \times \mathbf{a}') \times (\mathbf{b}' \times \mathbf{c}')$, obtaining

$$\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}' \mathbf{r} = \mathbf{r} \cdot \mathbf{b}' \times \mathbf{c}' \mathbf{a}' + \mathbf{r} \cdot \mathbf{c}' \times \mathbf{a}' \mathbf{b}' + \mathbf{r} \cdot \mathbf{a}' \times \mathbf{b}' \mathbf{c}'.$$

This reduces without difficulty to

$$\mathbf{r} = \mathbf{r} \cdot \mathbf{a}\,\mathbf{a}' + \mathbf{r} \cdot \mathbf{b}\,\mathbf{b}' + \mathbf{r} \cdot \mathbf{c}\,\mathbf{c}' \tag{7–12}$$

since

$$\mathbf{a} = \frac{\mathbf{b}' \times \mathbf{c}'}{\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}'}, \quad \mathbf{b} = \frac{\mathbf{c}' \times \mathbf{a}'}{\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}'}, \quad \mathbf{c} = \frac{\mathbf{a}' \times \mathbf{b}'}{\mathbf{a}' \cdot \mathbf{b}' \times \mathbf{c}'}, \tag{7–13}$$

from (7–11).

Evidently $\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$, while $\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0$. On account of these relations the sets of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are said to be *reciprocal* to each other. Reciprocal vectors are useful in the study of dyadics, to be undertaken later. Equations (7–10) and (7–11) signify that a given vector may be expressed in terms of any three non-coplanar vectors, the respective coefficients of these vectors being the scalar products of the given vector and the vectors of the reciprocal set.

Problem 7a. By expanding $\{\mathbf{r} \times (\mathbf{a} \times \mathbf{b})\} \times \mathbf{c}$ deduce (7–12).

8. Transformation of Components of a Vector. — By definition a vector is a quantity whose properties are represented completely by a directed segment of a straight line. Let x_0, y_0, z_0 be the coordinates of the origin and x_t, y_t, z_t the coordinates of the terminus of the linear segment representative of the vector $\mathbf{P}$. Then the rectangular components P_x, P_y, P_z of $\mathbf{P}$ parallel to the axes XYZ under consideration are given by the projections $x_t - x_0$, $y_t - y_0$ and $z_t - z_0$ of the linear segment on the X, Y and Z axes. Consider a second set of axes $X'Y'Z'$ parallel respectively to XYZ but with origin at a different point. The projections $x_t' - x_0'$, $y_t' - y_0'$ and $z_t' - z_0'$ of the linear segment on the X', Y' and Z' axes are equal respectively to its projections on the X, Y and Z axes. *Hence the magnitudes of the rectangular components of a vector are unaltered by a translation of the axes.*

On the other hand a rotation of the axes changes the values of the components of a vector. Since translation has no effect on the magnitudes of these quantities we may take the origin of coordinates at the origin of the linear segment representative of the vector $\mathbf{P}$ without any loss of generality. Then the components of $\mathbf{P}$ parallel to

the axes XYZ are given simply by the coordinates x_t, y_t, z_t of the terminus of the linear segment. Relative to a second set of axes, $X'Y'Z'$, with the same origin as XYZ but differently oriented, the components of $\mathbf{P}$ are given by the coordinates x_t', y_t', z_t' of the terminus of the linear segment with respect to these axes. So when we pass from one set of axes to another differently oriented, the components of a vector must transform exactly as do the coordinates of a point. This condition is necessary in order that the components P_x', P_y', P_z' parallel to $X'Y'Z'$ shall give the same resultant vector as do the components P_x, P_y, P_z parallel to XYZ.

If the direction cosines of the X' axis relative to XYZ are l_{11}, l_{12}, l_{13}, etc., in accord with the table below, then the coordinates of a

	X	Y	Z
X'	l_{11}	l_{12}	l_{13}
Y'	l_{21}	l_{22}	l_{23}
Z'	l_{31}	l_{32}	l_{33}

point transform according to the equations

$$\begin{aligned} x' &= l_{11}x + l_{12}y + l_{13}z, & x &= l_{11}x' + l_{21}y' + l_{31}z', \\ y' &= l_{21}x + l_{22}y + l_{23}z, & y &= l_{12}x' + l_{22}y' + l_{32}z', \\ z' &= l_{31}x + l_{32}y + l_{33}z, & z &= l_{13}x' + l_{23}y' + l_{33}z', \end{aligned} \Bigg\} \quad (8\text{–}1)$$

and the components of a vector $\mathbf{P}$ transform according to the equivalent equations

$$\begin{aligned} P_x' &= l_{11}P_x + l_{12}P_y + l_{13}P_z, & P_x &= l_{11}P_x' + l_{21}P_y' + l_{31}P_z', \\ P_y' &= l_{21}P_x + l_{22}P_y + l_{23}P_z, & P_y &= l_{12}P_x' + l_{22}P_y' + l_{32}P_z', \\ P_z' &= l_{31}P_x + l_{32}P_y + l_{33}P_z, & P_z &= l_{13}P_x' + l_{23}P_y' + l_{33}P_z'. \end{aligned} \Bigg\} \quad (8\text{–}2)$$

Expressed as the vector sum of its components,

$$\mathbf{P} = \boldsymbol{i}P_x + \boldsymbol{j}P_y + \boldsymbol{k}P_z = \boldsymbol{i}'P_x' + \boldsymbol{j}'P_y' + \boldsymbol{k}'P_z', \qquad (8\text{–}3)$$

where $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ are unit vectors parallel to the X', Y', Z' axes respectively. In fact, we can deduce the relations (8–2) immediately

from (8–3) merely by expressing $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in terms of $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$. As the direction cosines of $\boldsymbol{i}$ relative to $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ are l_{11}, l_{21}, l_{31}, etc.,

$$\left.\begin{aligned} \boldsymbol{i} &= \boldsymbol{i}'l_{11} + \boldsymbol{j}'l_{21} + \boldsymbol{k}'l_{31}, \\ \boldsymbol{j} &= \boldsymbol{i}'l_{12} + \boldsymbol{j}'l_{22} + \boldsymbol{k}'l_{32}, \\ \boldsymbol{k} &= \boldsymbol{i}'l_{13} + \boldsymbol{j}'l_{23} + \boldsymbol{k}'l_{33}, \end{aligned}\right\} \tag{8–4}$$

and therefore

$$\begin{aligned} \boldsymbol{i}P_x + \boldsymbol{j}P_y + \boldsymbol{k}P_z = {} & \boldsymbol{i}'(l_{11}P_x + l_{12}P_y + l_{13}P_z) \\ & + \boldsymbol{j}'(l_{21}P_x + l_{22}P_y + l_{23}P_z) + \boldsymbol{k}'(l_{31}P_x + l_{32}P_y + l_{33}P_z). \end{aligned}$$

Comparison with (8–3) gives P_x', P_y', P_z' in terms of P_x, P_y, P_z.

The direction cosines l_{11}, l_{12}, l_{13}, etc., can be expressed as partial derivatives of the one set of coordinates relative to the other by differentiating the relations (8–1). For instance

$$l_{11} = \frac{\partial x'}{\partial x} = \frac{\partial x}{\partial x'}, \quad l_{12} = \frac{\partial x'}{\partial y} = \frac{\partial y}{\partial x'}, \quad l_{13} = \frac{\partial x'}{\partial z} = \frac{\partial z}{\partial x'}. \tag{8–5}$$

As an example of the use of the transformations (8–2) let us find the components of the vector

$$\mathbf{V} \equiv \boldsymbol{i}\frac{\partial \Phi}{\partial x} + \boldsymbol{j}\frac{\partial \Phi}{\partial y} + \boldsymbol{k}\frac{\partial \Phi}{\partial z}$$

along the X', Y', and Z' axes. Here Φ is any scalar function of the coordinates. Making use of (8–5) we have

$$V_x' = \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial x'} = \frac{\partial \Phi}{\partial x'},$$

and similar expressions hold for V_y' and V_z'. Hence

$$\boldsymbol{i}\frac{\partial \Phi}{\partial x} + \boldsymbol{j}\frac{\partial \Phi}{\partial y} + \boldsymbol{k}\frac{\partial \Phi}{\partial z} = \boldsymbol{i}'\frac{\partial \Phi}{\partial x'} + \boldsymbol{j}'\frac{\partial \Phi}{\partial y'} + \boldsymbol{k}'\frac{\partial \Phi}{\partial z'}, \tag{8–6}$$

a relation we shall find useful later.

Again, if we take the scalar product of the vector

$$\mathbf{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z = \boldsymbol{i}'x' + \boldsymbol{j}'y' + \boldsymbol{k}'z'$$

and the vector $\mathbf{P}$ given by (8–3) we find that

$$\mathbf{r}\cdot\mathbf{P} = xP_x + yP_y + zP_z = x'P_x' + y'P_y' + z'P_z'. \tag{8–7}$$

9. Scalar Functions of Position in Space. — Many physical laws involve scalar functions of the coordinates, and often of the components of one or more constant vectors as well. Such a *scalar function of position in space* associates with each point in a spacial region a definite scalar magnitude, the aggregate of these scalar magnitudes constituting a *scalar field*. For instance, the potential

$$\frac{xA_x + yA_y + zA_z}{4\pi(x^2 + y^2 + z^2)^{3/2}}$$

due to an electric or magnetic dipole of moment **A** located at the origin constitutes a function of this type. Although **A** is a constant as regards the coordinates it may, of course, be a function of the time.

Consider a scalar function Φ of the coordinates and of the components of any number of constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$. At once the question arises whether any arbitrary function of x, y, z, A_{1x}, A_{1y}, A_{1z}, $\cdots A_{nx}$, A_{ny}, A_{nz} can enter a physical law. In order to answer this question let us transform from the axes XYZ to a new set of axes $X'Y'Z'$ with the same origin but oriented relative to XYZ in accord with the scheme on page 13. In general the transformation introduces into the function the direction cosines l_{11}, l_{12}, l_{13}, etc., which determine the orientation of the new axes relative to the old. But the presence of these direction cosines makes the function depend not only on the coordinates and the components of the constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$ but also on the particular orientation given to the axes $X'Y'Z'$. Now the use of axes is merely an artifice convenient for the description of physical phenomena and it is certain that the character of a law of nature can depend in no way upon the particular orientation given the axes. We conclude, then, that the only scalar functions of the coordinates and of the components of a number of constant vectors which can enter physical laws are those in which the direction cosines defining the relative orientation of the axes do not appear when we pass from one set of axes to another differently oriented. Such a function will be called a *proper scalar function* of the coordinates and of the components of the constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$, or a *scalar invariant.*

Since the direction cosines l_{11}, l_{12}, l_{13}, etc., do not appear in a proper scalar function when we transform from the axes XYZ to $X'Y'Z'$, such a scalar is the *same* function of x', y', z', A'_{1x}, A'_{1y},

$A'_{1z}, \cdots A'_{nx}, A'_{ny}, A'_{nz}$ whatever the orientation of $X'Y'Z'$ relative to XYZ may be. In the particular case where X' coincides with X, Y' with Y, and Z' with Z, x', y', z', $A'_{1x}, A'_{1y}, A'_{1z}, \cdots A'_{nx}, A'_{ny}, A'_{nz}$ are identical with $x, y, z, A_{1x}, A_{1y}, A_{1z}, \cdots A_{nx}, A_{ny}, A_{nz}$. Hence it follows that for any orientation of $X'Y'Z'$ relative to XYZ, a proper function Φ, when expressed in terms of the primed coordinates and primed vector components, must be the *same* function of these quantities as it is of the unprimed coordinates and unprimed vector components.

For example, we find that if we transform by means of (8–1) and (8–2),

$$f\{x+y+z\} = f\{(l_{11}+l_{12}+l_{13})x' \\ +(l_{21}+l_{22}+l_{23})y' + (l_{31}+l_{32}+l_{33})z'\}.$$

This function, therefore, is not a proper function, and cannot occur in a physical law. On the other hand

$$f\{(x^2+y^2+z^2),\ (xA_x+yA_y+zA_z),\ (A_x{}^2+A_y{}^2+A_z{}^2)\} \\ = f\{(x'^2+y'^2+z'^2),\ (x'A_x'+y'A_y'+z'A_z'),\ (A_x'^2+A_y'^2+A_z'^2)\}$$

is a scalar invariant and therefore a proper function of x, y, z, A_x, A_y, A_z.

Evidently the arguments which can appear in a proper function are limited by the condition that none of them may depend upon the particular orientation given to the axes. Now the number of independent variables appearing in a function Φ of the coordinates and of the components of the constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$ is at most $3(n+1)$, namely, $x, A_{1x}, \cdots A_{nx}, y, A_{1y}, \cdots A_{ny}, z, A_{1z}, \cdots A_{nz}$. Let us change these variables to the set

$$\begin{array}{lr} x,\ y,\ A_{1z}, & 3, \\ r^2,\ A_1{}^2,\ A_2{}^2,\ \cdots\ A_n{}^2, & n+1, \\ \mathbf{r}\cdot\mathbf{A}_1,\ \mathbf{r}\cdot\mathbf{A}_2,\ \cdots\ \mathbf{r}\cdot\mathbf{A}_n, & n, \\ \mathbf{A}_1\cdot\mathbf{A}_2,\ \cdots\ \mathbf{A}_1\cdot\mathbf{A}_n, & n-1, \end{array}$$

where $\mathbf{r} \equiv \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. Provided appropriate conventions are adopted for determining the signs of such radicals as $z = \sqrt{r^2 - x^2 - y^2}$, these $3(n+1)$ variables are equivalent to the original group, for x, y, r specify the coordinates; A_{1z}, $\mathbf{r}\cdot\mathbf{A}_1$, A_1

fix $\mathbf{A}_1$ relative to the Z axis and $\mathbf{r}$; and the remaining $3(n - 1)$ variables fix the vectors $\mathbf{A}_2, \cdots \mathbf{A}_n$ relative to $\mathbf{r}$ and $\mathbf{A}_1$. But the three variables in the first row cannot appear in a proper function, for their values depend upon the orientation of the axes XYZ. Consequently the most general proper scalar function of the coordinates and of the components of the n constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$ can be a function of only the $3n$ independent arguments listed in the last three rows of the table. These may appear explicitly, or implicitly in expressions formed from them, such as $\mathbf{A}_i \cdot \mathbf{A}_j$ or $\mathbf{r} \times \mathbf{A}_i \cdot \mathbf{A}_j$. It should be noted that the only arguments permitted are those specifying the magnitudes and *relative* orientations of the vectors $\mathbf{r}, \mathbf{A}_1, \cdots \mathbf{A}_n$.

Problem 9a. Express $\mathbf{A}_2 \cdot \mathbf{A}_3$ in terms of the arguments listed in the last three rows of the table.

$$\textit{Ans.}\quad \mathbf{A}_2 \cdot \mathbf{A}_3 = \alpha_2\alpha_3 r^2 + (\alpha_2\beta_3 + \alpha_3\beta_2)\mathbf{r}\cdot\mathbf{A}_1 + \beta_2\beta_3 A_1{}^2 + \gamma_2\gamma_3,$$

where
$$\alpha_2 \equiv \frac{\mathbf{r}\cdot\mathbf{A}_2 A_1{}^2 - \mathbf{r}\cdot\mathbf{A}_1\mathbf{A}_1\cdot\mathbf{A}_2}{r^2 A_1{}^2 - \overline{\mathbf{r}\cdot\mathbf{A}_1}^2}, \quad \beta_2 \equiv \frac{\mathbf{A}_1\cdot\mathbf{A}_2 r^2 - \mathbf{r}\cdot\mathbf{A}_1\mathbf{r}\cdot\mathbf{A}_2}{r^2 A_1{}^2 - \overline{\mathbf{r}\cdot\mathbf{A}_1}^2},$$

$$\gamma_2 \equiv \sqrt{A_2{}^2 - \alpha_2{}^2 r^2 - 2\alpha_2\beta_2\mathbf{r}\cdot\mathbf{A}_1 - \beta_2{}^2 A_1{}^2}, \text{ etc.}$$

10. Vector Functions of Position in Space. — If each component of a vector is a function of the coordinates the vector is called a *vector function of position in space.* The components of such a vector may in addition be functions of the components of one or more constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$. A vector function of position associates a definite vector with each point of a spacial region, the aggregate of these vectors constituting a *vector field.* The simplest vector function of the coordinates is the position vector $\mathbf{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$ of the point x, y, z relative to the origin of the axes XYZ. This function associates with every point a vector having the magnitude and direction of the line drawn from the origin to the point in question. The electric intensity

$$\boldsymbol{i}\left\{\frac{3x(xA_x + yA_y + zA_z) - r^2 A_x}{4\pi r^5}\right\}$$
$$+ \boldsymbol{j}\left\{\frac{3y(xA_x + yA_y + zA_z) - r^2 A_y}{4\pi r^5}\right\}$$
$$+ \boldsymbol{k}\left\{\frac{3z(xA_x + yA_y + zA_z) - r^2 A_z}{4\pi r^5}\right\}$$

in the field due to an electric dipole of moment **A** located at the origin is a vector function of the coordinates and of the components of the constant vector **A**.

Just as in the case of a scalar function, there are certain restrictions on the arguments which may appear in the components V_x, V_y, V_z of a vector function **V** entering a physical law. We shall suppose that V_x, V_y, V_z are functions of the components of certain constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$ as well as of the coordinates. Then if we transform to a set of axes $X'Y'Z'$ oriented relative to XYZ in accord with the scheme on page 13, the components V_x', V_y', V_z' of **V** relative to the new axes will in general be functions of the direction cosines l_{11}, l_{12}, l_{13}, etc., as well as of x', y', z', A'_{1x}, A'_{1y}, A'_{1z}, $\cdots$ A'_{nx}, A'_{ny}, A'_{nz}. As, however, no physical quantity can depend upon the particular orientation of the axes employed in its description, only those vector functions can occur in physical laws whose components relative to $X'Y'Z'$, obtained by transforming from XYZ in accord with (8–1) and (8–2), do not involve the direction cosines of the new axes relative to the old. Following the same reasoning as that employed in the case of a scalar function, we conclude that V_x', V_y', V_z' must be the *same* three functions of x', y', z', A'_{1x}, A'_{1y}, A'_{1z}, $\cdots$ A'_{nx}, A'_{ny}, A'_{nz} as V_x, V_y, V_z are of x, y, z, A_{1x}, A_{1y}, A_{1z}, $\cdots$ A_{nx}, A_{ny}, A_{nz}, respectively. A vector function whose components satisfy this condition we shall call a *proper vector function.*

A simple example of a vector function which does not satisfy these conditions is the two-dimensional vector whose components relative to XY are $V_x = y$, $V_y = x$. If the $X'Y'$ axes make an angle γ with the XY axes, $l_{11} = l_{22} = \cos\gamma$ and $l_{12} = -l_{21} = \sin\gamma$, and we find from (8–1) and (8–2) that

$$V_x' = x' \sin 2\gamma + y' \cos 2\gamma,$$

$$V_y' = x' \cos 2\gamma - y' \sin 2\gamma.$$

Consequently V_x and V_y are not the components of a proper vector. On the other hand, if $V_x = y$, $V_y = -x$, then $V_x' = y'$, $V_y' = -x'$, showing that these functions are the components of a proper two-dimensional vector.

Next we shall find the most general form of a proper vector function **V** of the coordinates and of the components of the constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$. As in (8–7) we have

$$V_x'x' + V_y'y' + V_z'z' = V_x x + V_y y + V_z z.$$

Therefore, as $\mathbf{V}$ is a proper vector function, the scalar product $\mathbf{V}\cdot\mathbf{r}$ is the same function of x', y', z', A'_{1x}, A'_{1y}, A'_{1z}, $\cdots$ as of x, y, z, A_{1x}, A_{1y}, A_{1z}, $\cdots$. Hence it is a proper scalar function. In similar fashion it is shown that $\mathbf{V}\cdot\mathbf{A}_1$ and $\mathbf{V}\cdot\mathbf{r}\times\mathbf{A}_1$ are also proper scalar functions.

Now, since $\mathbf{r}$, $\mathbf{A}_1$ and $\mathbf{r}\times\mathbf{A}_1$ are three non-coplanar vectors, we can write $\mathbf{V}$ in the form

$$\mathbf{V} = \alpha\mathbf{r} + \beta\mathbf{A}_1 + \gamma\mathbf{r}\times\mathbf{A}_1 \tag{10–1}$$

as shown in article 7. Therefore

$$\mathbf{V}\cdot\mathbf{r} = \alpha r^2 + \beta\mathbf{r}\cdot\mathbf{A}_1,$$

$$\mathbf{V}\cdot\mathbf{A}_1 = \alpha\mathbf{r}\cdot\mathbf{A}_1 + \beta A_1{}^2,$$

$$\mathbf{V}\cdot\mathbf{r}\times\mathbf{A}_1 = \gamma(r^2A_1{}^2 - \overline{\mathbf{r}_1\cdot\mathbf{A}_1}^2),$$

and

$$\alpha = \frac{A_1{}^2\mathbf{V}\cdot\mathbf{r} - \mathbf{r}\cdot\mathbf{A}_1\mathbf{V}\cdot\mathbf{A}_1}{r^2A_1{}^2 - \overline{\mathbf{r}\cdot\mathbf{A}_1}^2}, \quad \beta = \frac{r^2\mathbf{V}\cdot\mathbf{A}_1 - \mathbf{r}\cdot\mathbf{A}_1\mathbf{V}\cdot\mathbf{r}}{r^2A_1{}^2 - \overline{\mathbf{r}\cdot\mathbf{A}_1}^2},$$

$$\gamma = \frac{\mathbf{V}\cdot\mathbf{r}\times\mathbf{A}_1}{r^2A_1{}^2 - \overline{\mathbf{r}\cdot\mathbf{A}_1}^2}.$$

Since r^2, $A_1{}^2$ and $\mathbf{r}\cdot\mathbf{A}_1$ are proper scalars, it follows that α, β, γ must be proper scalar functions and therefore can contain only those independent arguments listed in the last three rows of the table in article 9. Consequently the most general proper vector function of the coordinates and of the components of the constant vectors $\mathbf{A}_1, \cdots \mathbf{A}_n$ must be expressible in the form (10–1), where the coefficients α, β, γ are three proper scalar functions.

Of course we can choose others of the constant vectors $\mathbf{A}_2, \cdots \mathbf{A}_n$ in place of either or both $\mathbf{r}$ and $\mathbf{A}_1$ in obtaining the most general expression for the proper vector $\mathbf{V}$. For instance we may write

$$\mathbf{V} = \lambda\mathbf{r} + \mu\mathbf{A}_2 + \nu\mathbf{r}\times\mathbf{A}_2, \tag{10–2}$$

or

$$\mathbf{V} = \xi\mathbf{A}_1 + \eta\mathbf{A}_2 + \zeta\mathbf{A}_1\times\mathbf{A}_2, \tag{10–3}$$

where λ, μ, ν and ξ, η, ζ are proper scalar functions; but these two forms or any others to which we may be led are reducible to the form (10–1). Obviously the two constant vectors appearing in (10–3) must not be collinear.

Finally we must consider the result of combining two proper vector functions **P** and **Q** of the coordinates and of the components of any number of constant vectors $\mathbf{A}_1$, etc. It is easily shown that the scalar product $\mathbf{P}\cdot\mathbf{Q}$ is a proper scalar function and that the vector product $\mathbf{P}\times\mathbf{Q}$ is a proper vector function. To prove this express each of the two vectors both in terms of its components along XYZ and in terms of its components along $X'Y'Z'$, as in (8–3). Then, if the scalar product is formed,

$$P_xQ_x + P_yQ_y + P_zQ_z = P_x'Q_x' + P_y'Q_y' + P_z'Q_z'. \qquad (10\text{–}4)$$

As **P** and **Q** are proper vector functions, P_x', P_y', P_z' and Q_x', Q_y', Q_z' are the same functions of x', y', z', A'_{1x}, A'_{1y}, A'_{1z}, etc., as P_x, P_y, P_z and Q_x, Q_y, Q_z are of x, y, z, A_{1x}, A_{1y}, A_{1z}, etc., respectively. Therefore the same is true of the two sides of (10–4), showing that the scalar product is a proper scalar function.

Forming the vector product of **P** and **Q**,

$$\begin{aligned} &\boldsymbol{i}(P_yQ_z - P_zQ_y) + \boldsymbol{j}(P_zQ_x - P_xQ_z) + \boldsymbol{k}(P_xQ_y - P_yQ_x) \\ &\quad = \boldsymbol{i}'(P_y'Q_z' - P_z'Q_y') + \boldsymbol{j}'(P_z'Q_x' - P_x'Q_z') + \boldsymbol{k}'(P_x'Q_y' - P_y'Q_x'). \end{aligned}$$

Hence, as **P** and **Q** are proper vectors, $P_y'Q_z' - P_z'Q_y'$ is the same function of the primed quantities as $P_yQ_z - P_zQ_y$ is of the unprimed quantities, and the same statement holds for the other corresponding components. Thus the vector product is a proper vector function.

Problem 10a. Reduce the two-dimensional vector $V_x = y$, $V_y = -x$ to the form (10–1). *Ans.* $\mathbf{V} = \mathbf{r}\times\mathbf{k}$.

Problem 10b. Reduce $\mathbf{A}_1\times\mathbf{A}_2$ to the form (10–1).

$$\textit{Ans.}\quad \alpha = \frac{A_1^2\mathbf{r}\cdot\mathbf{A}_1\times\mathbf{A}_2}{r^2A_1^2 - \overline{\mathbf{r}_1\cdot\mathbf{A}_1}^2},\quad \beta = -\frac{\mathbf{r}\cdot\mathbf{A}_1\mathbf{r}\cdot\mathbf{A}_1\times\mathbf{A}_2}{r^2A_1^2 - \overline{\mathbf{r}\cdot\mathbf{A}_1}^2},$$

$$\gamma = \frac{\mathbf{r}\cdot\mathbf{A}_1\mathbf{A}_1\cdot\mathbf{A}_2 - \mathbf{r}\cdot\mathbf{A}_2A_1^2}{r^2A_1^2 - \overline{\mathbf{r}\cdot\mathbf{A}_1}^2}.$$

Problem 10c. What is the most general vector function of position in space the components of which are not functions of the components of any constant vectors? *Ans.* $f(r)\,\mathbf{r}$.

Problem 10d. Express the vector function representing the electric intensity in the field of an electric dipole of moment **A** in the form (10–1).

$$\textit{Ans.}\quad \frac{3\mathbf{r}\cdot\mathbf{A}}{4\pi r^5}\mathbf{r} - \frac{1}{4\pi r^3}\mathbf{A}.$$

11. Differentiation and Integration of a Vector. — Consider a point P whose position vector referred to the origin of a set of fixed axes is

$$\mathbf{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z,$$

and whose coordinates x, y, z are functions of a parameter t. As t varies, P moves along some curved path such as that shown in Fig. 11.

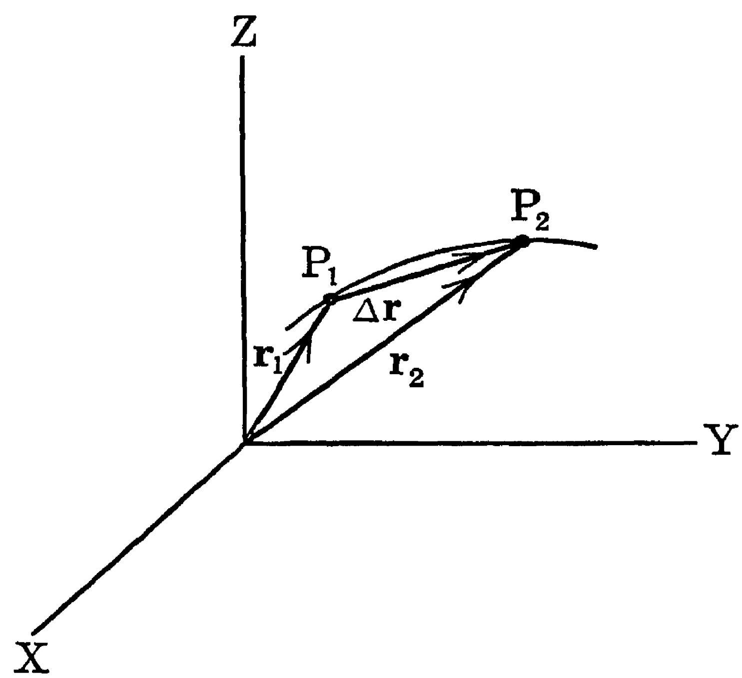

FIG. 11.

Let P_1 and P_2 be two successive positions of P, corresponding to a change Δt in t. Then the displacement of P in the interval Δt is

$$\mathbf{r}_2 - \mathbf{r}_1 = \Delta\mathbf{r} = \boldsymbol{i}\Delta x + \boldsymbol{j}\Delta y + \boldsymbol{k}\Delta z.$$

The direction of this vector element approaches that of the tangent to the curve as P_2 is made to approach P_1. Hence, dividing by Δt and passing to the limit, the *derivative* of $\mathbf{r}$ with respect to t, namely,

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{i}\frac{dx}{dt} + \boldsymbol{j}\frac{dy}{dt} + \boldsymbol{k}\frac{dz}{dt} \qquad (11\text{–}1)$$

is a vector having at every point along the curve the direction of the tangent to the curve.

Proceeding in the same way we obtain the second derivative

$$\frac{d^2\mathbf{r}}{dt^2} = \boldsymbol{i}\frac{d^2x}{dt^2} + \boldsymbol{j}\frac{d^2y}{dt^2} + \boldsymbol{k}\frac{d^2z}{dt^2}, \qquad (11\text{–}2)$$

and so on.

Usually the moving point P discussed above is a point in space and t represents the time, in which case (11–1) gives the velocity $\mathbf{v}$ of the point, and (11–2) gives its acceleration $\mathbf{f} = d\mathbf{v}/dt$.

In dealing with vector derivatives it is necessary to distinguish carefully between the derivative of the magnitude of a vector and the magnitude of the derivative of the vector. Thus,

$$\frac{d}{dt}|\mathbf{r}| = \frac{dr}{dt} = \frac{d}{dt}\sqrt{x^2+y^2+z^2}$$

is not the same as

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2},$$

and there is a similar distinction in the case of the second derivative.

Proceeding in the same way we see that the derivative of any vector function $\mathbf{Q}$ with respect to a parameter t is given by

$$\frac{d\mathbf{Q}}{dt} = \boldsymbol{i}\frac{dQ_x}{dt} + \boldsymbol{j}\frac{dQ_y}{dt} + \boldsymbol{k}\frac{dQ_z}{dt}. \tag{11–3}$$

Next consider the differentation of products involving vectors. Evidently

$$\left.\begin{aligned} \frac{d}{dt}(m\mathbf{Q}) &= m\frac{d\mathbf{Q}}{dt}, \; m \text{ constant,} \\ \frac{d}{dt}(m\mathbf{Q}) &= \frac{dm}{dt}\mathbf{Q}, \; \mathbf{Q} \text{ constant,} \end{aligned}\right\} \tag{11–4}$$

since $m\mathbf{Q}$ is a vector whose rectangular components are mQ_x, mQ_y and mQ_z. On the other hand if m and $\mathbf{Q}$ both are functions of t,

$$\begin{aligned} \frac{d}{dt}(m\mathbf{Q}) &= \boldsymbol{i}\frac{d}{dt}(mQ_x) + \boldsymbol{j}\frac{d}{dt}(mQ_y) + \boldsymbol{k}\frac{d}{dt}(mQ_z) \\ &= \frac{dm}{dt}(\boldsymbol{i}Q_x + \boldsymbol{j}Q_y + \boldsymbol{k}Q_z) + m\left(\boldsymbol{i}\frac{dQ_x}{dt} + \boldsymbol{j}\frac{dQ_y}{dt} + \boldsymbol{k}\frac{dQ_z}{dt}\right) \\ &= \frac{dm}{dt}\mathbf{Q} + m\frac{d\mathbf{Q}}{dt}. \end{aligned} \tag{11–5}$$

Expanding, differentiating and collecting terms in the same manner, we find that

$$\frac{d}{dt}(\mathbf{P}\cdot\mathbf{Q}) = \frac{d\mathbf{P}}{dt}\cdot\mathbf{Q} + \mathbf{P}\cdot\frac{d\mathbf{Q}}{dt}, \tag{11-6}$$

and

$$\frac{d}{dt}(\mathbf{P}\times\mathbf{Q}) = \frac{d\mathbf{P}}{dt}\times\mathbf{Q} + \mathbf{P}\times\frac{d\mathbf{Q}}{dt}. \tag{11-7}$$

Thus differentiation of products of vectors proceeds according to the same rules as differentiation of products of scalars. In (11–7) the order of the vectors must not be altered, of course, without proper change of sign.

The derivative of a vector of constant magnitude but variable direction has a direction perpendicular to that of the vector itself. As an example of this, return to the case of the point P moving along the curved path (Fig. 11) with velocity $\mathbf{v} = d\mathbf{r}/dt$. If $d\lambda$ represents an element of distance measured *along* the curve and $\mathbf{t}_1$ is a unit vector tangent to the curve at P we may write

$$\mathbf{v} = \frac{d\lambda}{dt}\mathbf{t}_1 = v\mathbf{t}_1. \tag{11-8}$$

Here $\mathbf{t}_1$ has the constant magnitude unity, but varies in direction as P moves along the curve. Then

$$\mathbf{f} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{t}_1 + v\frac{d\mathbf{t}_1}{dt}.$$

The change in $\mathbf{t}_1$ as P moves from P_1 to P_2 is shown in Fig. 12, where ρ is the radius of curvature. Evidently the direction of $\Delta\mathbf{t}_1$ is toward the center of curvature C. From similar triangles its magnitude is $\Delta\lambda/\rho$, since $|\mathbf{t}_1| = |\mathbf{t}_1 + \Delta\mathbf{t}_1| = 1$. Thus, if $\boldsymbol{\rho}_1$ is a unit vector directed toward the center of curvature, $\Delta\mathbf{t}_1 = (\Delta\lambda/\rho)\boldsymbol{\rho}_1$. Dividing by Δt and passing to the limit

$$\frac{d\mathbf{t}_1}{dt} = \frac{1}{\rho}\frac{d\lambda}{dt}\boldsymbol{\rho}_1 = \frac{v}{\rho}\boldsymbol{\rho}_1,$$

so that

$$\mathbf{f} = \frac{dv}{dt}\mathbf{t}_1 + \frac{v^2}{\rho}\boldsymbol{\rho}_1. \tag{11-9}$$

The second term on the right of (11–9) is the well-known *centripetal acceleration.*

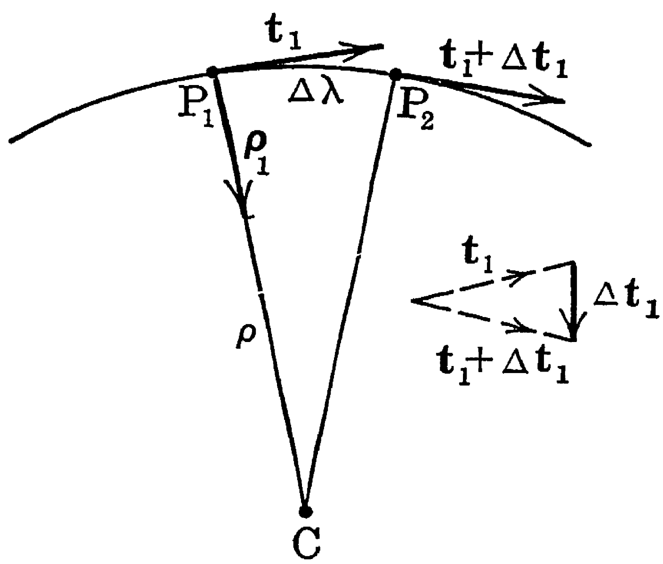

Fig. 12.

Finally, if a vector is a function of several variables such as x, y, z, its *partial derivatives* with respect to these variables have the same significance as in the case of a scalar function, being calculated in each instance with all independent variables except the one in question held constant.

Vector integration is the inverse of differentiation as in scalar analysis. It is usually performed by expressing the vector or vectors involved in terms of their components, and, as in the scalar case, it may be regarded as a process of summation. Thus, consider the integral of $\mathbf{V}$ through the volume τ,

$$\int_\tau \mathbf{V}d\tau = \boldsymbol{i}\int_\tau V_x d\tau + \boldsymbol{j}\int_\tau V_y d\tau + \boldsymbol{k}\int_\tau V_z d\tau. \qquad (11\text{–}10)$$

Now

$$\int_\tau V_x d\tau = \sum_i V_{xi}\Delta\tau_i$$

where V_{xi} is the value of V_x in the ith element of volume $\Delta\tau_i$ and the summation is taken over all elements of volume in τ. There are similar expressions for $\int_\tau V_y d\tau$ and $\int_\tau V_z d\tau$. Hence

$$\begin{aligned}\int_\tau \mathbf{V}d\tau &= \boldsymbol{i}\sum_i V_{xi}\Delta\tau_i + \boldsymbol{j}\sum_i V_{yi}\Delta\tau_i + \boldsymbol{k}\sum_i V_{zi}\Delta\tau_i \\ &= \sum_i(\boldsymbol{i}V_{xi} + \boldsymbol{j}V_{yi} + \boldsymbol{k}V_{zi})\Delta\tau_i \\ &= \sum_i \mathbf{V}_i\Delta\tau_i.\end{aligned}$$

Problem 11a. Show that

$$\frac{d}{dt}(\mathbf{P}\times\mathbf{Q}\cdot\mathbf{R}) = \frac{d\mathbf{P}}{dt}\times\mathbf{Q}\cdot\mathbf{R} + \mathbf{P}\times\frac{d\mathbf{Q}}{dt}\cdot\mathbf{R} + \mathbf{P}\times\mathbf{Q}\cdot\frac{d\mathbf{R}}{dt}.$$

Problem 11b. Show that

$$\frac{d}{dt}\{(\mathbf{P}\times\mathbf{Q})\times\mathbf{R}\} =$$
$$\left(\frac{d\mathbf{P}}{dt}\times\mathbf{Q}\right)\times\mathbf{R} + \left(\mathbf{P}\times\frac{d\mathbf{Q}}{dt}\right)\times\mathbf{R} + (\mathbf{P}\times\mathbf{Q})\times\frac{d\mathbf{R}}{dt}.$$

Problem 11c. If $d\boldsymbol{\lambda}$ is an element of a closed curve in space show that $\oint d\boldsymbol{\lambda} = 0$, where $\oint$ indicates integration completely around the curve.

12. The Gradient. — Let $\Phi\,(x, y, z)$ be a proper scalar function of the coordinates, according to the definition given in article 9. If C is a constant,

$$\Phi(x, y, z) = C \tag{12–1}$$

represents a surface at every point of which the function has the value C. By assigning to C a succession of values we obtain a family of equi-valued surfaces. Confining ourselves to single-valued functions, it is clear that no two of these surfaces can intersect.

Now let us investigate the change in Φ as we move from the point $P_1(x, y, z)$ on the equi-valued surface C to any nearby point $P_2(x + dx, y + dy, z + dz)$, which may be on the surface C, or not, as we choose. If the distance from P_1 to P_2 is $d\lambda$, the displacement involved can be represented by $d\boldsymbol{\lambda} = \boldsymbol{i}\,dx + \boldsymbol{j}\,dy + \boldsymbol{k}\,dz$. The change in Φ is, then,

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = \mathbf{F}\cdot d\boldsymbol{\lambda} \tag{12–2}$$

where $\mathbf{F}$ is defined by

$$\mathbf{F} \equiv \boldsymbol{i}\frac{\partial\Phi}{\partial x} + \boldsymbol{j}\frac{\partial\Phi}{\partial y} + \boldsymbol{k}\frac{\partial\Phi}{\partial z}. \tag{12–3}$$

In article 8 it is shown that changing from the axes XYZ to a new set $X'Y'Z'$ gives

$$\mathbf{F} = \boldsymbol{i}\frac{\partial\Phi}{\partial x} + \boldsymbol{j}\frac{\partial\Phi}{\partial y} + \boldsymbol{k}\frac{\partial\Phi}{\partial z} = \boldsymbol{i}'\frac{\partial\Phi}{\partial x'} + \boldsymbol{j}'\frac{\partial\Phi}{\partial y'} + \boldsymbol{k}'\frac{\partial\Phi}{\partial z'}.$$

Now, since Φ is the same function of x', y', z' and the components A_x', A_y', A_z', etc., of any constant vectors involved, as it is of x, y, z,

A_x, A_y, A_z, etc., each component of **F**, referred to $X'Y'Z'$, is the same function of x', y', z', A_x', A_y', A_z', etc., as it is of x, y, z, A_x, A_y, A_z, etc., when referred to XYZ. Hence **F** is a proper vector function.

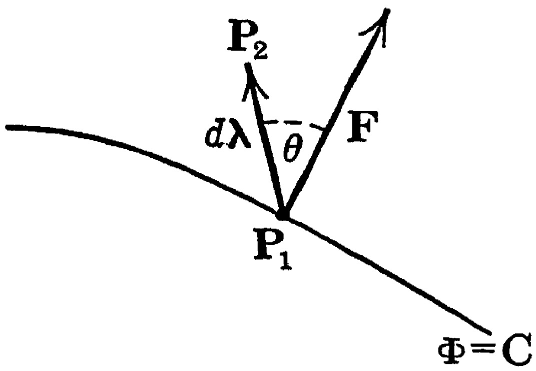

FIG. 13.

We see that the vector **F** has the direction of the normal to C at the point P_1. For suppose P_2 lies in C, so that $d\boldsymbol{\lambda}$ is tangent to the surface. Then $d\Phi = \mathbf{F}\cdot d\boldsymbol{\lambda} = 0$, showing that **F** is perpendicular to $d\boldsymbol{\lambda}$, whatever orientation the latter may have in the tangent plane. To determine the magnitude of **F**, on the other hand, choose P_2 in such a way (Fig. 13) that $d\boldsymbol{\lambda}$ makes the angle θ with the normal to C, that is, with **F**. Then $d\Phi = \mathbf{F}\cdot d\boldsymbol{\lambda} = F\cos\theta\, d\lambda$, giving

$$F\cos\theta = \frac{\partial\Phi}{\partial\lambda}. \qquad (12\text{–}4)$$

Thus, the component of **F** in any direction equals the space rate of increase of Φ in that direction. If $\theta = 0$, so that $d\lambda$ becomes an element dn of the normal, we have

$$F = \frac{\partial\Phi}{\partial n}, \qquad (12\text{–}5)$$

the maximum space rate of increase of Φ.

Because **F** coincides with the maximum space rate of increase of Φ, both in direction and in magnitude, it is called the *gradient* of Φ. The gradient is particularly important in the study of static electric and magnetic fields.

The gradient of Φ is usually symbolized by $\nabla\Phi$, where ∇ (read *del*) is an operator defined from (12–3) as

$$\nabla \equiv \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}. \qquad (12\text{–}6)$$

On changing axes the form of this operator remains unchanged as shown in (8–6), that is,

$$\nabla' = \boldsymbol{i}'\frac{\partial}{\partial x'} + \boldsymbol{j}'\frac{\partial}{\partial y'} + \boldsymbol{k}'\frac{\partial}{\partial z'}. \qquad (12\text{–}7)$$

Hence we may call ∇ a *proper vector operator.*

Evidently we may write (12–4) in the form

$$\boldsymbol{\lambda}_1 \cdot \nabla\Phi = \frac{\partial\Phi}{\partial\lambda}, \tag{12–8}$$

where $\boldsymbol{\lambda}_1$ is a unit vector in the direction of $d\boldsymbol{\lambda}$. Similarly (12–5) becomes

$$\mathbf{n}_1 \cdot \nabla\Phi = \frac{\partial\Phi}{\partial n}, \tag{12–9}$$

where $\mathbf{n}_1$ is a unit vector along the normal.

Problem 12a. Find the gradient of $\Phi = A/\sqrt{x^2 + y^2 + z^2}$ where A is a constant.

Ans. $\nabla\Phi = -\dfrac{A}{(x^2 + y^2 + z^2)^{3/2}}\,(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z).$

13. **The Divergence.** — Since the operator ∇ defined in the preceding article has the formal properties of a vector we may form its product with any vector according to the usual rules. Let

$$\mathbf{V}(x, y, z) = \mathbf{i}V_x(x, y, z) + \mathbf{j}V_y(x, y, z) + \mathbf{k}V_z(x, y, z)$$

be a proper vector function of the coordinates. If we form the scalar product

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{13–1}$$

we obtain a proper scalar function called the *divergence* of $\mathbf{V}$. That it is a *proper* scalar follows from the fact that on changing to a new set of axes,

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x'}{\partial x'} + \frac{\partial V_y'}{\partial y'} + \frac{\partial V_z'}{\partial z'} \tag{13–2}$$

by (12–7) and (8–3). Then, since V_x', V_y', V_z' are the same functions, respectively, of x', y', z' and the components A_x', A_y', A_z', etc., of any constant vectors involved, as V_x, V_y, V_z are of x, y, z, A_x, A_y, A_z, etc., (13–2) is the same function of the primed quantities as (13–1) is of the unprimed quantities.

The physical significance of the divergence is best understood by considering a common case in which $\mathbf{V} \equiv \rho\mathbf{v}$ represents the current per unit cross-section of some material of density ρ moving through space with velocity $\mathbf{v}$. For example, take ρ to be density of electric charge, so that $\rho\mathbf{v}$ is electric current per unit area. Now let us calculate the rate at which charge passes out of a small volume $\Delta\tau = \Delta x\Delta y\Delta z$ (Fig. 14). Consider first the two faces perpendicular to the X axis. As the outward flux through the left-hand face is $-\rho v_x\Delta y\Delta z$, and that through the right-hand face is

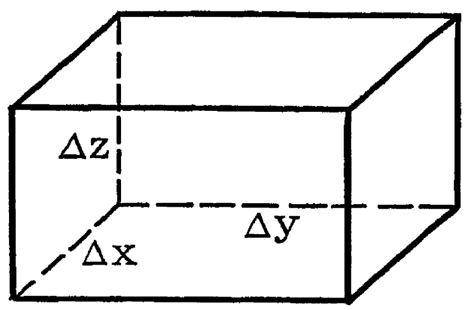

FIG. 14.

$$\left\{\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\Delta x\right\}\Delta y\Delta z,$$

the net outward flux through this pair of faces is

$$\left\{-\rho v_x + \rho v_x + \frac{\partial}{\partial x}(\rho v_x)\Delta x\right\}\Delta y\Delta z = \frac{\partial}{\partial x}(\rho v_x)\Delta x\Delta y\Delta z.$$

Calculating the outward flux through the other two pairs of faces and adding, the total outward flux is

$$\left\{\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right\}\Delta x\Delta y\Delta z = \nabla\cdot\overline{\rho\mathbf{v}}\Delta x\Delta y\Delta z. \quad (13\text{–}3)$$

We see then that $\nabla\cdot\overline{\rho\mathbf{v}}$ represents the rate per unit volume at which charge *diverges* from the region $\Delta\tau$.

Assuming that charge is indestructible, (13–3) must be exactly equal to the rate of diminution of charge in $\Delta\tau$, namely, $-\dfrac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z$. Thus

$$\nabla\cdot\overline{\rho\mathbf{v}} + \frac{\partial\rho}{\partial t} = 0, \quad (13\text{–}4)$$

a relation, known as the *equation of continuity*, which is inherent in Maxwell's electromagnetic equations. The *bar* or *vinculum* over the product $\rho\mathbf{v}$ indicates that ∇ as a *differential* operator acts on both ρ and $\mathbf{v}$.

As implied in the preceding paragraph it is necessary to specify the quantities which are to be differentiated when more than one

function of the coordinates follows the operator ∇. If no vinculum (or parentheses) appears in such a case we shall understand that ∇ acts differentially only on the quantity immediately following it. If a vinculum does appear, on the other hand, all quantities under it are to be differentiated after the vector expansion has been performed. Thus

$$\nabla \cdot \rho \mathbf{v} = \mathbf{v} \cdot \nabla \rho = v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z},$$

$$\nabla \cdot \mathbf{v} \rho = \rho \nabla \cdot \mathbf{v} = \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right),$$

while, as in (13–4),

$$\nabla \cdot \overline{\rho \mathbf{v}} = \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}.$$

Similarly in evaluating expressions such as $\nabla \cdot \mathbf{P} \times \mathbf{Q}$ or $\nabla \cdot \mathbf{PQ}$ only derivatives of P_x, P_y and P_z are taken, while in the case of $\nabla \cdot \overline{\mathbf{P} \times \mathbf{Q}}$ or $\nabla \cdot \overline{\mathbf{PQ}}$, the components of both $\mathbf{P}$ and $\mathbf{Q}$ are differentiated.

Problem 13a. Show by actual transformation that on changing axes

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \frac{\partial V_x'}{\partial x'} + \frac{\partial V_y'}{\partial y'} + \frac{\partial V_z'}{\partial z'}.$$

Problem 13b. Find the equation of continuity for an incompressible fluid.

Ans. $\nabla \cdot \mathbf{v} = 0$.

Problem 13c. Show that $\nabla \cdot \overline{\mathbf{P} \times \mathbf{Q}} = \nabla \cdot \mathbf{P} \times \mathbf{Q} - \nabla \cdot \mathbf{Q} \times \mathbf{P}$.

Problem 13d. Show that $\nabla \cdot \overline{\mathbf{PQ}} = \mathbf{Q} \nabla \cdot \mathbf{P} + \mathbf{P} \cdot \nabla \mathbf{Q}$.

14. The Curl.—If $\mathbf{V}(x, y, z)$ is a proper vector function of the coordinates we may form the vector product of ∇ and $\mathbf{V}$ so as to obtain another proper vector function known as the *curl* or *rotation* of $\mathbf{V}$. This is

$$\nabla \times \mathbf{V} = \boldsymbol{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \boldsymbol{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \boldsymbol{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}. \tag{14–1}$$

It is seen at once that this is a *proper* vector function, for, transforming to a new set of axes as in the case of (13–1) and (13–2), the form of $\nabla \times \mathbf{V}$ remains the same. So each of its components, when referred to the axes $X'Y'Z'$, is the same function of all primed quantities as it is of the corresponding unprimed quantities when referred to XYZ.

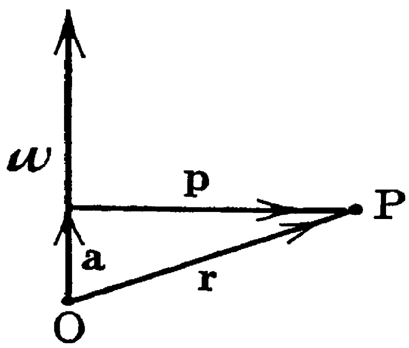

Fig. 15.

The designation *rotation* is due to an application of (14–1) to mechanics. Suppose a rigid body rotates with angular velocity $\boldsymbol{\omega}$ about a fixed point O (Fig. 15) which we will take as origin of a set of rectangular axes. The linear velocity of any point P having a position vector $\mathbf{r}$ is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{p} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a}) = \boldsymbol{\omega} \times \mathbf{r}$. Taking the curl of both sides of this equation

$$\nabla \times \mathbf{v} = \nabla \times (\overline{\boldsymbol{\omega} \times \mathbf{r}}) = \nabla \cdot \overline{\mathbf{r}\boldsymbol{\omega}} - \nabla \cdot \overline{\boldsymbol{\omega}\mathbf{r}} \tag{14–2}$$

from (7–9), where the vinculum indicates, as in the preceding article, that ∇ acts differentially on both following vectors. However, in this case $\boldsymbol{\omega}$ is not a function of the coordinates so that (14–2) reduces to

$$\nabla \times \mathbf{v} = \boldsymbol{\omega}\nabla \cdot \mathbf{r} - \boldsymbol{\omega} \cdot \nabla \mathbf{r}.$$

Now

$$\boldsymbol{\omega}\nabla \cdot \mathbf{r} = \boldsymbol{\omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3\boldsymbol{\omega},$$

and

$$\boldsymbol{\omega} \cdot \nabla \mathbf{r} = \omega_x \frac{\partial \mathbf{r}}{\partial x} + \omega_y \frac{\partial \mathbf{r}}{\partial y} + \omega_z \frac{\partial \mathbf{r}}{\partial z} = \boldsymbol{i}\omega_x + \boldsymbol{j}\omega_y + \boldsymbol{k}\omega_z = \boldsymbol{\omega},$$

so that, finally,

$$\nabla \times \mathbf{v} = 3\boldsymbol{\omega} - \boldsymbol{\omega} = 2\boldsymbol{\omega}. \tag{14–3}$$

Thus, the curl or rotation of the linear velocity is twice the angular velocity.

Problem 14a. Show by actual transformation that

$$\boldsymbol{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$
$$= \boldsymbol{i}'\left(\frac{\partial V_z'}{\partial y'} - \frac{\partial V_y'}{\partial z'}\right) + \boldsymbol{j}'\left(\frac{\partial V_x'}{\partial z'} - \frac{\partial V_z'}{\partial x'}\right) + \boldsymbol{k}'\left(\frac{\partial V_y'}{\partial x'} - \frac{\partial V_x'}{\partial y'}\right).$$

Problem 14b. Establish the identity $\nabla \times (\overline{\mathbf{P} \times \mathbf{Q}}) = \nabla\cdot\overline{\mathbf{QP}} - \nabla\cdot\overline{\mathbf{PQ}}$ by actual expansion.

15. Successive Applications of ∇. — It is possible to form scalar and vector products in which the operator ∇ appears more than once. For example, the divergence of the gradient of a proper scalar function is

$$\nabla\cdot\nabla\Phi = \nabla\cdot\left(\boldsymbol{i}\frac{\partial\Phi}{\partial x} + \boldsymbol{j}\frac{\partial\Phi}{\partial y} + \boldsymbol{k}\frac{\partial\Phi}{\partial z}\right) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}. \qquad (15\text{–}1)$$

As the same result is obtained by allowing the scalar product

$$\nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (15\text{–}2)$$

to act on Φ as an operator, we may think of $\nabla\cdot\nabla$, which is called the *Laplacian*, as a *proper scalar operator*. This allows us to write

$$\nabla\cdot\nabla\mathbf{V} = \frac{\partial^2\mathbf{V}}{\partial x^2} + \frac{\partial^2\mathbf{V}}{\partial y^2} + \frac{\partial^2\mathbf{V}}{\partial z^2}, \qquad (15\text{–}3)$$

an expression which appears in the analysis of wave motion.

Proceeding in a similar manner the curl of $\nabla\Phi$ is

$$\nabla \times \nabla\Phi = \nabla \times \left(\boldsymbol{i}\frac{\partial\Phi}{\partial x} + \boldsymbol{j}\frac{\partial\Phi}{\partial y} + \boldsymbol{k}\frac{\partial\Phi}{\partial z}\right) = 0. \qquad (15\text{–}4)$$

As before the result may be obtained by regarding $\nabla \times \nabla$ as an operator. Since the vector product of any vector by itself vanishes, we have $\nabla \times \nabla\Phi = 0$ at once.

A proper vector function of position in space whose curl vanishes in a region τ is said to be *irrotational* in that region. Evidently the gradient of any proper scalar function is an irrotational vector.

Again, we may take the divergence of the curl of any proper vector function $\mathbf{V}$. Thus

$$\nabla\cdot\nabla \times \mathbf{V} = \nabla \times \nabla\cdot\mathbf{V} = 0, \qquad (15\text{–}5)$$

since the dot and cross may be interchanged in a triple scalar product.

A proper vector function whose divergence vanishes in a region τ is said to be *solenoidal* in that region. Thus the curl of any proper vector function is a solenoidal vector.

Finally, there remains to be considered the curl of the curl of $\mathbf{V}$.

As this is a triple vector product it may be expanded in the usual manner and we have

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla \mathbf{V}. \tag{15-6}$$

The left-hand side of this equation is usually written $\nabla \times \nabla \times \mathbf{V}$ without parentheses, since $(\nabla \times \nabla) \times \mathbf{V}$, with which it might be confused, is identically zero and has no importance.

Products in which ∇ appears more than twice require discussion only to the extent of pointing out that, since the order in which partial derivatives are taken is immaterial, ∇ and $\nabla \cdot \nabla$ are commutative. Thus

$$\left.\begin{aligned} \nabla(\nabla \cdot \nabla \Phi) &= \nabla \cdot \nabla(\nabla \Phi), \\ \nabla \cdot (\nabla \cdot \nabla \mathbf{V}) &= \nabla \cdot \nabla(\nabla \cdot \mathbf{V}), \\ \nabla \times (\nabla \cdot \nabla \mathbf{V}) &= \nabla \cdot \nabla(\nabla \times \mathbf{V}). \end{aligned}\right\} \tag{15-7}$$

The parentheses have been retained for the sake of clarity, although they are not essential as no real ambiguity can arise through their omission.

Problem 15a. Establish the identities (15–5) and (15–6) by actual expansion.

Problem 15b. Find the value of $\nabla \cdot \nabla \left(\frac{1}{r}\right)$ where $r = \sqrt{x^2 + y^2 + z^2}$.

Ans. 0.

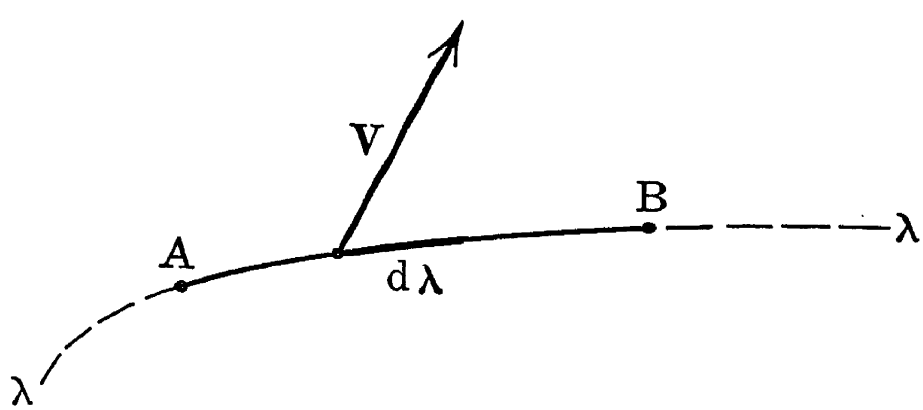

FIG. 16.

16. Line and Surface Integrals. — Let $\mathbf{V}$ be a proper vector function of the coordinates. The integral of the tangential component of $\mathbf{V}$ along any curve λ (Fig. 16) is called the *line integral* of $\mathbf{V}$. Thus, if $d\boldsymbol{\lambda} = \boldsymbol{i}\,dx + \boldsymbol{j}\,dy + \boldsymbol{k}\,dz$ is a vector element of the curve, the line integral of $\mathbf{V}$ from A to B is

$$\int_A^B \mathbf{V} \cdot d\boldsymbol{\lambda} = \int_A^B (V_x\,dx + V_y\,dy + V_z\,dz). \tag{16-1}$$

The path of integration may, of course, be a closed curve, in which case the line integral, written as $\oint \mathbf{V}\cdot d\boldsymbol{\lambda}$, becomes a *loop integral.*

In general, the value of the line integral depends not only on the end points A and B but also on the exact path followed in the integration, and in the case of a closed path $\oint \mathbf{V}\cdot d\boldsymbol{\lambda}$ does not vanish. However, if $\mathbf{V}$ is the gradient of some scalar function Φ, so that

$$V_x = \frac{\partial\Phi}{\partial x}, \quad V_y = \frac{\partial\Phi}{\partial y}, \quad V_z = \frac{\partial\Phi}{\partial z},$$

then

$$\int_A^B \mathbf{V}\cdot d\boldsymbol{\lambda} = \int_A^B \left(\frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz\right)$$
$$= \int_A^B d\Phi = \Phi_B - \Phi_A. \tag{16–2}$$

In this case the value of the line integral depends on the end points alone, and, provided Φ is single-valued,

$$\oint \mathbf{V}\cdot d\boldsymbol{\lambda} = 0. \tag{16–3}$$

Conversely, suppose the line integral of $\mathbf{V}$ around *every* closed path vanishes. Then for any path $ACBD$ (Fig. 17)

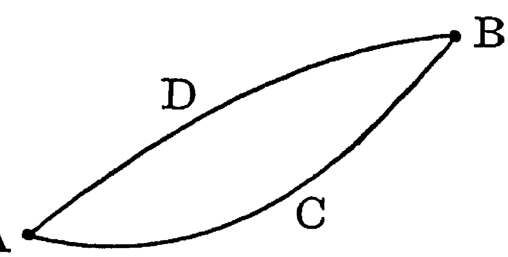

Fig. 17.

$$\oint \mathbf{V}\cdot d\boldsymbol{\lambda} = \int_{ACB} \mathbf{V}\cdot d\boldsymbol{\lambda} + \int_{BDA} \mathbf{V}\cdot d\boldsymbol{\lambda}$$
$$= \int_{ACB} \mathbf{V}\cdot d\boldsymbol{\lambda} - \int_{ADB} \mathbf{V}\cdot d\boldsymbol{\lambda} = 0,$$

and

$$\int_{ACB} \mathbf{V}\cdot d\boldsymbol{\lambda} = \int_{ADB} \mathbf{V}\cdot d\boldsymbol{\lambda}.$$

Since the integral from A to B is independent of the path it can depend only on the end points and we must have

$$\int_A^B \mathbf{V}\cdot d\boldsymbol{\lambda} = \Phi_B - \Phi_A = \int_A^B d\Phi, \tag{16–4}$$

where Φ is some single-valued function of the coordinates. Furthermore, as $d\Phi = (\nabla\Phi)\cdot d\boldsymbol{\lambda}$, this reduces to

$$\int_A^B (\mathbf{V} - \nabla\Phi)\cdot d\boldsymbol{\lambda} = 0. \tag{16–5}$$

Now, as the points A and B are arbitrary, the line integral of $(\mathbf{V} - \nabla\Phi)$ must vanish for every possible path, even for one of infinitesimal length. Hence (16–5) can be true only if $(\mathbf{V} - \nabla\Phi)$ itself vanishes everywhere. Thus

$$\mathbf{V} = \nabla\Phi, \tag{16–6}$$

which shows that when the line integral of $\mathbf{V}$ around every closed path is zero, $\mathbf{V}$ must be the gradient of some scalar function.

A common line integral is the electromotive force

$$\mathcal{E} = \int_A^B \mathbf{E}\cdot d\boldsymbol{\lambda},$$

which is the work done by an electric field $\mathbf{E}$ on a unit positive charge moving through the field from A to B. In general the electromotive force depends on the path chosen, but, if the field is electrostatic, $\mathbf{E}$ is the gradient of a scalar potential and $\mathcal{E}$ is then equal to the *potential difference* of the end points, regardless of the path followed.

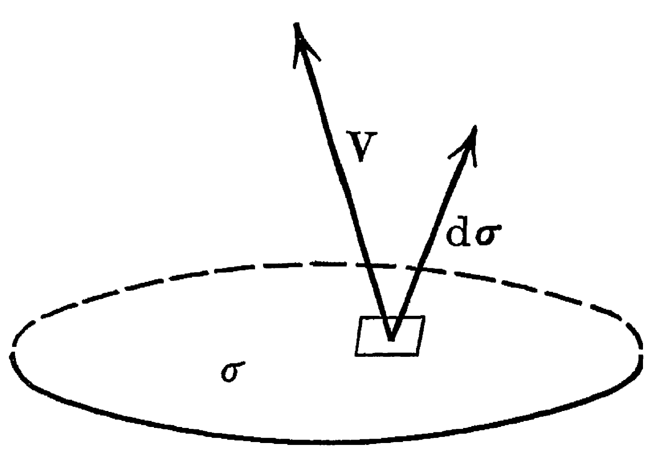

Fig. 18.

Next let us consider the integral of the normal component of a proper vector function $\mathbf{V}$ over a surface σ (Fig. 18). This is known as the *surface integral* of $\mathbf{V}$. Thus, if $d\boldsymbol{\sigma} = \mathbf{i}d\sigma_x + \mathbf{j}d\sigma_y + \mathbf{k}d\sigma_z$ is a vector element of the surface (Art. 3), the surface integral of $\mathbf{V}$ over σ is

$$\int_\sigma \mathbf{V}\cdot d\boldsymbol{\sigma} = \int_\sigma (V_x d\sigma_x + V_y d\sigma_y + V_z d\sigma_z). \tag{16–7}$$

The foregoing integral is also called the *flux* of $\mathbf{V}$ through the

surface σ. For suppose that $\mathbf{V} = \rho\mathbf{v}$ gives the quantity of material of density ρ passing through unit cross-section in unit time as in article 13. Then $\mathbf{V}\cdot d\boldsymbol{\sigma}$ is the rate of passage of material through the element of area $d\boldsymbol{\sigma}$ and $\int_\sigma \mathbf{V}\cdot d\boldsymbol{\sigma}$ is the total flux through σ. If ρ is charge density, $\rho\mathbf{v}$ is current density and the integral then gives the total current through the surface.

17. Gauss' Theorem. — This theorem states that the volume integral of the divergence of a finite, continuous and single-valued vector function $\mathbf{V}$ of position in space, taken over any volume τ, is equal to the surface integral of $\mathbf{V}$ taken over the closed surfaces bounding the volume τ, that is,

$$\int_\tau \nabla\cdot\mathbf{V}d\tau = \int_\sigma \mathbf{V}\cdot d\boldsymbol{\sigma}, \tag{17–1}$$

where $d\tau = dx\,dy\,dz$ is an element of the volume τ, and the vector element of surface $d\boldsymbol{\sigma}$ has the direction of the outward-drawn normal in accord with the convention of article 3. This theorem implies that, if the integrand has the form of the divergence of a vector, the value of a volume integral depends only on the values of the vector over the surfaces bounding the volume, and not at all on the values of the vector at points in the interior.

We shall prove Gauss' theorem first for a simply connected region inside which the derivatives of $\mathbf{V}$ as well as the function itself are continuous. We have

$$\int_\tau \nabla\cdot\mathbf{V}d\tau = \iiint \frac{\partial V_x}{\partial x}\,dx\,dy\,dz$$
$$+\iiint \frac{\partial V_y}{\partial y}\,dx\,dy\,dz + \iiint \frac{\partial V_z}{\partial z}\,dx\,dy\,dz.$$

Consider the first integral on the right. Integrating with respect to x, that is, along a strip of cross-section $dy\,dz$ extending from P_1 to P_2 (Fig. 19),

$$\iiint \frac{\partial V_x}{\partial x}\,dx\,dy\,dz = \iint \{V_x(x_2, y, z) - V_x(x_1, y, z)\}dy\,dz,$$

where x_1, y, z are the coordinates of P_1 and x_2, y, z those of P_2. Now at P_1, $d\sigma_x = -dy\,dz$, while at P_2, $d\sigma_x = dy\,dz$. So

$$\iiint \frac{\partial V_x}{\partial x}\,dx\,dy\,dz = \int V_x(x_2, y, z)d\sigma_x + \int V_x(x_1, y, z)d\sigma_x,$$

where the first surface integral is taken over the right-hand part of the surface σ and the second over the left-hand part. Therefore

$$\iiint \frac{\partial V_x}{\partial x}\,dx\,dy\,dz = \int_\sigma V_x d\sigma_x,$$

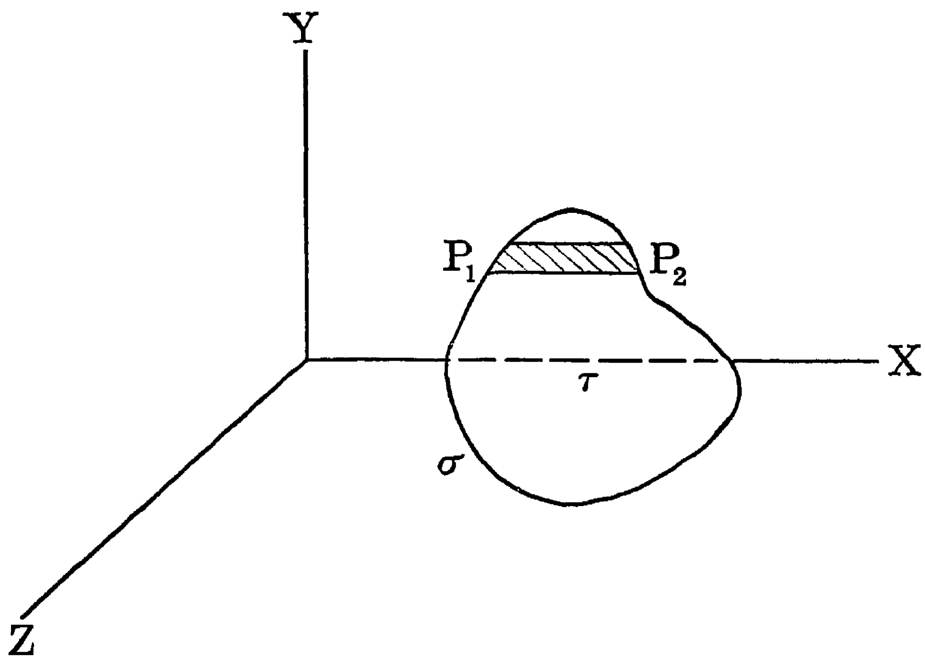

FIG. 19.

where the surface integral is evaluated over the entire surface. Similarly

$$\iiint \frac{\partial V_y}{\partial y}\,dx\,dy\,dz = \int_\sigma V_y d\sigma_y$$

and

$$\iiint \frac{\partial V_z}{\partial z}\,dx\,dy\,dz = \int_\sigma V_z d\sigma_z.$$

Adding these three equations we get Gauss' theorem

$$\int_\tau \nabla\cdot\mathbf{V}d\tau = \int_\sigma (V_x d\sigma_x + V_y d\sigma_y + V_z d\sigma_z) = \int_\sigma \mathbf{V}\cdot d\boldsymbol{\sigma}.$$

If the volume τ is bounded by two surfaces σ_1 and σ_2, as in Fig. 20, we can divide it into two simply connected regions τ_1 and τ_2 by means

of the surface *ABCD*. Applying Gauss' theorem to each of these regions and adding, we find that the volume integral over τ is equal to the sum of two surface integrals. But the portion of the one surface integral over *ABCD* is annulled by that of the other, since the outward drawn normal in the first is opposite to that in the second. Hence we are left with the sum of the surface integrals over σ_1 and σ_2, the positive normal in each being that directed outward from the volume τ. Evidently this device may be applied to a volume bounded by any finite number of surfaces, in which case the surface integral must be evaluated over all the surfaces bounding the volume.

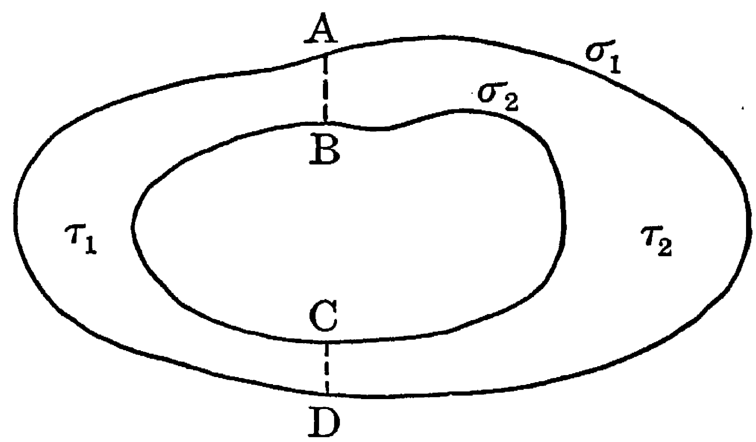

FIG. 20.

Finally, if the derivatives of **V** are discontinuous across a surface σ' lying inside the simply connected region τ (Fig. 21), we may divide τ into two parts, τ_1 and τ_2, separated by the surface σ'. Applying Gauss' theorem to τ_1 we obtain the sum of a surface integral over σ and one over σ'. Similarly the volume integral over τ_2 is equal to a surface integral over σ'. Provided **V** is continuous across σ' the one integral over this surface is the negative of the other, since the outward drawn normals in the two cases have opposite directions. Hence the volume integral over the entire region $\tau = \tau_1 + \tau_2$ is equal to the surface integral over σ.

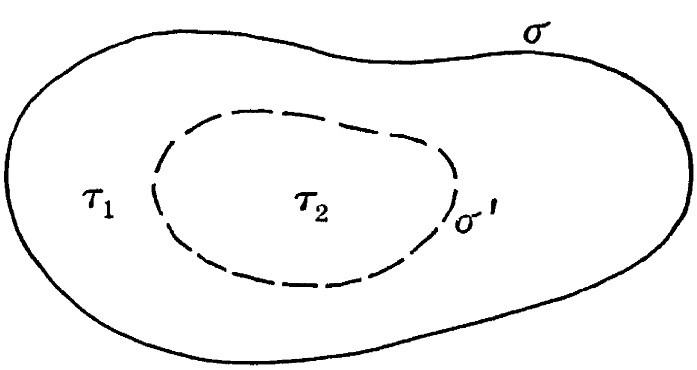

FIG. 21.

Since Gauss' theorem holds for a vector function **V** the derivatives of which are discontinuous over a surface lying in the region of integration, we are at liberty to apply it to a region divided into subregions in each of which **V** is represented by a different function, provided the functions employed in two adjoining subregions give identical values of **V** on the surface separating them. Such an instance occurs in the electric field of a sphere of radius a

with a charge q uniformly distributed throughout its volume. For $r \leqslant a$, the electric intensity is $(q/4\pi a^3)\mathbf{r}$, while for $r \geqslant a$ it is $(q/4\pi r^3)\mathbf{r}$. In fact, whenever we speak of a point charge or a surface charge, we shall understand a small sphere or a thin layer inside which the charge density is large but finite. Hence the field will be everywhere finite and continuous, so that Gauss' theorem can be applied to the electric intensity.

As noted in article 16, the product $\mathbf{V}\cdot d\boldsymbol{\sigma}$ is called the flux of the vector $\mathbf{V}$ through the surface element $d\boldsymbol{\sigma}$. If $\mathbf{E}$ represents the electric intensity, $\mathbf{E}\cdot d\boldsymbol{\sigma}$ is the *electric flux*, and if $\mathbf{H}$ represents the magnetic intensity, $\mathbf{H}\cdot d\boldsymbol{\sigma}$ is the *magnetic flux* through $d\boldsymbol{\sigma}$.

A useful corollary of Gauss' theorem known as Green's theorem states that

$$\int_\tau (u\nabla\cdot\nabla v - v\nabla\cdot\nabla u)d\tau = \int_\sigma (u\nabla v - v\nabla u)\cdot d\boldsymbol{\sigma}, \qquad (17\text{–}2)$$

where u and v are finite, continuous and single-valued scalar functions of the coordinates having derivatives possessing the same properties. To prove Green's theorem we start with the identities

$$\nabla\cdot\overline{u\nabla v} = \nabla u\cdot\nabla v + u\nabla\cdot\nabla v,$$

$$\nabla\cdot\overline{v\nabla u} = \nabla v\cdot\nabla u + v\nabla\cdot\nabla u.$$

Subtracting, integrating over the volume τ and converting the integral on the left into a surface integral by Gauss' theorem, we have Green's theorem.

Other corollaries of Gauss' theorem, which hold for vector functions to which Gauss' theorem is applicable, are the following:

$$\int_\tau \nabla\Phi d\tau = \int_\sigma \Phi d\boldsymbol{\sigma}, \qquad (17\text{–}3)$$

$$\int_\tau \nabla\cdot\overline{\mathbf{UV}}d\tau = \int_\sigma \mathbf{VU}\cdot d\boldsymbol{\sigma}, \qquad (17\text{–}4)$$

$$\int_\tau \nabla\times\mathbf{V}d\tau = -\int_\sigma \mathbf{V}\times d\boldsymbol{\sigma}. \qquad (17\text{–}5)$$

In the first, Φ may be any scalar function, such as $\mathbf{U}\cdot\mathbf{V}$.

To prove the first relation,

$$\int_\tau \nabla \Phi d\tau = i\int_\tau \nabla \cdot \overline{i\Phi} d\tau + \cdots$$

$$= i\int_\sigma \Phi i \cdot d\boldsymbol{\sigma} + \cdots$$

$$= \int_\sigma \Phi d\boldsymbol{\sigma};$$

o prove the second,

$$\int_\tau \nabla \cdot \overline{\mathbf{UV}} d\tau = i\int_\tau \nabla \cdot \overline{\mathbf{U}V_x} d\tau + \cdots$$

$$= i\int_\sigma V_x \mathbf{U} \cdot d\boldsymbol{\sigma} + \cdots$$

$$= \int_\sigma \mathbf{VU} \cdot d\boldsymbol{\sigma};$$

.nd to prove the third,

$$\int_\tau \nabla \times \mathbf{V} d\tau = i\int_\tau \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) d\tau + \cdots$$

$$= i\left\{\int_\tau \nabla \cdot \overline{jV_z} d\tau - \int_\tau \nabla \cdot \overline{kV_y} d\tau\right\} + \cdots$$

$$= i\int_\sigma (V_z d\sigma_y - V_y d\sigma_z) + \cdots$$

$$= -\int_\sigma \mathbf{V} \times d\boldsymbol{\sigma}.$$

18. Stokes' Theorem. — This theorem states that the surface ntegral of the curl of a finite, continuous and single-valued vector unction **V** of position in space, taken over any surface σ, is equal to

the line integral of **V** along the closed curve or curves bounding the surface σ, that is,

$$\int_\sigma \nabla \times \mathbf{V} \cdot d\boldsymbol{\sigma} = \oint \mathbf{V} \cdot d\boldsymbol{\lambda}, \tag{18-1}$$

where $d\boldsymbol{\lambda}$ is a vector element of the periphery taken in the sense of rotation of a right-handed screw advancing from the negative to the positive side of the surface, in accord with the convention of article 3. This theorem implies that, if the integrand has the form of the curl of a vector, the value of the surface integral depends only upon the values of the vector along the periphery, and therefore that the integral has the same value over all surfaces having the same periphery.

We shall prove Stokes' theorem first for a simply connected surface on which the derivatives of **V** as well as the function itself are continuous. Since

$$\nabla \times \mathbf{V} = i\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + j\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + k\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right),$$

it follows that

$$\int_\sigma \nabla \times \mathbf{V} \cdot d\boldsymbol{\sigma} = \int_\sigma \left(\frac{\partial V_x}{\partial z}\, d\sigma_y - \frac{\partial V_x}{\partial y}\, d\sigma_z\right)$$
$$+ \int_\sigma \left(\frac{\partial V_y}{\partial x}\, d\sigma_z - \frac{\partial V_y}{\partial z}\, d\sigma_x\right) + \int_\sigma \left(\frac{\partial V_z}{\partial y}\, d\sigma_x - \frac{\partial V_z}{\partial x}\, d\sigma_y\right).$$

Let P_1P_2 (Fig. 22) be the trace of the surface σ on a plane parallel to the YZ coordinate plane at a distance x from the origin. We will integrate the first integral on the right of the equation above along a strip of the surface extending from P_1 to P_2 lying between planes parallel to the YZ coordinate plane at distances x and $x + dx$ from the origin. The figure is drawn so that both y and z increase as we proceed from P_1 to P_2. The Y component of $d\boldsymbol{\sigma}$ is positive, and therefore $d\sigma_y = dz\, dx$, but the Z component is negative and so $d\sigma_z = -\, dy\, dx$. Therefore

$$\int_\sigma \left(\frac{\partial V_x}{\partial z}\, d\sigma_y - \frac{\partial V_x}{\partial y}\, d\sigma_z\right) = \iint \left(\frac{\partial V_x}{\partial z}\, dz + \frac{\partial V_x}{\partial y}\, dy\right) dx.$$

But, as x remains constant for integration along the strip P_1P_2,

$$dV_x = \frac{\partial V_x}{\partial y}\,dy + \frac{\partial V_x}{\partial z}\,dz.$$

So

$$\int_\sigma \left(\frac{\partial V_x}{\partial z}\,d\sigma_y - \frac{\partial V_x}{\partial y}\,d\sigma_z\right) = \int dx \int_{P_1}^{P_2} dV_x$$

$$= \int V_x(x, y_2, z_2)dx - \int V_x(x, y_1, z_1)dx,$$

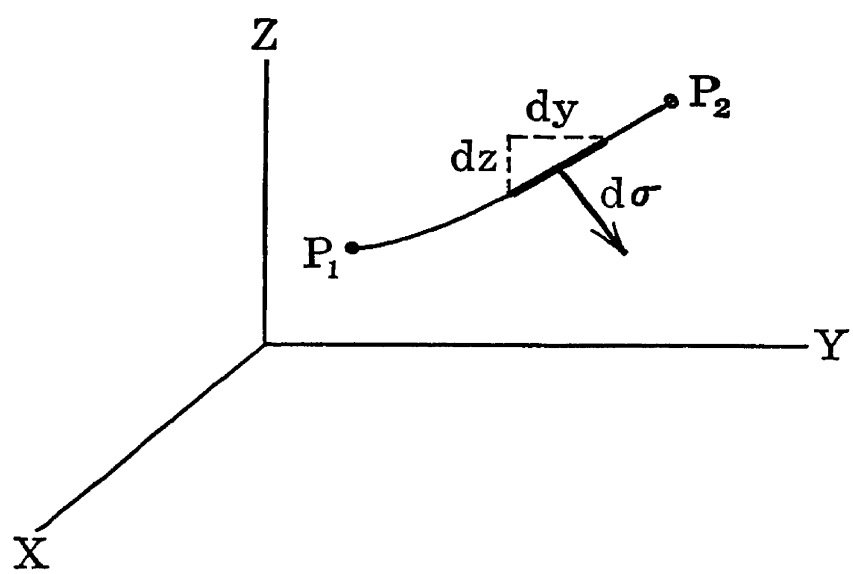

FIG. 22.

where x, y_1, z_1 are the coordinates of P_1 and x, y_2, z_2 those of P_2. At P_1 the sense in which the periphery is described is *into* the paper, whereas at P_2 it is *out from* the paper. So at P_1, $d\lambda_x = -\,dx$ and at P_2, $d\lambda_x = dx$. Therefore

$$\int_\sigma \left(\frac{\partial V_x}{\partial z}\,d\sigma_y - \frac{\partial V_x}{\partial y}\,d\sigma_z\right) = \int V_x(x, y_2, z_2)d\lambda_x + \int V_x(x, y_1, z_1)d\lambda_x,$$

where the first line integral is taken over the right-hand portion of the periphery and the second over the left-hand portion. Consequently

$$\int_\sigma \left(\frac{\partial V_x}{\partial z}\,d\sigma_y - \frac{\partial V_x}{\partial y}\,d\sigma_z\right) = \oint V_x d\lambda_x,$$

the line integral being taken all the way around the periphery. Similarly

$$\int_\sigma \left(\frac{\partial V_y}{\partial x} d\sigma_z - \frac{\partial V_y}{\partial z} d\sigma_x \right) = \oint V_y d\lambda_y,$$

and

$$\int_\sigma \left(\frac{\partial V_z}{\partial y} d\sigma_x - \frac{\partial V_z}{\partial x} d\sigma_y \right) = \oint V_z d\lambda_z.$$

Adding these three equations we obtain Stokes' theorem

$$\int_\sigma \nabla \times \mathbf{V} \cdot d\boldsymbol{\sigma} = \oint (V_x d\lambda_x + V_y d\lambda_y + V_z d\lambda_z) = \oint \mathbf{V} \cdot d\boldsymbol{\lambda}.$$

If the surface σ is bounded by two curves λ_1 and λ_2, as in Fig. 23, we can divide it into two simply connected surfaces σ_1 and σ_2 by the

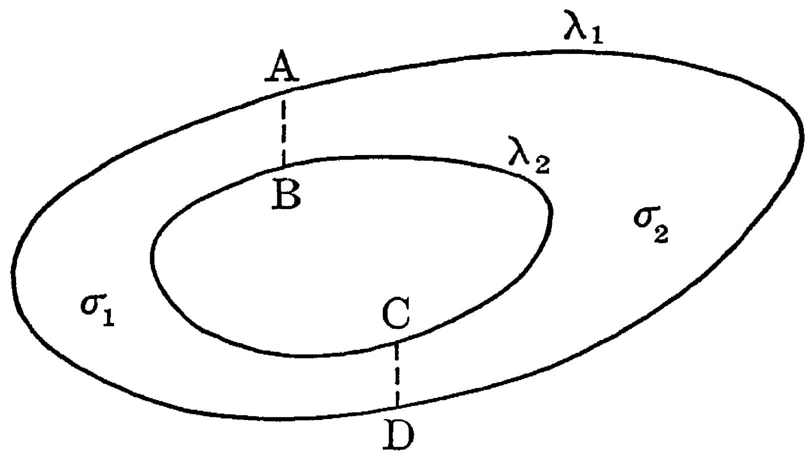

FIG. 23.

curves AB and CD. Applying Stokes' theorem to each of these surfaces and adding, we find that the surface integral over σ is equal to the sum of two line integrals. But the portions of the one line integral over AB and CD are annulled by those of the other, since these curves are described in opposite senses in the two cases. Hence we are left with the sum of the line integrals over λ_1 and λ_2, both being described in the positive sense relative to the surface. Evidently this device may be applied to a surface bounded by any finite number of closed curves. Stokes' theorem holds in any such case, provided the line integral is evaluated over all the curves bounding the surface σ in the positive sense relative to the adjacent surface.

Finally, if the derivatives of **V** are discontinuous over a curve λ' lying on the surface σ (Fig. 24) we may divide σ into two parts, σ_1 and σ_2, separated by the curve λ'. Applying Stokes' theorem to σ_1 we obtain the sum of a line integral over λ and one over λ'. Similarly the surface integral over σ_2 is equal to a line integral over λ'. Provided **V** is continuous across λ' the one integral along this curve is the negative of the other, since they are described in opposite senses. Hence the surface integral over the entire surface $\sigma = \sigma_1 + \sigma_2$ is equal to the line integral over λ.

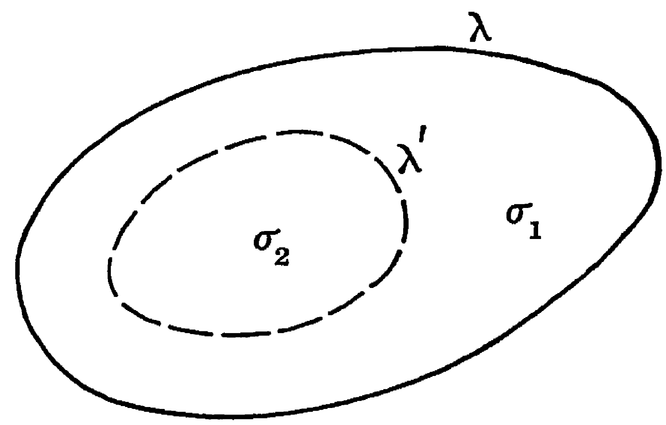

FIG. 24.

Since Stokes' theorem holds for a vector function **V** the derivatives of which are discontinuous, we are at liberty to apply it to a region divided into subregions in each of which **V** is represented by a different function, provided the functions employed in two adjoining subregions give identical values of **V** on the surface separating them. Such an instance occurs in the magnetic field of a straight current **i** of cross-sectional radius a. For $r \leqslant a$, the magnetic intensity is $2\,\mathbf{i} \times \mathbf{r}/4\pi a^2 c$, while for $r \geqslant a$ it is $2\,\mathbf{i} \times \mathbf{r}/4\pi r^2 c$. In fact whenever we speak of a linear current we shall understand a tube of small cross-section inside which the current density is large but finite. Hence the field will be everywhere finite and continuous, so that Stokes' theorem can be applied to the magnetic intensity.

The product $\mathbf{V}\cdot d\boldsymbol{\lambda}$ may be called the *vectormotive force* of the vector **V** along the line element $d\boldsymbol{\lambda}$. If **E** represents the electric intensity, $\mathbf{E}\cdot d\boldsymbol{\lambda}$ is the *electromotive force*, and if **H** represents the magnetic intensity, $\mathbf{H}\cdot d\boldsymbol{\lambda}$ is the *magnetomotive force*, along $d\boldsymbol{\lambda}$.

It should be noticed that when the surface to which Stokes' law is applied is a closed surface, the length of the periphery is zero, and hence

$$\int_\sigma \nabla \times \mathbf{V}\cdot d\boldsymbol{\sigma} = 0.$$

If $\nabla \times \mathbf{V}$ vanishes everywhere in a spacial region, **V** can be expressed as the gradient of a scalar function of position in space in that region. For under these circumstances the line integral of **V** around every closed curve vanishes, and hence **V** is the gradient of

some scalar function, as was shown in article 16. Now an irrotational vector is, by definition, one whose curl vanishes in the region under consideration. So every irrotational vector is the gradient of some scalar function of position in space.

Two corollaries of Stokes' theorem, which hold for vector functions to which this theorem is applicable, are sometimes useful. The first states that

$$\oint \mathbf{V} \times d\boldsymbol{\lambda} = \int_\sigma \nabla \cdot \mathbf{V} d\boldsymbol{\sigma} - \int_\sigma \nabla \mathbf{V} \cdot d\boldsymbol{\sigma}. \qquad (18\text{–}2)$$

To prove this we have

$$\begin{aligned}
\oint \mathbf{V} \times d\boldsymbol{\lambda} &= \boldsymbol{i}\oint (V_y d\lambda_z - V_z d\lambda_y) + \cdots \\
&= \boldsymbol{i}\left\{\oint V_y \boldsymbol{k}\cdot d\boldsymbol{\lambda} - \oint V_z \boldsymbol{j}\cdot d\boldsymbol{\lambda}\right\} + \cdots \\
&= \boldsymbol{i}\left\{\int_\sigma \nabla \times (\boldsymbol{k}V_y)\cdot d\boldsymbol{\sigma} - \int_\sigma \nabla \times (\boldsymbol{j}V_z)\cdot d\boldsymbol{\sigma}\right\} + \cdots \\
&= \boldsymbol{i}\int_\sigma \left\{\left(\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) d\sigma_x - \frac{\partial V_y}{\partial x} d\sigma_y - \frac{\partial V_z}{\partial x} d\sigma_z\right\} + \cdots \\
&= \boldsymbol{i}\int_\sigma \left\{\nabla \cdot \mathbf{V} d\sigma_x - \frac{\partial \mathbf{V}}{\partial x} \cdot d\boldsymbol{\sigma}\right\} + \cdots \\
&= \int_\sigma \nabla \cdot \mathbf{V} d\boldsymbol{\sigma} - \int_\sigma \nabla \mathbf{V} \cdot d\boldsymbol{\sigma}.
\end{aligned}$$

The second corollary states that

$$\oint \mathbf{V} \times (\mathbf{U} \times d\boldsymbol{\lambda}) = \int_\sigma \nabla \times \overline{\mathbf{V}\mathbf{U}} \cdot d\boldsymbol{\sigma} + \int_\sigma \nabla \cdot \overline{\mathbf{U}\mathbf{V}} \times d\boldsymbol{\sigma}, \qquad (18\text{–}3)$$

where ∇, in each of the surface integrals, operates as a differential operator on the entire quantity under the vinculum. In proving this we note first that

$$\oint \mathbf{V} \times (\mathbf{U} \times d\boldsymbol{\lambda}) = \oint \mathbf{U}\mathbf{V} \cdot d\boldsymbol{\lambda} - \oint \mathbf{U} \cdot \mathbf{V} d\boldsymbol{\lambda}.$$

Now

$$\oint \mathbf{U}\mathbf{V}\cdot d\boldsymbol{\lambda} = \boldsymbol{i}\oint U_x\mathbf{V}\cdot d\boldsymbol{\lambda} + \cdots$$

$$= \boldsymbol{i}\int_\sigma \nabla \times \overline{\mathbf{V}U_x}\cdot d\boldsymbol{\sigma} + \cdots$$

$$= \int_\sigma d\boldsymbol{\sigma} \times \nabla\cdot\overline{\mathbf{V}\mathbf{U}},$$

and

$$\oint \mathbf{U}\cdot\mathbf{V}d\boldsymbol{\lambda} = \boldsymbol{i}\oint \mathbf{U}\cdot\mathbf{V}\boldsymbol{i}\cdot d\boldsymbol{\lambda} + \cdots$$

$$= \boldsymbol{i}\int_\sigma \nabla \times (\overline{\boldsymbol{i}\mathbf{U}\cdot\mathbf{V}})\cdot d\boldsymbol{\sigma} + \cdots$$

$$= \boldsymbol{i}\int_\sigma \left\{\frac{\partial}{\partial z}\overline{\mathbf{U}\cdot\mathbf{V}}d\sigma_y - \frac{\partial}{\partial y}\overline{\mathbf{U}\cdot\mathbf{V}}d\sigma_z\right\} + \cdots$$

$$= \int_\sigma d\boldsymbol{\sigma} \times \nabla\overline{\mathbf{U}\cdot\mathbf{V}}.$$

Hence

$$\oint \mathbf{V} \times (\mathbf{U} \times d\boldsymbol{\lambda}) = \int_\sigma d\boldsymbol{\sigma} \times \{\nabla\cdot\overline{\mathbf{V}\mathbf{U}} - \nabla\overline{\mathbf{U}\cdot\mathbf{V}}\}$$

$$= \int_\sigma \{(d\boldsymbol{\sigma} \times \nabla) \times \overline{\mathbf{U}\} \times \mathbf{V}}.$$

But, if we expand the triple vector product inside the braces,

$$\{(d\boldsymbol{\sigma} \times \nabla) \times \overline{\mathbf{U}\} \times \mathbf{V}} = \nabla \times \overline{\mathbf{V}\mathbf{U}}\cdot d\boldsymbol{\sigma} + \nabla\cdot\overline{\mathbf{U}\mathbf{V}} \times d\boldsymbol{\sigma},$$

which proves the corollary.

19. Orthogonal Curvilinear Coordinates. — Frequently we need to express the gradient, divergence or curl in other than rectangular coordinates. Let us take as coordinate surfaces any three orthogonal families, $u(x, y, z) = \text{const.}$, $v(x, y, z) = \text{const.}$, $w(x, y, z) = \text{const.}$, and introduce the unit vectors $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{w}_1$, in the directions of increasing u, v, w respectively, u, v, w being so ordered that $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{w}_1$ constitute a right-handed set. Although $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{w}_1$ are unit vectors, it must be borne in mind that their directions may change as we pass from point to point. Let us designate by a_u, a_v and a_w the three functions of u, v, w which represent the distances along the normals

to the three coordinate surfaces per unit increment in the coordinates u, v and w, respectively. Then a vector element of distance is given in terms of du, dv, dw by

$$d\boldsymbol{\lambda} = \mathbf{u}_1 a_u du + \mathbf{v}_1 a_v dv + \mathbf{w}_1 a_w dw, \tag{19-1}$$

a vector element of area by

$$d\boldsymbol{\sigma} = \mathbf{u}_1 a_v a_w dv\, dw + \mathbf{v}_1 a_w a_u dw\, du + \mathbf{w}_1 a_u a_v\, du\, dv, \tag{19-2}$$

and an element of volume by

$$d\tau = a_u a_v a_w\, du\, dv\, dw. \tag{19-3}$$

A vector $\mathbf{P}$ may be expressed in terms of its components P_u, P_v, P_w in the directions of increasing u, v, w respectively by

$$\mathbf{P} = \mathbf{u}_1 P_u + \mathbf{v}_1 P_v + \mathbf{w}_1 P_w. \tag{19-4}$$

Evidently

$$\mathbf{P}\cdot\mathbf{Q} = P_u Q_u + P_v Q_v + P_w Q_w, \tag{19-5}$$

and

$$\mathbf{P}\times\mathbf{Q} = \begin{vmatrix} \mathbf{u}_1 & \mathbf{v}_1 & \mathbf{w}_1 \\ P_u & P_v & P_w \\ Q_u & Q_v & Q_w \end{vmatrix}\cdot \tag{19-6}$$

The differential operator ∇ becomes in the new coordinates

$$\nabla = \mathbf{u}_1 \frac{\partial}{a_u \partial u} + \mathbf{v}_1 \frac{\partial}{a_v \partial v} + \mathbf{w}_1 \frac{\partial}{a_w \partial w}, \tag{19-7}$$

and the gradient of a scalar function $\Phi(u, v, w)$ is

$$\nabla\Phi = \mathbf{u}_1 \frac{\partial\Phi}{a_u \partial u} + \mathbf{v}_1 \frac{\partial\Phi}{a_v \partial v} + \mathbf{w}_1 \frac{\partial\Phi}{a_w \partial w}. \tag{19-8}$$

The expansion of the divergence and of the curl of a vector function

$$\mathbf{V}(u, v, w) = \mathbf{u}_1 V_u(u, v, w) + \mathbf{v}_1 V_v(u, v, w) + \mathbf{w}_1 V_w(u, v, w) \tag{19-9}$$

can be obtained by taking the scalar and the vector products of (19–7) and (19–9), but in carrying out the indicated differentiation the derivatives of the unit vectors $\mathbf{u}_1$, $\mathbf{v}_1$, $\mathbf{w}_1$ must not be overlooked, since the derivative of a vector of constant magnitude but variable direction is not zero. However, we shall employ a simpler method making use of Gauss' theorem and of Stokes' theorem.

To obtain the divergence we apply Gauss' theorem to an elementary rectangular parallelopiped $\Delta\tau$ bounded by surfaces over which u, v and w are constant. Then

$$\nabla\cdot\mathbf{V}\Delta\tau = \int_\sigma \mathbf{V}\cdot d\boldsymbol{\sigma}.$$

The net outward flux of $\mathbf{V}$ through the two faces for which u is constant is

$$V_u a_v a_w \Delta v\Delta w + \frac{\partial}{\partial u}(V_u a_v a_w)\Delta u\Delta v\Delta w - V_u a_v a_w \Delta v\Delta w$$
$$= \frac{\partial}{\partial u}(a_v a_w V_u)\Delta u\Delta v\Delta w,$$

and similar expressions hold for the two remaining pairs of surfaces. Hence Gauss' theorem gives

$$\nabla\cdot\mathbf{V}a_u a_v a_w \Delta u\Delta v\Delta w$$
$$= \left\{\frac{\partial}{\partial u}(a_v a_w V_u) + \frac{\partial}{\partial v}(a_w a_u V_v) + \frac{\partial}{\partial w}(a_u a_v V_w)\right\}\Delta u\Delta v\Delta w$$

and the divergence of $\mathbf{V}$ is

$$\nabla\cdot\mathbf{V} = \frac{1}{a_u a_v a_w}\left\{\frac{\partial}{\partial u}(a_v a_w V_u) + \frac{\partial}{\partial v}(a_w a_u V_v) + \frac{\partial}{\partial w}(a_u a_v V_w)\right\}. \quad (19\text{–}10)$$

To obtain the u component of the curl we apply Stokes' theorem to an elementary rectangle $\Delta\sigma_u$ in a surface u = const., the edges of the rectangle being lines along which either v or w is constant. Then

$$|\nabla\times\mathbf{V}|_u\Delta\sigma_u = \oint \mathbf{V}\cdot d\boldsymbol{\lambda}.$$

Now the line integral of $\mathbf{V}$ along the two edges for which v is constant is

$$V_w a_w \Delta w + \frac{\partial}{\partial v}(V_w a_w)\Delta v\Delta w - V_w a_w\Delta w = \frac{\partial}{\partial v}(a_w V_w)\Delta v\Delta w,$$

and along the two edges for which w is constant

$$-\left\{V_v a_v \Delta v + \frac{\partial}{\partial w}(V_v a_v)\Delta w\Delta v - V_v a_v\Delta v\right\} = -\frac{\partial}{\partial w}(a_v V_v)\Delta v\Delta w.$$

Hence Stokes' theorem gives

$$|\nabla\times\mathbf{V}|_u a_v a_w\Delta v\Delta w = \left\{\frac{\partial}{\partial v}(a_w V_w) - \frac{\partial}{\partial w}(a_v V_v)\right\}\Delta v\Delta w.$$

In similar manner the other components are obtained. Altogether the three components of the curl are

$$\left.\begin{aligned} |\nabla \times \mathbf{V}|_u &= \frac{1}{a_v a_w}\left\{\frac{\partial}{\partial v}(a_w V_w) - \frac{\partial}{\partial w}(a_v V_v)\right\}, \\ |\nabla \times \mathbf{V}|_v &= \frac{1}{a_w a_u}\left\{\frac{\partial}{\partial w}(a_u V_u) - \frac{\partial}{\partial u}(a_w V_w)\right\}, \\ |\nabla \times \mathbf{V}|_w &= \frac{1}{a_u a_v}\left\{\frac{\partial}{\partial u}(a_v V_v) - \frac{\partial}{\partial v}(a_u V_u)\right\}. \end{aligned}\right\} \quad (19\text{–}11)$$

Since the Laplacian is the divergence of the gradient, its expansion in curvilinear coordinates is obtained by substituting the components of (19–8) in (19–10). Thus

$$\nabla \cdot \nabla \Phi = \frac{1}{a_u a_v a_w}\left\{\frac{\partial}{\partial u}\left(\frac{a_v a_w}{a_u}\frac{\partial \Phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{a_w a_u}{a_v}\frac{\partial \Phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{a_u a_v}{a_w}\frac{\partial \Phi}{\partial w}\right)\right\}. \quad (19\text{–}12)$$

The functions a_u, a_v, a_w are determined most easily from the expression for the square of the linear element,

$$d\lambda^2 = a_u{}^2 du^2 + a_v{}^2 dv^2 + a_w{}^2 dw^2. \quad (19\text{–}13)$$

In spherical coordinates, for instance,

$$d\lambda^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (19\text{–}14)$$

and therefore $a_r = 1$, $a_\theta = r$, $a_\phi = r \sin \theta$, while in cylindrical coordinates

$$d\lambda^2 = dr^2 + r^2 d\phi^2 + dz^2 \quad (19\text{–}15)$$

and $a_r = 1$, $a_\phi = r$, $a_z = 1$.

Other orthogonal curvilinear coordinates will be introduced as occasion arises.

Problem 19a. Find the Laplacian of Φ in spherical coordinates.

Problem 19b. Find the three components of the curl of $\mathbf{V}$ in spherical coordinates.

20. The Potential Operator. — We shall now define an operator which is as important in integration as the vector operator ∇ is in differentiation. This operator is called the *potential*, and is symbolized by the abbreviation *Pot.*

Let $u(x_2, y_2, z_2)$ be a finite, continuous and single-valued function of the coordinates x_2, y_2, z_2 of a point P_2 (Fig. 25), and let r_{12} be the distance $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ of the point P_2 from another point P_1 whose coordinates are x_1, y_1, z_1. Furthermore let u either vanish identically for all $r_{12} > p$, or become equal to $f(\theta, \phi)/r_{12}^n$, where θ and ϕ designate polar angle and azimuth respectively and $n > 0$. We shall assume the second alternative in our

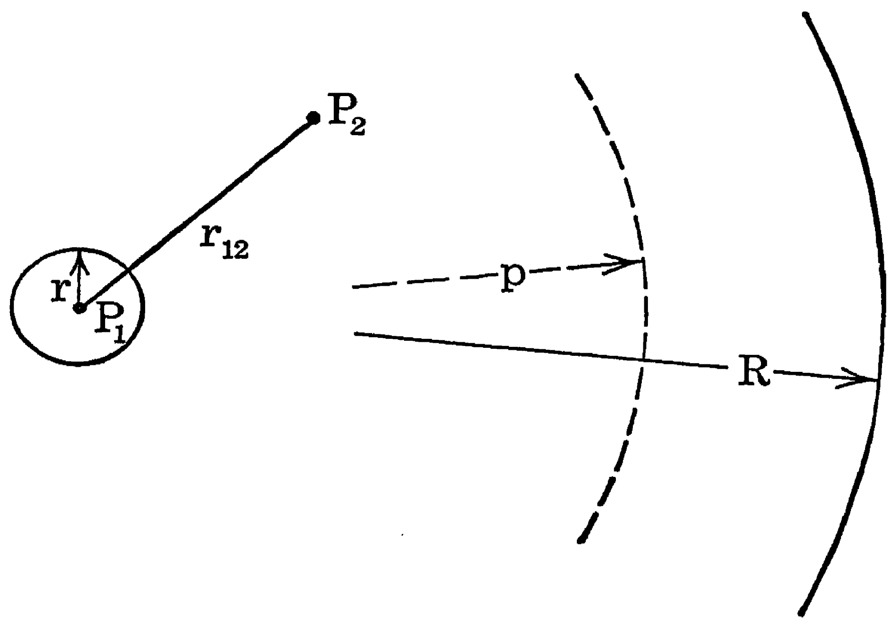

FIG. 25.

subsequent treatment, since the results obtained for it hold *a fortiori* for the first.

We define Pot* u at P_1 by †

$$\text{Pot}^* u = \int_r^R \frac{u(x_2, y_2, z_2)}{4\pi r_{12}} d\tau_2, \tag{20–1}$$

where $d\tau_2$ is the volume element $dx_2dy_2dz_2$, and the integral is taken over the region lying between a small sphere of radius r and a large sphere of radius $R \gg p$ surrounding P_1. Then we define the potential of u at P_1 by

$$\text{Pot}\, u = \Big|_{\substack{r \to 0 \\ R \to \infty}}^{t} \text{Pot}^* u. \tag{20–2}$$

Evidently both Pot* u and Pot u are functions of x_1, y_1, z_1.

† The factor 4π is inserted in the denominator to make this quantity agree in numerical coefficient with the electromagnetic potentials to be introduced later.

To show that Pot u, as so defined, exists, we can divide the region of integration between the spheres r and R into three parts by describing a small fixed sphere of radius $a > r$ and a large fixed sphere of radius A, where $R > A > p$, about P_1. Then

$$\text{Pot}^* u = \int_r^a \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2 + \int_a^A \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2 + \int_A^R \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2.$$

For brevity put u_1 for the value of u at the point P_1. Then, as u is continuous at P_1, we can choose a small enough so that the first integral does not differ in absolute value from

$$u_1 \int_r^a \frac{d\tau_2}{4\pi r_{12}} = \tfrac{1}{2} u_1 (a^2 - r^2)$$

by more than a previously assigned small quantity ϵ, for any $r < a$. The second integral obviously converges. The third becomes

$$\int_A^R f(\theta, \phi) \frac{d\tau_2}{4\pi {r_{12}}^{n+1}} = \frac{1}{4\pi(n-2)} \left(\frac{1}{A^{n-2}} - \frac{1}{R^{n-2}} \right) \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \sin\, \theta d\theta d\phi.$$

Hence, if we put $4\pi k$ for the integral with respect to θ and ϕ on the right,

$$\text{Pot}^* u = \tfrac{1}{2} u_1 (a^2 - r^2) + \int_a^A \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2 + \frac{k}{n-2} \left(\frac{1}{A^{n-2}} - \frac{1}{R^{n-2}} \right)$$

to any desired degree of precision, and evidently a definite limit is approached as $r \to 0$ and $R \to \infty$, provided $n > 2$. Hence the potential of u exists if $n > 2$. However, as the first and second derivatives of $1/r_{12}^n$ are of the order of $1/r_{12}^{n+1}$ and $1/r_{12}^{n+2}$, respectively, it follows that the potential of the first derivative of u exists if $n > 1$ and of the second derivative if $n > 0$.

In the case of a vector function $\mathbf{V}(x_2, y_2, z_2)$ we have

$$\text{Pot}\, \mathbf{V} = \boldsymbol{i}\, \text{Pot}\, V_x + \boldsymbol{j}\, \text{Pot}\, V_y + \boldsymbol{k}\, \text{Pot}\, V_z. \qquad (20\text{–}3)$$

If V_x, V_y, V_z satisfy the conditions required in the case of u, their potentials exist and therefore the potential of **V** exists. Evidently Pot **V** is a vector function of x_1, y_1, z_1. It is known as a *vector potential* in contradistinction to the *scalar potential* (20–2).

Since r_{12} is a permissible argument of a proper scalar or vector function, it follows that the potential of a proper scalar function is a proper scalar function, and the potential of a proper vector function is a proper vector function, of the coordinates and of the components of any constant vectors which may appear.

21. Commutation of Pot and ∇. — We shall now show that, if u and its first derivatives are finite, continuous and single-valued functions of the coordinates such that their potentials exist,

$$\nabla_1 \operatorname{Pot} u = \operatorname{Pot} \nabla_2 u, \tag{21–1}$$

where

$$\nabla_1 \equiv \boldsymbol{i}\frac{\partial}{\partial x_1} + \boldsymbol{j}\frac{\partial}{\partial y_1} + \boldsymbol{k}\frac{\partial}{\partial z_1},$$

and

$$\nabla_2 \equiv \boldsymbol{i}\frac{\partial}{\partial x_2} + \boldsymbol{j}\frac{\partial}{\partial y_2} + \boldsymbol{k}\frac{\partial}{\partial z_2}.$$

It must be borne in mind that u is a function of x_2, y_2, z_2 but not of x_1, y_1, z_1, whereas Pot u is a function of x_1, y_1, z_1 but not of x_2, y_2, z_2.

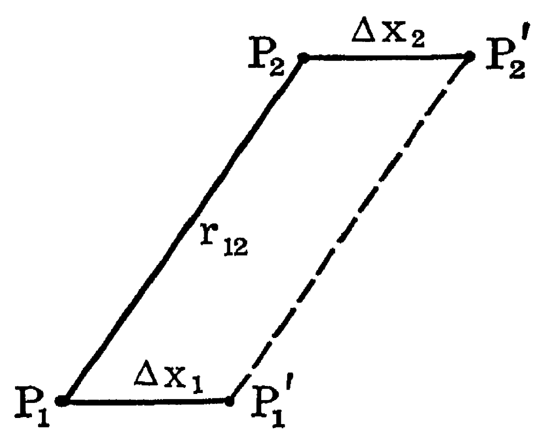

FIG. 26.

Let P_1 (Fig. 26) be the point x_1, y_1, z_1 and P_1' the point $x_1 + \Delta x_1$, y_1, z_1. If the potential at P_1 is indicated by $\{\operatorname{Pot} u\}_{x_1, y_1, z_1}$ and that at P_1' by $\{\operatorname{Pot} u\}_{x_1+\Delta x_1, y_1, z_1}$,

$$\frac{\partial}{\partial x_1} \operatorname{Pot} u = \underset{\Delta x_1 \to 0}{\Big|^{t}} \frac{\{\operatorname{Pot} u\}_{x_1+\Delta x_1, y_1, z_1} - \{\operatorname{Pot} u\}_{x_1, y_1, z_1}}{\Delta x_1}. \tag{21–2}$$

Now suppose that we associated with the volume element $d\tau_2$ at P_2, not the value $u(x_2, y_2, z_2)$ of u at P_2, but the value $u(x_2 + \Delta x_2, y_2, z_2)$ of u at P_2', where $\Delta x_2 = \overline{P_2P_2'}$ is taken equal to $\Delta x_1 = \overline{P_1P_1'}$. This represents a translation of the distribution of u to the left through a distance Δx_2. But a shift of the distribution of u to the left by the amount Δx_2 evidently gives the same potential at P_1 as the original distribution of u gives at the point P_1' at an equal distance Δx_1 to the right of P_1. Therefore

$$\{\text{Pot}\ u(x_2, y_2, z_2)\}_{x_1+\Delta x_1,\ y_1,\ z_1} = \{\text{Pot}\ u(x_2 + \Delta x_2, y_2, z_2)\}_{x_1,\ y_1,\ z_1}$$

$$= \{\text{Pot}\ u(x_2, y_2, z_2)\}_{x_1,\ y_1,\ z_1} + \left\{\text{Pot}\ \frac{\partial u}{\partial x_2}\right\}_{x_1,\ y_1,\ z_1} \Delta x_2 + \cdots.$$

Since $\Delta x_1 = \Delta x_2$, we get on substituting in (21–2) and letting $\Delta x_1 \rightarrow 0$,

$$\frac{\partial}{\partial x_1} \text{Pot}\ u = \text{Pot}\ \frac{\partial u}{\partial x_2}.$$

Similarly

$$\frac{\partial}{\partial y_1} \text{Pot}\ u = \text{Pot}\ \frac{\partial u}{\partial y_2},$$

and

$$\frac{\partial}{\partial z_1} \text{Pot}\ u = \text{Pot}\ \frac{\partial u}{\partial z_2}.$$

Multiplying these three equations by $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ respectively and adding, we have (21–1).

If the second derivatives of u as well as the first satisfy the conditions for the existence of the potential,

$$\frac{\partial^2}{\partial x_1^{\,2}} \text{Pot}\ u = \frac{\partial}{\partial x_1} \text{Pot}\ \frac{\partial u}{\partial x_2} = \text{Pot}\ \frac{\partial^2 u}{\partial x_2^{\,2}}$$

and consequently

$$\nabla_1 \cdot \nabla_1\ \text{Pot}\ u = \text{Pot}\ \nabla_2 \cdot \nabla_2 u. \tag{21–3}$$

Similar relations hold for vector functions satisfying the necessary conditions. Thus

$$\nabla_1 \cdot \text{Pot}\ \mathbf{V} = \frac{\partial}{\partial x_1} \text{Pot}\ V_x + \frac{\partial}{\partial y_1} \text{Pot}\ V_y + \frac{\partial}{\partial z_1} \text{Pot}\ V_z$$

$$= \text{Pot}\ \frac{\partial V_x}{\partial x_2} + \text{Pot}\ \frac{\partial V_y}{\partial y_2} + \text{Pot}\ \frac{\partial V_z}{\partial z_2}$$

$$= \text{Pot}\ \nabla_2 \cdot \mathbf{V}, \tag{21–4}$$

and

$$\nabla_1 \times \text{Pot}\,\mathbf{V} = \boldsymbol{i}\left(\frac{\partial}{\partial y_1}\text{Pot}\,V_z - \frac{\partial}{\partial z_1}\text{Pot}\,V_y\right) + \cdots$$

$$= \boldsymbol{i}\,\text{Pot}\left(\frac{\partial V_z}{\partial y_2} - \frac{\partial V_y}{\partial z_2}\right) + \cdots$$

$$= \text{Pot}\,\nabla_2 \times \mathbf{V}. \qquad (21\text{–}5)$$

In like fashion we may show that

$$\nabla_1 \cdot \nabla_1\,\text{Pot}\,\mathbf{V} = \text{Pot}\,\nabla_2 \cdot \nabla_2 \mathbf{V}, \qquad (21\text{–}6)$$

$$\nabla_1 \nabla_1 \cdot \text{Pot}\,\mathbf{V} = \text{Pot}\,\nabla_2 \nabla_2 \cdot \mathbf{V}, \qquad (21\text{–}7)$$

$$\nabla_1 \times \nabla_1 \times \text{Pot}\,\mathbf{V} = \text{Pot}\,\nabla_2 \times \nabla_2 \times \mathbf{V}. \qquad (21\text{–}8)$$

22. Poisson's Theorem. — This theorem states that, if u and its first and second derivatives are finite, continuous and single-valued functions of the coordinates and u either vanishes identically for $r_{12} > p$ or becomes equal to $f(\theta, \phi)/r_{12}^n$ where $n > 2$, then

$$\nabla_1 \cdot \nabla_1\,\text{Pot}\,u = -\,u_1, \qquad (22\text{–}1)$$

where u_1 is the value of $u(x_2, y_2, z_2)$ at the point x_1, y_1, z_1.

Although it is sometimes convenient to deal with functions which are discontinuous across certain surfaces, it is never necessary to do so, since we can always replace such a surface of discontinuity by a thin layer in which the function and its derivatives change rapidly but continuously from their values on the one side to those on the other. Thus if u is the charge density of an electrified body, we may suppose that u decreases rapidly but continuously as we pass through the surface of the body from the value which exists in the interior to a value effectly zero outside. Even in the case of a surface charge, such as exists on a conductor in an electrostatic field, we may suppose the charge to reside in a thin layer instead of on a mathematical surface, such that when we pass through the surface of the conductor the charge density increases rapidly but continuously from zero to a finite maximum value, and then decreases rapidly but continuously to the value zero outside the conductor. Indeed, this concept of a surface charge is undoubtedly more in accord with reality than that of a charge spread over a mathematical surface. Although we shall frequently employ the phrase "point charge," we shall generally understand thereby a charge concentrated in a small but finite region in such a way that the charge density is everywhere finite and con-

tinuous. By these devices we are able to avoid many analytical complications in the development of the general theory, while still reserving the right to introduce surfaces of discontinuity into specific problems where it is clear that their use does not vitiate the calculation.

As

$$\nabla_1\cdot\nabla_1 \operatorname{Pot} u = \nabla_1\cdot\nabla_1 \Big\lfloor_{\substack{r\to 0\\ R\to\infty}}^{t} \int_r^R \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2,$$

we might be tempted to differentiate under the sign of integration, treating the limits of the integral as constants. However, that procedure, which would yield zero in view of the fact that $\nabla_1\cdot\nabla_1(1/r_{12}) = 0$, is not legitimate since the lower limit is a function of x_1, y_1, z_1. In fact we shall see later that it is permissible to differentiate Pot u once under the sign of integration without regard to the limits, but not twice. The correct evaluation of the second derivative will lead us to Poisson's theorem.

At present, however, we shall employ another method making use of the commutation property of ∇ and Pot, and of Green's theorem. We have from (21–3)

$$\nabla_1\cdot\nabla_1 \operatorname{Pot} u = \operatorname{Pot} \nabla_2\cdot\nabla_2 u = \Big\lfloor_{\substack{r\to 0\\ R\to\infty}}^{t} \operatorname{Pot}^* \nabla_2\cdot\nabla_2 u. \qquad (22\text{–}2)$$

By Green's theorem (17–2)

$$\int_\tau \left\{\frac{1}{r_{12}}\nabla_2\cdot\nabla_2 u - u\nabla_2\cdot\nabla_2\left(\frac{1}{r_{12}}\right)\right\} d\tau_2$$
$$= \int_\sigma \left\{\frac{1}{r_{12}}\nabla_2 u - u\nabla_2\left(\frac{1}{r_{12}}\right)\right\}\cdot d\sigma_2.$$

As, however, $\nabla_2\cdot\nabla_2(1/r_{12})$ is identically zero,

$$\operatorname{Pot}^* \nabla_2\cdot\nabla_2 u = \int_r^R \frac{1}{4\pi r_{12}} \nabla_2\cdot\nabla_2 u\, d\tau_2$$
$$= \int_\sigma \frac{1}{4\pi r_{12}} \nabla_2 u\cdot d\sigma_2 - \int_\sigma \frac{u}{4\pi}\nabla_2\left(\frac{1}{r_{12}}\right)\cdot d\sigma_2. \qquad (22\text{–}3)$$

The region over which the volume integral is taken is that included between a small sphere of radius r about the point P_1, whose coordi-

nates are x_1, y_1, z_1, and a large sphere of radius R. Therefore the surface integrals must be evaluated over each of these spheres. Now let R be taken large enough so that u, if it does not vanish identically over the outer surface, is equal to $f(\theta, \phi)/r_{12}^n$, where $n > 0$. Hence the normal component $\dfrac{\partial u}{\partial r_{12}}$ of $\nabla_2 u$ is $-nf(\theta, \phi)/r_{12}^{n+1}$. Moreover the normal component of $\nabla_2(1/r_{12})$ is $-1/r_{12}^2$. Consequently the sum of the two surface integrals over the sphere R is

$$\frac{1 - n}{4\pi R^n}\int_0^{2\pi}\int_0^{\pi} f(\theta, \phi) \sin \theta d\theta d\phi,$$

which approaches zero as R increases without limit, provided $n > 0$.

Over the small sphere the normal component of $\nabla_2(1/r_{12})$ is $1/r_{12}^2$ since the positive normal to the surface is inward. Let χ be the angle which $\mathbf{r}_{12}$ makes with the value $(\nabla_2 u)_1$ of $\nabla_2 u$ at P_1. Then we can choose r small enough so that the sum of the two surface integrals over the sphere r does not differ in absolute value from

$$(\nabla_2 u)_1 \cdot \int_\sigma \frac{d\sigma_2}{4\pi r} - u_1 \int_\sigma \frac{d\sigma_2}{4\pi r^2}$$
$$= -\,|\,\nabla_2 u\,|_1 \int_0^{\pi} \tfrac{1}{2} r \cos \chi \sin \chi \, d\chi - u_1 = -u_1$$

by more than a previously assigned small quantity ϵ. Passing to the limit,

$$\operatorname{Pot} \nabla_2 \cdot \nabla_2 u = -u_1, \quad (22\text{–}4)$$

a relation which is valid *provided* $n > 0$. If, however, $n > 2$, Pot u exists, and we obtain Poisson's theorem (22–1) by combining (22–4) with (22–2).

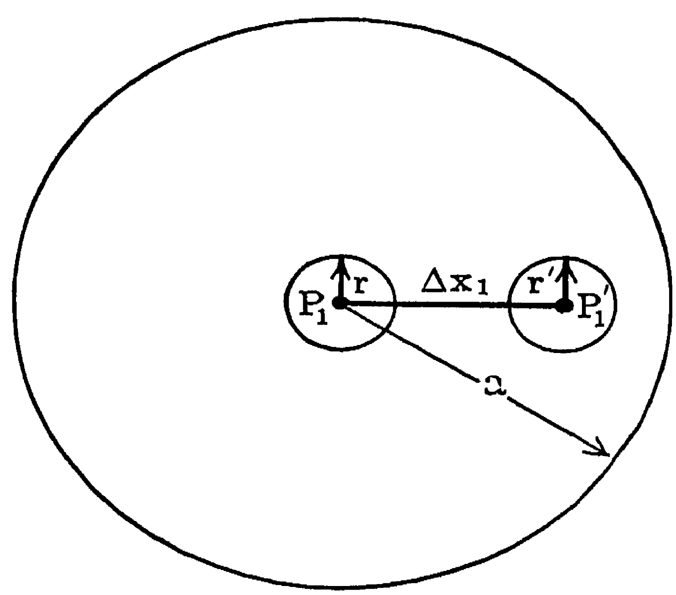

FIG. 27.

Now let us consider the differentiation of Pot u under the sign of integration. Let P_1 (Fig. 27) be the point whose coordinates are x_1, y_1, z_1, and P_1' a point whose coordinates are $x_1 + \Delta x_1$, y_1, z_1. First we describe a

fixed sphere of radius a about P_1 so small that we can replace the continuous function u by u_1 at all points inside this sphere without appreciable error. About P_1 and P_1' we have the two spheres r and r' of the same radius, small enough so that they lie entirely inside a. Next we describe a fixed sphere $A > p$ about P_1, outside of which lie the two spheres R and R' of the same radius described about P_1 and P_1'. Then

$$(\text{Pot}^* \, u)_{x_1,\, y_1,\, z_1} = u_1 \int_r^a \frac{d\tau_2}{4\pi r_{12}} + \int_a^A \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2 + \int_A^R \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2.$$

As the spheres a and A are fixed, we can differentiate the second integral under the sign of integration. Hence

$$\begin{aligned}\frac{\partial}{\partial x_1}(\text{Pot}^* \, u) = u_1 \underset{\Delta x_1 \to 0}{\Big\lfloor^t} \frac{1}{\Delta x_1}\left\{\int_{r'}^a \frac{d\tau_2}{4\pi r'_{12}} - \int_r^a \frac{d\tau_2}{4\pi r_{12}}\right\} \\ + \int_a^A \frac{1}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r_{12}}\right) u(x_2, y_2, z_2)\, d\tau_2 \\ + \underset{\Delta x_1 \to 0}{\Big\lfloor^t} \frac{1}{\Delta x_1}\left\{\int_A^{R'} \frac{u(x_2, y_2, z_2)}{4\pi r'_{12}}\, d\tau_2 - \int_A^R \frac{u(x_2, y_2, z_2)}{4\pi r_{12}}\, d\tau_2\right\}, \quad (22\text{--}5)\end{aligned}$$

where r'_{12} is the radius vector drawn from P_1'. Now

$$\int_r^a \frac{d\tau_2}{r_{12}} = 2\pi(a^2 - r^2).$$

To get the corresponding integral about P_1', we recall that the potential in the usual units due to a spherical shell of mass m and radius b at a point distant ρ from its center is m/b if $\rho \leqslant b$ and m/ρ if $\rho \geqslant b$. Now the integral under consideration is equal to the integral throughout the volume bounded by a less that through the volume bounded by r'. Consequently, expressing the first as the sum of the integral over the region bounded by the sphere of radius Δx_1 about P_1

and the integral over the region between this sphere and the sphere of radius a,

$$\int_{r'}^{a} \frac{d\tau_2}{r'_{12}} = \tfrac{4}{3}\pi\overline{\Delta x_1}^2 + \int_{\Delta x_1}^{a} \frac{4\pi r_{12}^{\,2} dr_{12}}{r_{12}} - \int_0^{r'} \frac{4\pi r'^{\,2}_{12} dr'_{12}}{r'_{12}}$$

$$= \tfrac{4}{3}\pi\overline{\Delta x_1}^2 + 2\pi(a^2 - \overline{\Delta x_1}^2) - 2\pi r'^2.$$

Hence, as $r' = r$,

$$\int_{r'}^{a} \frac{d\tau_2}{4\pi r'_{12}} - \int_{r}^{a} \frac{d\tau_2}{4\pi r_{12}} = -\tfrac{1}{6}\overline{\Delta x_1}^2.$$

The expression in the last brace on the right of (22–5) is easily found to be of the order of $\Delta x_1/A^{n-1}$, and therefore the last term is negligible if A is taken sufficiently large, provided $n > 1$.

Now, proceeding to the limit $\Delta x_1 \to 0$, $r \to 0$, $R \to \infty$ in (22–5), in any order whatsoever,

$$\frac{\partial}{\partial x_1}(\text{Pot } u) = \int_a^A \frac{1}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r_{12}}\right) u(x_2, y_2, z_2)d\tau_2. \qquad (22\text{–}6)$$

By taking a sufficiently small and A sufficiently large at the start, we can ensure that the difference

$$\Big|^{t}_{\substack{a \to 0 \\ A \to \infty}} \int_a^A \frac{1}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r_{12}}\right) u(x_2, y_2, z_2)d\tau_2 - \int_a^A \frac{1}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r_{12}}\right) u(x_2, y_2, z_2)d\tau_2$$

is less than any previously assigned small quantity ϵ. Hence we may write

$$\frac{\partial}{\partial x_1}(\text{Pot } u) = \int_0^\infty \frac{1}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r_{12}}\right) u(x_2, y_2, z_2)d\tau_2, \qquad (22\text{–}7)$$

which shows that we may differentiate Pot u once under the sign of integration without regard to the limits.

Now we shall proceed to the second derivative. Indicating by an asterisk that the integral extends only from the sphere r to the sphere R, we obtain by differentiation

$$\left\{\frac{\partial}{\partial x_1}(\text{Pot } u)\right\}^* = u_1\int_r^a \frac{x_2 - x_1}{4\pi r_{12}^3}\,d\tau_2$$

$$+\int_a^A \frac{1}{4\pi}\frac{\partial}{\partial x_1}\left(\frac{1}{r_{12}}\right)u(x_2, y_2, z_2)d\tau_2 + \int_A^R \frac{x_2 - x_1}{4\pi r_{12}^3}\,u(x_2, y_2, z_2)d\tau_2,$$

and

$$\frac{\partial}{\partial x_1}\left\{\frac{\partial}{\partial x_1}(\text{Pot } u)\right\}^* = u_1 \mathop{\Big|}_{\Delta x_1\to 0}^{t} \frac{1}{\Delta x_1}\left\{\int_{r'}^a \frac{x_2 - x_1'}{4\pi r_{12}'^3}\,d\tau_2 - \int_r^a \frac{x_2 - x_1}{4\pi r_{12}^3}\,d\tau_2\right\}$$

$$+\int_a^A \frac{1}{4\pi}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{r_{12}}\right)u(x_2, y_2, z_2)d\tau_2$$

$$+\mathop{\Big|}_{\Delta x_1\to 0}^{t} \frac{1}{\Delta x_1}\left\{\int_A^{R'} \frac{x_2 - x_1'}{4\pi r_{12}'^3}\,u(x_2, y_2, z_2)d\tau_2\right.$$

$$\left.-\int_A^R \frac{x_2 - x_1}{4\pi r_{12}^3}\,u(x_2, y_2, z_2)d\tau_2\right\}. \qquad (22\text{–}8)$$

We notice that $\{(x_2 - x_1)/r_{12}^3\}d\tau_2$ is just the X component of the force at P_1 due to attracting matter of unit density filling the volume $d\tau_2$. As the force due to a spherical shell of such matter is zero for an interior point, and the same as if the matter were concentrated at its geometrical center for an exterior point,

$$\int_r^a \frac{x_2 - x_1}{r_{12}^3}\,d\tau_2 = 0$$

and

$$\int_{r'}^a \frac{x_2 - x_1'}{r_{12}'^3}\,d\tau_2 = -\tfrac{4}{3}\pi\Delta x_1.$$

The expression in the second brace on the right of (22–8) is of the order of $\Delta x_1/A^n$, and therefore the last term is negligible if A is taken sufficiently large, provided $n > 0$.

Hence, proceeding to the limit $\Delta x_1 \to 0$, $r \to 0$, $R \to \infty$ in (22–8), in any order whatsoever,

$$\frac{\partial^2}{\partial x_1^2}(\text{Pot } u) = -\tfrac{1}{3}u_1 + \int_a^A \frac{1}{4\pi}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{r_{12}}\right)u(x_2, y_2, z_2)d\tau_2. \qquad (22\text{–}9)$$

As in (22–7) we may write this

$$\frac{\partial^2}{\partial x_1^2}(\text{Pot } u) = -\tfrac{1}{3}u_1 + \int_0^\infty \frac{1}{4\pi}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{r_{12}}\right) u(x_2, y_2, z_2) d\tau_2, \qquad (22\text{–}10)$$

which shows that we may *not* differentiate Pot u twice under the sign of integration without regard to the limits.

Adding to (22–10) similar expressions for the second derivatives with respect to y and z, we get Poisson's theorem

$$\nabla_1 \cdot \nabla_1 \text{ Pot } u = -u_1, \qquad (22\text{–}11)$$

since $\nabla_1 \cdot \nabla_1 (1/r_{12})$ vanishes identically.

If each component of a vector function $\mathbf{V}$ of the coordinates satisfies the conditions imposed on the scalar function u just considered, Poisson's theorem evidently holds for $\mathbf{V}$, that is,

$$\nabla_1 \cdot \nabla_1 \text{ Pot } \mathbf{V} = -\mathbf{V}_1. \qquad (22\text{–}12)$$

Hereafter we shall drop the subscripts appearing in the equations expressing the results of this and of the previous article. However, it must be remembered that ∇ appearing before Pot represents ∇_1, whereas ∇ following Pot represents ∇_2. In fact, when ∇ commutes with Pot it changes from ∇_1 to ∇_2 or *vice versa*. Both Pot u and $\nabla \cdot \nabla$ Pot u are functions of x_1, y_1, z_1, and hence it follows of necessity that the u appearing on the right of (22–11) is the value of u at x_1, y_1, z_1, that is, a function of x_1, y_1, z_1. In future we shall generally omit the subscripts on the coordinates x, y, z, save in those cases where both sets of coordinates appear explicitly.

23. Poisson's and Laplace's Equations. — An important equation of theoretical physics, known as *Poisson's equation*, is

$$\nabla \cdot \nabla \Phi = -\rho, \qquad (23\text{–}1)$$

where ρ is a finite, continuous and single-valued function of the coordinates which either vanishes or becomes equal to $f(\theta, \phi)/r^n$, where $n > 2$, for all $r > p$.

By means of Poisson's theorem we can solve Poisson's equation for Φ. For, if we take the potential of each side,

$$\text{Pot } \nabla \cdot \nabla \Phi = -\text{Pot } \rho.$$

Using (21–3) this becomes

$$\nabla \cdot \nabla \text{ Pot } \Phi = -\text{Pot } \rho,$$

and, applying Poisson's theorem (22–1),

$$\Phi = \text{Pot } \rho. \tag{23-2}$$

Similarly if

$$\nabla \cdot \nabla \mathbf{V} = -\mathbf{w}, \tag{23-3}$$

where $\mathbf{w}$ is a vector function of the coordinates whose components satisfy the conditions stated above for ρ, we get the solution

$$\mathbf{V} = \text{Pot } \mathbf{w}. \tag{23-4}$$

If $\rho = 0$ everywhere inside a volume τ, Poisson's equation (23–1) reduces to *Laplace's equation*

$$\nabla \cdot \nabla \Phi = 0 \tag{23-5}$$

inside this region. If Φ is given over the surface σ bounding τ, a solution of this equation which satisfies the boundary conditions is unique. For let Φ_1 be one such solution and Φ_2 another distinct solution, if such exist. Then $\Phi_0 \equiv \Phi_2 - \Phi_1$ satisfies Laplace's equation. Integrating the identity

$$\begin{aligned} \nabla \Phi_0 \cdot \nabla \Phi_0 &= \nabla \cdot \overline{\Phi_0 \nabla \Phi_0} - \Phi_0 \nabla \cdot \nabla \Phi_0 \\ &= \nabla \cdot \overline{\Phi_0 \nabla \Phi_0} \end{aligned}$$

over τ we get

$$\begin{aligned} \int_\tau \nabla \Phi_0 \cdot \nabla \Phi_0 d\tau &= \int_\tau \nabla \cdot \overline{\Phi_0 \nabla \Phi_0} d\tau \\ &= \int_\sigma \Phi_0 \nabla \Phi_0 \cdot d\boldsymbol{\sigma} \end{aligned}$$

by Gauss' theorem. But $\Phi_2 = \Phi_1$ and hence $\Phi_0 = 0$ everywhere on σ. Consequently

$$\int_\tau \nabla \Phi_0 \cdot \nabla \Phi_0 d\tau = 0.$$

Now the volume integral of the square of a vector can vanish only if the vector vanishes identically everywhere in the region of integration. Hence $\nabla \Phi_0 = 0$ everywhere in τ, and $\Phi_0 = C$, a constant. But $\Phi_0 \equiv \Phi_2 - \Phi_1$ vanishes everywhere on σ. Consequently $C = 0$ and $\Phi_2 = \Phi_1$ everywhere in τ.

If the dependent variable is a vector function, Laplace's equation becomes

$$\nabla \cdot \nabla \mathbf{V} = 0. \tag{23-6}$$

This is equivalent to three scalar equations in which the dependent variables are V_x, V_y and V_z. Consequently a solution which satisfies assigned boundary conditions over the surface σ bounding τ is unique.

If Laplace's equation holds over all space, (23–2) shows that $\Phi = 0$ is the only possible solution. So in any physical problem Laplace's equation can hold only in a limited region or regions.

24. Resolution of a Vector Function into Irrotational and Solenoidal Parts. — Let **V** and its first and second derivatives be finite, continuous, single-valued functions of the coordinates. Then

$$\nabla \times \nabla \times \mathbf{V} = \nabla\nabla \cdot \mathbf{V} - \nabla \cdot \nabla\mathbf{V},$$

and, provided **V** vanishes identically for all $r > p$ or becomes equal to $f(\theta, \phi)/r^n$ where $n > 0$,

$$\text{Pot}\, \nabla \cdot \nabla\mathbf{V} = \text{Pot}\, \nabla\nabla \cdot \mathbf{V} - \text{Pot}\, \nabla \times \nabla \times \mathbf{V}.$$

Under the same conditions $\text{Pot}\, \nabla \cdot \nabla\mathbf{V} = -\mathbf{V}$, in accord with (22–4). Hence

$$\mathbf{V} = -\, \text{Pot}\, \nabla\nabla \cdot \mathbf{V} + \text{Pot}\, \nabla \times \nabla \times \mathbf{V}. \tag{24–1}$$

The first term on the right is irrotational and the second solenoidal. This relation shows us that any well-behaved vector function, which vanishes, however slowly, at infinity, can be expressed as the sum of an irrotational and a solenoidal vector function, and is completely determined by its divergence and its curl.

If **V** is entirely irrotational and vanishes more rapidly than $1/r$ at infinity,

$$\mathbf{V} = -\, \nabla\, \text{Pot}\, \nabla \cdot \mathbf{V}, \tag{24–2}$$

and hence **V** can be expressed as the gradient of a scalar potential function, as proved otherwise in article 18.

If **V** is entirely solenoidal and vanishes more rapidly than $1/r$ at infinity,

$$\mathbf{V} = \nabla \times \text{Pot}\, \nabla \times \mathbf{V}, \tag{24–3}$$

and therefore **V** can be expressed as the curl of a vector potential function.

While a vector function **V** may be both irrotational and solenoidal in a limited region, it cannot possess both properties everywhere, for then (24–1) would require that it be zero throughout space.

Problem 24a. If $\mathbf{V}$ is a constant vector parallel to the X axis, show that it can be expressed either as $\nabla\Phi$ or $\nabla \times \mathbf{A}$, and find Φ and $\mathbf{A}$.

Ans. $\Phi = Vx + \text{const}$; $\mathbf{A} = \frac{1}{2}(\mathbf{V} \times \mathbf{r}) + \nabla u$, where u is an arbitrary function of the coordinates.

25. Dyadics or Tensors. — Many physical laws are represented analytically by equating one vector to the product of another vector by a scalar. Such a law is the second law of motion, $\mathbf{F} = m\mathbf{f}$, where $\mathbf{F}$ is the force acting on the mass m and $\mathbf{f}$ the resulting acceleration. On the contrary, the law relating the electric displacement $\mathbf{D}$ to the electric intensity $\mathbf{E}$ in a homogeneous anisotropic medium, while linear, is not so simple, for each component of displacement is in general a linear function of all three components of electric intensity. Here we must write

$$D_x = a_{11}E_x + a_{12}E_y + a_{13}E_z,$$

$$D_y = a_{21}E_x + a_{22}E_y + a_{23}E_z,$$

$$D_z = a_{31}E_x + a_{32}E_y + a_{33}E_z.$$

It is obviously desirable to express this law in vector form. We can do so if we replace E_x by $\boldsymbol{i}\cdot\mathbf{E}$, E_y by $\boldsymbol{j}\cdot\mathbf{E}$, E_z by $\boldsymbol{k}\cdot\mathbf{E}$, multiply the three equations by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ respectively, and add. Then we have

$$\mathbf{D} = \left\{\begin{aligned} &a_{11}\boldsymbol{ii} + a_{12}\boldsymbol{ij} + a_{13}\,\boldsymbol{ik} \\ &+a_{21}\boldsymbol{ji} + a_{22}\boldsymbol{jj} + a_{23}\,\boldsymbol{jk} \\ &+a_{31}\boldsymbol{ki} + a_{32}\boldsymbol{kj} + a_{33}\boldsymbol{kk} \end{aligned}\right\}\cdot\mathbf{E}, \tag{25–1}$$

where the dot on the right-hand side indicates that the scalar product is to be taken between the second unit vector in each term inside the braces and $\mathbf{E}$.

The operator

$$\begin{aligned} \Psi \equiv\ & a_{11}\boldsymbol{ii} + a_{12}\boldsymbol{ij} + a_{13}\,\boldsymbol{ik} \\ & + a_{21}\boldsymbol{ji} + a_{22}\boldsymbol{jj} + a_{23}\boldsymbol{jk} \\ & + a_{31}\boldsymbol{ki} + a_{32}\boldsymbol{kj} + a_{33}\boldsymbol{kk} \end{aligned} \tag{25–2}$$

is known as a *dyadic*, or *tensor of second rank*. In tensor language a vector is called a *tensor of first rank*, and a scalar a *tensor of zero rank*. Equation (25–1) may now be written

$$\mathbf{D} = \Psi\cdot\mathbf{E},$$

which expresses $\mathbf{D}$ as a *linear vector function* of $\mathbf{E}$. In addition to the product $\mathbf{\Psi}\cdot\mathbf{E}$ of $\mathbf{\Psi}$ and $\mathbf{E}$ we may form the product $\mathbf{E}\cdot\mathbf{\Psi}$ by taking the scalar product of $\mathbf{E}$ by the first unit vector in each term of $\mathbf{\Psi}$. In the first, $\mathbf{E}$ is said to be a *postfactor* to $\mathbf{\Psi}$, in the second, a *prefactor*. Note that the vector $\mathbf{E}\cdot\mathbf{\Psi}$ is not in general equal to the vector $\mathbf{\Psi}\cdot\mathbf{E}$.

If $a_{ij} = a_{ji}$ in (25–2) the dyadic $\mathbf{\Psi}$ is said to be *symmetric*, whereas if $a_{ij} = - a_{ji}$ (and therefore $a_{ii} = 0$) the dyadic is *skew-symmetric*. If $\mathbf{P}$ is a vector, evidently $\mathbf{P}\cdot\mathbf{\Psi} = \mathbf{\Psi}\cdot\mathbf{P}$ if $\mathbf{\Psi}$ is symmetric, and $\mathbf{P}\cdot\mathbf{\Psi} = - \mathbf{\Psi}\cdot\mathbf{P}$ if $\mathbf{\Psi}$ is skew-symmetric.

Clearly each term in the dyadic $\mathbf{\Psi}$ is formed by the juxtaposition of two vectors, such as $a_{21}\boldsymbol{j}$ and $\boldsymbol{i}$, or $\boldsymbol{j}$ and $a_{21}\boldsymbol{i}$, in the term $a_{21}\boldsymbol{ji}$. If $\mathbf{a}$ and $\mathbf{l}$ are any two vectors, the undetermined product $\mathbf{al}$ is called a *dyad*, the vector $\mathbf{a}$ being known as the *antecedent* and $\mathbf{l}$ as the *consequent*. If either antecedent or consequent is expressed as the sum of two or more vectors, the distributive law holds, for

$$(\mathbf{a} + \mathbf{b})\mathbf{l}\cdot\mathbf{P} = (\mathbf{al} + \mathbf{bl})\cdot\mathbf{P}.$$

On the contrary, the commutative law does not hold in general for the two vectors constituting a dyad, since

$$\mathbf{al}\cdot\mathbf{P} \neq \mathbf{la}\cdot\mathbf{P}$$

unless $\mathbf{a}$ has the same direction as $\mathbf{l}$.

Evidently a dyadic is just a sum of dyads obeying the commutative and associative laws of addition. We shall now show that any dyadic may be expressed as the sum of three dyads. For let

$$\mathbf{\Psi} = \mathbf{al} + \mathbf{bm} + \mathbf{cn} + \mathbf{do}.$$

If $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are not coplanar, we may put $\mathbf{o} = f\mathbf{l} + g\mathbf{m} + h\mathbf{n}$ so that

$$\mathbf{\Psi} = (\mathbf{a} + f\mathbf{d})\mathbf{l} + (\mathbf{b} + g\mathbf{d})\mathbf{m} + (\mathbf{c} + h\mathbf{d})\mathbf{n},$$

whereas, if $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ are coplanar, we have $\mathbf{n} = p\mathbf{l} + q\mathbf{m}$ and

$$\mathbf{\Psi} = (\mathbf{a} + p\mathbf{c})\mathbf{l} + (\mathbf{b} + q\mathbf{c})\mathbf{m} + \mathbf{do}.$$

Hence, the most general form of a dyadic is

$$\mathbf{\Psi} = \mathbf{al} + \mathbf{bm} + \mathbf{cn}. \tag{25–3}$$

Furthermore, if the antecedents or consequents of the dyads constituting a dyadic are coplanar, the dyadic may be reduced to the sum of two dyads, and if they are collinear, to a single dyad. The first is called a *planar* dyadic and the second a *linear* dyadic.

If the vector $\mathbf{P}$ is perpendicular to the plane of the consequents of a planar dyadic $\boldsymbol{\Psi}$, then $\boldsymbol{\Psi}\cdot\mathbf{P} = 0$. Conversely if $\boldsymbol{\Psi}\cdot\mathbf{P} = 0$ for some vector $\mathbf{P}$, $\boldsymbol{\Psi}$ must be at least planar with its consequents at right angles to $\mathbf{P}$.

If a dyadic is given in the form (25–3), it may at once be put in the *nonian form* (25–2) by expressing each of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{l}, \mathbf{m}, \mathbf{n}$ in terms of its components. For let $\mathbf{a} = \boldsymbol{i}a_x + \boldsymbol{j}a_y + \boldsymbol{k}a_z$, etc. Then

$$a_{11} = a_xl_x + b_xm_x + c_xn_x,$$

$$a_{12} = a_xl_y + b_xm_y + c_xn_y,$$

$$a_{21} = a_yl_x + b_ym_x + c_yn_x,$$

etc. The scalars a_{ij} appearing in the form (25–2) are called the *elements* of the dyadic.

Let $\boldsymbol{\Psi}$ be a dyadic with elements a_{ij} as in (25–2), and $\boldsymbol{\Phi}$ a dyadic with elements b_{ij}. Then, if $\boldsymbol{\Psi} = \boldsymbol{\Phi}$, each element of $\boldsymbol{\Psi}$ is equal to the corresponding element of $\boldsymbol{\Phi}$. To prove this, take the scalar product of each dyadic with $\boldsymbol{i}$. We have then

$$\boldsymbol{\Psi}\cdot\boldsymbol{i} = a_{11}\boldsymbol{i} + a_{21}\boldsymbol{j} + a_{31}\boldsymbol{k},$$

$$\boldsymbol{\Phi}\cdot\boldsymbol{i} = b_{11}\boldsymbol{i} + b_{21}\boldsymbol{j} + b_{31}\boldsymbol{k}.$$

As these two vectors are equal, $a_{11} = b_{11}$, $a_{21} = b_{21}$, $a_{31} = b_{31}$. Multiplication by $\boldsymbol{j}$ and $\boldsymbol{k}$ shows the remaining corresponding elements to be equal.

Let the dyadic $\mathbf{X}$ with elements c_{ij} be the sum of the dyadics $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ with elements a_{ij} and b_{ij} respectively. Then $c_{ij} = a_{ij} + b_{ij}$. For we have

$$\mathbf{X}\cdot\boldsymbol{i} = \boldsymbol{\Psi}\cdot\boldsymbol{i} + \boldsymbol{\Phi}\cdot\boldsymbol{i}$$

and similar relations with $\boldsymbol{j}$ and $\boldsymbol{k}$ replacing $\boldsymbol{i}$. Writing these in terms of the elements of the dyadics involved, the desired relations are obtained at once.

The antecedents and consequents of a dyadic $\boldsymbol{\Psi}$ appearing in a physical law may be either constant vectors or proper vector functions of position in space. Then if $\mathbf{P}$ is a proper vector, $\boldsymbol{\Psi}\cdot\mathbf{P}$ and $\mathbf{P}\cdot\boldsymbol{\Psi}$ are proper vectors.

To find the transformation for the elements of a dyadic when we pass from one set of rectangular axes XYZ to another set $X'Y'Z'$ differently oriented, we note that $\mathbf{P}\cdot\boldsymbol{\Psi}\cdot\mathbf{Q}$ represents the same scalar relative to either set of axes, whatever the vectors $\mathbf{P}$ and $\mathbf{Q}$ may be.

Letting **P** and **Q** represent successively the unit vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$, the elements a'_{ij} of the dyadic when referred to $X'Y'Z'$ can be immediately determined in terms of the elements a_{ij} referred to XYZ. Thus, using the notation of (8–4),

$$\begin{aligned} a'_{21} &= \boldsymbol{j}' \cdot \boldsymbol{\Psi} \cdot \boldsymbol{i}' = (\boldsymbol{i}l_{21} + \boldsymbol{j}l_{22} + \boldsymbol{k}l_{23}) \cdot \boldsymbol{\Psi} \cdot (\boldsymbol{i}l_{11} + \boldsymbol{j}l_{12} + \boldsymbol{k}l_{13}) \\ &= \sum_{ij} l_{2i} a_{ij} l_{1j}. \end{aligned}$$

Since the distributive law holds for the product of a dyadic by a vector, this process is equivalent to substituting for $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in the nonian form (25–2) their values in terms of $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ as given by (8–4) and collecting like terms.

We shall now prove two important integral theorems for dyadics which follow from Gauss' theorem. Let $\boldsymbol{\Psi}$ be the dyadic (25–3) with antecedents and consequents which are functions of the coordinates, and $d\boldsymbol{\sigma}$ a vector element of area. Then

$$d\boldsymbol{\sigma} \cdot \boldsymbol{\Psi} = \mathbf{la} \cdot d\boldsymbol{\sigma} + \mathbf{mb} \cdot d\boldsymbol{\sigma} + \mathbf{nc} \cdot d\boldsymbol{\sigma},$$

and, if we integrate over any closed surface σ,

$$\int_\sigma d\boldsymbol{\sigma} \cdot \boldsymbol{\Psi} = \int_\sigma \mathbf{la} \cdot d\boldsymbol{\sigma} + \cdots = \int_\tau \nabla \cdot \overline{\mathbf{al}} d\tau + \cdots = \int_\tau \nabla \cdot \boldsymbol{\Psi} d\tau \qquad (25\text{–}4)$$

by (17–4). Next, if $\mathbf{r} \equiv \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$,

$$\begin{aligned} \int_\sigma \mathbf{r} \times (d\boldsymbol{\sigma} \cdot \boldsymbol{\Psi}) &= \int_\sigma \mathbf{r} \times \mathbf{la} \cdot d\boldsymbol{\sigma} + \cdots = \int_\tau \nabla \cdot \overline{\mathbf{ar} \times \mathbf{l}} d\tau + \cdots \\ &= \int_\tau \mathbf{r} \times (\nabla \cdot \overline{\mathbf{al}}) d\tau + \cdots + \int_\tau \mathbf{a} \cdot \nabla \mathbf{r} \times \mathbf{l} d\tau + \cdots. \end{aligned}$$

Now, as $\mathbf{a} \cdot \nabla = a_x \dfrac{\partial}{\partial x} + a_y \dfrac{\partial}{\partial y} + a_z \dfrac{\partial}{\partial z}$ and $\mathbf{r} \times \mathbf{l} = \boldsymbol{i}(yl_z - zl_y) + \boldsymbol{j}(zl_x - xl_z) + \boldsymbol{k}(xl_y - yl_x)$, it follows that

$$\begin{aligned} \mathbf{a} \cdot \nabla \mathbf{r} \times \mathbf{l} &= \boldsymbol{i}(a_y l_z - a_z l_y) + \boldsymbol{j}(a_z l_x - a_x l_z) + \boldsymbol{k}(a_x l_y - a_y l_x) \\ &= \mathbf{a} \times \mathbf{l}. \end{aligned}$$

Therefore

$$\int_\sigma \mathbf{r} \times (d\boldsymbol{\sigma} \cdot \boldsymbol{\Psi}) = \int_\tau \mathbf{r} \times (\nabla \cdot \boldsymbol{\Psi}) d\tau + \int_\tau (\mathbf{a} \times \mathbf{l} + \mathbf{b} \times \mathbf{m} + \mathbf{c} \times \mathbf{n}) d\tau.$$

The integrand of the last term is the vector formed by taking the sum of the vector products of the antecedent and the consequent in each dyad comprised in the dyadic $\mathbf{\Psi}$. This vector is called the *vector of the dyadic* and is designated by $\mathbf{\Psi}_*$. Since $(\mathbf{P} + \mathbf{Q}) \times \mathbf{R} = \mathbf{P} \times \mathbf{R} + \mathbf{Q} \times \mathbf{R}$, its value is independent of the form in which the dyadic is expressed. So finally we have

$$\int_\sigma \mathbf{r} \times (d\boldsymbol{\sigma} \cdot \mathbf{\Psi}) = \int_\tau \mathbf{r} \times (\nabla \cdot \mathbf{\Psi}) d\tau + \int_\tau \mathbf{\Psi}_* d\tau. \qquad (25\text{–}5)$$

Evidently if $\mathbf{\Psi}$ is symmetric, $\mathbf{\Psi}_* = 0$ and the last term in (25–5) vanishes. We shall have occasion to make an important application of these two theorems in the study of electromagnetic stresses.

26. Conjugate Dyadics. — If

$$\mathbf{\Psi} = \mathbf{al} + \mathbf{bm} + \mathbf{cn}$$

the *conjugate* of $\mathbf{\Psi}$ is defined as the dyadic

$$\mathbf{\Psi}_c = \mathbf{la} + \mathbf{mb} + \mathbf{nc}$$

obtained by interchanging the antecedents and the consequents of $\mathbf{\Psi}$. Evidently $\mathbf{P} \cdot \mathbf{\Psi} = \mathbf{\Psi}_c \cdot \mathbf{P}$ and $\mathbf{\Psi} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{\Psi}_c$ where $\mathbf{P}$ is any vector.

Inspection of (25–2) shows that if a dyadic is symmetric it is equal to its conjugate, and if it is skew-symmetric it is equal to the negative of its conjugate. Consequently if $\mathbf{\Psi}$ is symmetric, $\mathbf{P} \cdot \mathbf{\Psi} = \mathbf{\Psi} \cdot \mathbf{P}$ where $\mathbf{P}$ is any vector. Conversely, if $\mathbf{P} \cdot \mathbf{\Psi} = \mathbf{\Psi} \cdot \mathbf{P}$ where $\mathbf{P}$ is any vector, then $\mathbf{\Psi}$ is symmetric. For suppose $\mathbf{\Psi}$ to be expressed in the nonian form (25–2) and let $\mathbf{P} = \boldsymbol{i}$. Then

$$\boldsymbol{i} \cdot \mathbf{\Psi} = a_{11}\boldsymbol{i} + a_{12}\boldsymbol{j} + a_{13}\boldsymbol{k},$$

$$\mathbf{\Psi} \cdot \boldsymbol{i} = a_{11}\boldsymbol{i} + a_{21}\boldsymbol{j} + a_{31}\boldsymbol{k},$$

showing that $a_{12} = a_{21}$, $a_{31} = a_{13}$, and similarly for the other pairs of elements. Hence, as $a_{ij} = a_{ji}$, the dyadic is symmetric.

If $\mathbf{\Psi}$ is skew-symmetric, $\mathbf{P} \cdot \mathbf{\Psi} = - \mathbf{\Psi} \cdot \mathbf{P}$ where $\mathbf{P}$ is any vector. Conversely, if $\mathbf{P} \cdot \mathbf{\Psi} = - \mathbf{\Psi} \cdot \mathbf{P}$ where $\mathbf{P}$ is any vector, then $\mathbf{\Psi}$ is skew-symmetric. For, if $\mathbf{P} = \boldsymbol{i}$, we get, just as in the last paragraph, $a_{11} = - a_{11} = 0$, $a_{12} = - a_{21}$, $a_{31} = - a_{13}$, or in general $a_{ij} = - a_{ji}$, $a_{ii} = 0$.

Since $\mathbf{P} \cdot \mathbf{\Psi} = \mathbf{\Psi} \cdot \mathbf{P}$ is a relation between vectors which is independent of the orientation of the axes, a symmetric dyadic remains symmetric if we refer it to a new set of rectangular axes oriented

differently from the original set. The same statement holds for a skew-symmetric dyadic. The symmetry properties of dyadics, therefore, are unaffected by a rotation of the axes.

Any dyadic $\mathbf{\Psi}$ can be expressed as the sum of a symmetric and of a skew-symmetric part by writing it in the form

$$\mathbf{\Psi} = \tfrac{1}{2}(\mathbf{\Psi} + \mathbf{\Psi}_c) + \tfrac{1}{2}(\mathbf{\Psi} - \mathbf{\Psi}_c), \tag{26–1}$$

where $\frac{1}{2}(\mathbf{\Psi} + \mathbf{\Psi}_c)$ is symmetric since

$$(\mathbf{\Psi} + \mathbf{\Psi}_c)_c = \mathbf{\Psi}_c + (\mathbf{\Psi}_c)_c = \mathbf{\Psi}_c + \mathbf{\Psi} = \mathbf{\Psi} + \mathbf{\Psi}_c,$$

and $\frac{1}{2}(\mathbf{\Psi} - \mathbf{\Psi}_c)$ is skew-symmetric since

$$(\mathbf{\Psi} - \mathbf{\Psi}_c)_c = \mathbf{\Psi}_c - (\mathbf{\Psi}_c)_c = \mathbf{\Psi}_c - \mathbf{\Psi} = -(\mathbf{\Psi} - \mathbf{\Psi}_c).$$

27. **Normal Form of Dyadic.** — First we shall show that any three non-coplanar vectors may be chosen as antecedents or as consequents of a dyadic. Suppose that the dyadic is

$$\mathbf{\Psi} = \mathbf{al} + \mathbf{bm} + \mathbf{cn},$$

and that we wish to express it in terms of the consequents $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$. Provided these three vectors are not coplanar, we can write

$$\mathbf{l} = l_1\mathbf{p} + l_2\mathbf{q} + l_3\mathbf{r},$$

$$\mathbf{m} = m_1\mathbf{p} + m_2\mathbf{q} + m_3\mathbf{r},$$

$$\mathbf{n} = n_1\mathbf{p} + n_2\mathbf{q} + n_3\mathbf{r},$$

and therefore

$$\mathbf{\Psi} = \mathbf{ep} + \mathbf{fq} + \mathbf{gr}, \tag{27–1}$$

where

$$\mathbf{e} = l_1\mathbf{a} + m_1\mathbf{b} + n_1\mathbf{c},$$

$$\mathbf{f} = l_2\mathbf{a} + m_2\mathbf{b} + n_2\mathbf{c},$$

$$\mathbf{g} = l_3\mathbf{a} + m_3\mathbf{b} + n_3\mathbf{c}.$$

Similarly, if we wish to use $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ for antecedents,

$$\mathbf{\Psi} = \mathbf{pu} + \mathbf{qv} + \mathbf{rw}, \tag{27–2}$$

where

$$\mathbf{u} = a_1\mathbf{l} + b_1\mathbf{m} + c_1\mathbf{n},$$

$$\mathbf{v} = a_2\mathbf{l} + b_2\mathbf{m} + c_2\mathbf{n},$$

$$\mathbf{w} = a_3\mathbf{l} + b_3\mathbf{m} + c_3\mathbf{n},$$

the subscripts signifying components along $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ as before.

Now we shall prove that any dyadic may be put in the *normal form*

$$\Psi = a\mathbf{i}_1\mathbf{i}_2 + b\mathbf{j}_1\mathbf{j}_2 + c\mathbf{k}_1\mathbf{k}_2, \tag{27-3}$$

where $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ and $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$ are, in general, two differently oriented right-handed sets of orthogonal unit vectors. We can see at once from the nonian form (25–2) of the dyadic that it would be expected that the dyadic could be put in this form by suitably orienting the two sets of unit vectors. For suppose that we express the antecedent unit vectors in (25–2) in terms of unit vectors $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ of undetermined orientation, and the consequent unit vectors in terms of unit vectors $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$ of undetermined orientation. The resulting form of the dyadic will contain nine elements which are functions of the direction cosines of $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ and of $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$. Equating the six of these elements not on the principal diagonal to zero, we have just the right number of equations to determine the directions of $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ and $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$, since each of these sets of unit vectors has three degrees of freedom.

The detailed proof makes use of a unit vector $\boldsymbol{\alpha}$ of variable direction laid off from the origin. As $\boldsymbol{\alpha}$ takes on all possible directions, its terminus describes a unit sphere around the origin. A second vector $\boldsymbol{\beta}$, also laid off from the origin, is defined by the equation

$$\boldsymbol{\beta} = \Psi\cdot\boldsymbol{\alpha}.$$

As $\boldsymbol{\alpha}$ varies in direction, $\boldsymbol{\beta}$ varies in magnitude as well as in direction. Nevertheless, since all the elements of the dyadic Ψ are supposed to be finite, $\boldsymbol{\beta}$ never becomes infinite. Hence there must be some direction of $\boldsymbol{\alpha}$ for which the magnitude of $\boldsymbol{\beta}$ assumes a maximum value, or at least a value as great as any other. Let the fixed unit vector $\mathbf{i}_2$ be the value of $\boldsymbol{\alpha}$ for which $\boldsymbol{\beta}$ assumes this stationary value $\mathbf{a}$. Now consider all values of $\boldsymbol{\alpha}$ lying in the plane perpendicular to $\mathbf{i}_2$. Let the fixed unit vector $\mathbf{j}_2$ be the value of $\boldsymbol{\alpha}$ in this plane for which $\boldsymbol{\beta}$ assumes its greatest stationary value $\mathbf{b}$. Finally let $\mathbf{k}_2$ be a fixed unit vector perpendicular to $\mathbf{i}_2$ and $\mathbf{j}_2$ in the sense that makes $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$ a right-handed set. Then we can write Ψ in the form

$$\Psi = \mathbf{e}\mathbf{i}_2 + \mathbf{f}\mathbf{j}_2 + \mathbf{g}\mathbf{k}_2,$$

and

$$\boldsymbol{\beta} = \mathbf{e}\mathbf{i}_2\cdot\boldsymbol{\alpha} + \mathbf{f}\mathbf{j}_2\cdot\boldsymbol{\alpha} + \mathbf{g}\mathbf{k}_2\cdot\boldsymbol{\alpha}.$$

But, $\boldsymbol{\beta} = \mathbf{a}$ when $\boldsymbol{\alpha} = \mathbf{i}_2$. Therefore $\mathbf{e} = \mathbf{a}$. Similarly $\mathbf{f} = \mathbf{b}$. So

$$\Psi = \mathbf{a}\mathbf{i}_2 + \mathbf{b}\mathbf{j}_2 + \mathbf{g}\mathbf{k}_2. \tag{27-4}$$

All that remains is to prove that the three vectors **a, b, g** are orthogonal. As

$$\boldsymbol{\beta} = (\mathbf{a}i_2 + \mathbf{b}j_2 + \mathbf{g}k_2)\cdot\boldsymbol{\alpha},$$

$$d\boldsymbol{\beta} = (\mathbf{a}i_2 + \mathbf{b}j_2 + \mathbf{g}k_2)\cdot d\boldsymbol{\alpha},$$

and

$$\boldsymbol{\beta}\cdot d\boldsymbol{\beta} = \boldsymbol{\beta}\cdot\mathbf{a}i_2\cdot d\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot\mathbf{b}j_2\cdot d\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot\mathbf{g}k_2\cdot d\boldsymbol{\alpha}.$$

As $\boldsymbol{\beta}$ has the stationary value **a** when $\boldsymbol{\alpha} = i_2$, $\boldsymbol{\beta}\cdot d\boldsymbol{\beta} = 0$, and we have

$$\mathbf{a}\cdot\mathbf{b}j_2\cdot d\boldsymbol{\alpha} + \mathbf{a}\cdot\mathbf{g}k_2\cdot d\boldsymbol{\alpha} = 0,$$

since $d\boldsymbol{\alpha}$ must be perpendicular to i_2. Now $d\boldsymbol{\alpha}$ may have any direction parallel to the plane of j_2 and k_2. If it is taken parallel to j_2 we get $\mathbf{a}\cdot\mathbf{b} = 0$, whereas if it is taken parallel to k_2 we find $\mathbf{a}\cdot\mathbf{g} = 0$.

Next restrict $\boldsymbol{\alpha}$ to the j_2k_2 plane. Then, since $\boldsymbol{\beta}$ has the stationary value **b** when $\boldsymbol{\alpha} = j_2$,

$$\mathbf{b}\cdot\mathbf{g}k_2\cdot d\boldsymbol{\alpha} = 0,$$

which requires that $\mathbf{b}\cdot\mathbf{g} = 0$.

As **a**, **b**, **g** are orthogonal, we may select a right-handed set of orthogonal unit vectors i_1, j_1, k_1 by putting $\mathbf{a} = ai_1$, $\mathbf{b} = bj_1$, $\mathbf{g} = ck_1$, where c equals g or $-g$ according as **a**, **b**, **g** are right-handed or left-handed. Then (27–4) reduces to (27–3).

28. Normal Form of Symmetric Dyadic. — We have seen in article 27 that any dyadic can be put in the normal form

$$\Psi = ei_1i_2 + fj_1j_2 + gk_1k_2, \tag{28–1}$$

where i_1, j_1, k_1 and i_2, j_2, k_2 are two right-handed sets of orthogonal unit vectors. We shall now show that if Ψ is symmetric, (28–1) reduces to the yet simpler normal form

$$\Psi = ai_1i_1 + bj_1j_1 + ck_1k_1. \tag{28–2}$$

That it would be expected that the dyadic could be put in this form is evident from the considerations advanced in article 27 immediately following equation (27–3). For, since a symmetric dyadic is equal to its conjugate, the process described there results in only three equations of condition, which can be satisfied by a single set of suitably oriented orthogonal unit vectors.

To give a detailed proof for the case where e, f, g in (28–1) are all different, we note that, since $\mathbf{\Psi}$ is equal to its conjugate,

$$\mathbf{\Psi} = e\mathbf{i}_1\mathbf{i}_2 + f\mathbf{j}_1\mathbf{j}_2 + g\mathbf{k}_1\mathbf{k}_2 = e\mathbf{i}_2\mathbf{i}_1 + f\mathbf{j}_2\mathbf{j}_1 + g\mathbf{k}_2\mathbf{k}_1. \quad (28\text{–}3)$$

Without loss of generality we may assume $e > f > g$. Putting as before

$$\boldsymbol{\beta} = \mathbf{\Psi}\cdot\boldsymbol{\alpha}, \quad (28\text{–}4)$$

where $\boldsymbol{\alpha}$ is a unit vector of variable direction, we know that β^2 assumes its greatest stationary value e^2 when $\boldsymbol{\alpha} = \mathbf{i}_2$. But $\beta^2 = e^2$ from the second form of (28–3) only when $\boldsymbol{\alpha} = \pm\,\mathbf{i}_1$. Therefore $\mathbf{i}_2 = \pm\,\mathbf{i}_1$. Consequently $\mathbf{j}_1$ and $\mathbf{k}_1$, since they are perpendicular to $\mathbf{i}_1$, are also perpendicular to $\mathbf{i}_2$.

Next restrict $\boldsymbol{\alpha}$ to the plane perpendicular to $\mathbf{i}_2$. When $\boldsymbol{\alpha}$ is so restricted, β^2 assumes its greatest stationary value f^2 for $\boldsymbol{\alpha} = \mathbf{j}_2$. But $\beta^2 = f^2$ only when $\boldsymbol{\alpha}$, restricted to the plane perpendicular to $\mathbf{i}_1$, is equal to $\pm\,\mathbf{j}_1$. Hence $\mathbf{j}_2 = \pm\,\mathbf{j}_1$. Consequently we may put $e\mathbf{i}_2 = a\mathbf{i}_1$, $f\mathbf{j}_2 = b\mathbf{j}_1$, $g\mathbf{k}_2 = c\mathbf{k}_1$, where a equals e or $-e$ accordingly as $\mathbf{i}_2$ equals $\mathbf{i}_1$ or $-\mathbf{i}_1$, etc., thus obtaining (28–2).

This proof fails if two of the coefficients e, f, g in (28–1) are equal, for then one of the stationary values of β^2 exists for more than a single pair of opposite directions of $\boldsymbol{\alpha}$. However, this case can be included in the treatment given if it is considered as the limiting case reached when one of the originally different coefficients e, f, g is allowed to approach another in value.

The directions of the unit vectors $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ which give a symmetric dyadic $\mathbf{\Psi}$ its normal form (28–2) are known as the *principal axes* of the dyadic. It remains, now, to determine the directions of the principal axes and the magnitudes of the coefficients a, b, c when the dyadic is given in the nonian form (25–2) relative to an arbitrary set of unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Putting (28–2) in (28–4) we see that $\boldsymbol{\beta}$ has the direction of $\boldsymbol{\alpha}$ only when $\boldsymbol{\alpha}$ is parallel to one of the principal axes, $\mathbf{i}_1, \mathbf{j}_1$ or $\mathbf{k}_1$, and that then $\boldsymbol{\beta}$ is equal in magnitude to a, b, or c respectively. So the determination of the directions of the principal axes reduces to the solution of the equation

$$\mathbf{\Psi}\cdot\boldsymbol{\alpha} = \beta\boldsymbol{\alpha} \quad (28\text{–}5)$$

where β, as usual, represents the magnitude of the vector $\boldsymbol{\beta}$. Designating by l, m, n the direction cosines of $\boldsymbol{\alpha}$ relative to the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ with respect to which $\mathbf{\Psi}$ is given by (25–2),

$$\boldsymbol{\alpha} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k},$$

and the three scalar equations to which the vector equation (28–5) is equivalent are

$$\left.\begin{aligned}(a_{11} - \beta)l + a_{12}m + a_{31}n &= 0,\\ a_{12}l + (a_{22} - \beta)m + a_{23}n &= 0,\\ a_{31}l + a_{23}m + (a_{33} - \beta)n &= 0,\end{aligned}\right\} \tag{28–6}$$

where we have made use of the relation $a_{ij} = a_{ji}$.

Eliminating l, m, n we have

$$\begin{vmatrix} a_{11} - \beta & a_{12} & a_{31} \\ a_{12} & a_{22} - \beta & a_{23} \\ a_{31} & a_{23} & a_{33} - \beta \end{vmatrix} = 0. \tag{28–7}$$

Since we know that the dyadic can be put in the form (28–2), this cubic in β must have three real roots equal respectively to a, b and c. Using each of these roots in turn in (28–6) and making use as well of the relation $l^2 + m^2 + n^2 = 1$, we find the direction cosines of the principal axes relative to the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in terms of which the dyadic is expressed by (25–2).

We have already noted that if the unit vector $\boldsymbol{\alpha}$ in (28–4) is laid off from the origin, its terminus describes a unit sphere with center at the origin. Now we shall show that if the vector $\boldsymbol{\beta}$ is laid off from the origin as well, its terminus describes an ellipsoid with semi-axes a, b, c parallel to the principal axes of the dyadic. For let

$$\boldsymbol{\alpha} = l_1\boldsymbol{i}_1 + m_1\boldsymbol{j}_1 + n_1\boldsymbol{k}_1,$$

$$\boldsymbol{\beta} = \boldsymbol{i}_1 x + \boldsymbol{j}_1 y + \boldsymbol{k}_1 z,$$

where $\boldsymbol{i}_1$, $\boldsymbol{j}_1$, $\boldsymbol{k}_1$ are parallel to the principal axes of $\boldsymbol{\Psi}$ and l_1, m_1, n_1 are the direction cosines of $\boldsymbol{\alpha}$ relative to these axes. Then, putting (28–2) in (28–4) we get

$$x = al_1, \quad y = bm_1, \quad z = cn_1,$$

and remembering that $l_1{}^2 + m_1{}^2 + n_1{}^2 = 1$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{28–8}$$

Consequently the effect of the symmetric dyadic $\boldsymbol{\Psi}$ operating on the unit vector $\boldsymbol{\alpha}$ of variable direction is to distort the unit sphere

described by the terminus of the latter into an ellipsoid with semi-axes a, b, c along the principal axes of the dyadic.

Problem 28a. Reduce the symmetric dyadic

$$\begin{aligned}\Psi = 6\boldsymbol{ii} &+ 0\boldsymbol{ij} + 0\boldsymbol{ik}\\ + 0\boldsymbol{ji} &+ 34\boldsymbol{jj} + 12\boldsymbol{jk}\\ + 0\boldsymbol{ki} &+ 12\boldsymbol{kj} + 41\boldsymbol{kk}\end{aligned}$$

to normal form and find the direction cosines of the principal axes.

Ans. $\Psi = 6\boldsymbol{i}_1\boldsymbol{i}_1 + 50\boldsymbol{j}_1\boldsymbol{j}_1 + 25\boldsymbol{k}_1\boldsymbol{k}_1$. $l, m, n = 1, 0, 0; 0, \pm\frac{4}{5}, \mp\frac{3}{5}; 0, \pm\frac{3}{5}, \pm\frac{4}{5}$.

Problem 28b. Reduce the symmetric dyadic

$$\begin{aligned}\Psi = 53\boldsymbol{ii} &+ 6\boldsymbol{ij} + 12\boldsymbol{ik}\\ + 6\boldsymbol{ji} &+ 58\boldsymbol{jj} + 18\boldsymbol{jk}\\ + 12\boldsymbol{ki} &+ 18\boldsymbol{kj} + 85\boldsymbol{kk}\end{aligned}$$

to normal form and find the direction cosines of the principal axes.

Ans. $\Psi = 98\boldsymbol{i}_1\boldsymbol{i}_1 + 49\boldsymbol{j}_1\boldsymbol{j}_1 + 49\boldsymbol{k}_1\boldsymbol{k}_1$. $l, m, n = \frac{2}{7}, \frac{3}{7}, \frac{6}{7}$ for $\boldsymbol{i}_1$; others not determined because two coefficients are equal.

29. Normal Form of Skew-Symmetric Dyadic. — Let

$$\begin{aligned}\Psi = 0\boldsymbol{ii} &+ a_{12}\boldsymbol{ij} - a_{31}\boldsymbol{ik}\\ - a_{12}\boldsymbol{ji} &+ 0\boldsymbol{jj} + a_{23}\boldsymbol{jk}\\ + a_{31}\boldsymbol{ki} &- a_{23}\boldsymbol{kj} + 0\boldsymbol{kk}\end{aligned} \tag{29–1}$$

be a skew-symmetric dyadic, satisfying the condition $a_{ij} = - a_{ji}$. Then, if $\mathbf{P}$ is any vector,

$$\Psi\cdot\mathbf{P} = \boldsymbol{i}(P_y a_{12} - P_z a_{31}) + \boldsymbol{j}(P_z a_{23} - P_x a_{12}) + \boldsymbol{k}(P_x a_{31} - P_y a_{23}),$$

and if $\boldsymbol{\lambda}$ is the vector

$$\boldsymbol{\lambda} \equiv \boldsymbol{i}a_{23} + \boldsymbol{j}a_{31} + \boldsymbol{k}a_{12}$$

whose components are the three independent elements of Ψ, it appears that

$$\left.\begin{aligned}\Psi\cdot\mathbf{P} &= \mathbf{P}\times\boldsymbol{\lambda},\\ \mathbf{P}\cdot\Psi &= \boldsymbol{\lambda}\times\mathbf{P}.\end{aligned}\right\} \tag{29–2}$$

The direction of $\boldsymbol{\lambda}$ is called the *axis* of the skew-symmetric dyadic Ψ. If, now, the unit vector $\boldsymbol{i}_1$ is given the direction of $\boldsymbol{\lambda}$, we can

write $\boldsymbol{\lambda} = \lambda \boldsymbol{i}_1$, where $\lambda \equiv \sqrt{a_{23}{}^2 + a_{31}{}^2 + a_{12}{}^2}$. Choosing unit vectors $\boldsymbol{j}_1$ and $\boldsymbol{k}_1$ in any directions at right angles to $\boldsymbol{i}_1$ so as to constitute a right-handed rectangular set, we can in any event write

$$\begin{aligned}\boldsymbol{\Psi} = \; & 0\boldsymbol{i}_1\boldsymbol{i}_1 + a'_{12}\boldsymbol{i}_1\boldsymbol{j}_1 - a'_{31}\boldsymbol{i}_1\boldsymbol{k}_1 \\ & - a'_{12}\boldsymbol{j}_1\boldsymbol{i}_1 + 0\boldsymbol{j}_1\boldsymbol{j}_1 + a'_{23}\boldsymbol{j}_1\boldsymbol{k}_1 \\ & + a'_{31}\boldsymbol{k}_1\boldsymbol{i}_1 - a'_{23}\boldsymbol{k}_1\boldsymbol{j}_1 + 0\boldsymbol{k}_1\boldsymbol{k}_1\end{aligned}$$

for the dyadic $\boldsymbol{\Psi}$ referred to the new axes. But, on account of (29–2),

$$\boldsymbol{\Psi}\cdot\boldsymbol{i}_1 = - a'_{12}\boldsymbol{j}_1 + a'_{31}\boldsymbol{k}_1 = 0,$$

giving $a'_{12} = a'_{31} = 0$. Furthermore

$$\boldsymbol{\Psi}\cdot\boldsymbol{j}_1 = - a'_{23}\boldsymbol{k}_1 = - \lambda\boldsymbol{k}_1$$

giving $a'_{23} = \lambda$. So finally $\boldsymbol{\Psi}$ reduces to

$$\boldsymbol{\Psi} = \lambda(\boldsymbol{j}_1\boldsymbol{k}_1 - \boldsymbol{k}_1\boldsymbol{j}_1) \tag{29–3}$$

relative to the new axes. This is the normal form of a skew-symmetric dyadic. We see from (29–2) that operating with a skew-symmetric dyadic $\boldsymbol{\Psi}$ on a vector $\mathbf{P}$ is equivalent to taking the cross product of $\mathbf{P}$ by the vector $\boldsymbol{\lambda}$ formed from the elements of the dyadic. It is worthy of note that a skew-symmetric dyadic is always planar, as indicated by (29–3).

Problem 29a. Reduce the skew-symmetric dyadic

$$\begin{aligned}\boldsymbol{\Psi} = \; & 0ii + 2ij - 6ik \\ & - 2ji + 0jj + 3jk \\ & + 6ki - 3kj + 0kk\end{aligned}$$

to normal form and find the direction of its axis.

Ans. $\boldsymbol{\Psi} = 7(j_1k_1 - k_1j_1)$, $i_1 = \frac{3}{7}i + \frac{6}{7}j + \frac{2}{7}k$.

Problem 29b. Prove that the vector $\boldsymbol{\lambda}$ defined in this article is half the vector of the dyadic as defined in article 25.

Problem 29c. Let $\boldsymbol{\Psi}$ be the dyadic

$$\boldsymbol{\Psi} = \{ii + \cos\phi\,(jj + kk)\} + \{\sin\phi\,(kj - jk)\},$$

the first part of which is symmetric and the second skew-symmetric. Show that $\boldsymbol{\Psi}\cdot\mathbf{P}$, where $\mathbf{P}$ is any vector, is a vector obtained from $\mathbf{P}$ by a rotation through an angle ϕ about i. This dyadic is called a *versor*.

30. The Unit Dyadic. — The symmetric dyadic

$$\mathbf{I} = ii + jj + kk \tag{30–1}$$

is known as the *unit dyadic*, since, if $\mathbf{P}$ is any vector,

$$\left.\begin{aligned} \mathbf{I}\cdot\mathbf{P} &= ii\cdot\mathbf{P} + jj\cdot\mathbf{P} + kk\cdot\mathbf{P} = \mathbf{P}, \\ \mathbf{P}\cdot\mathbf{I} &= \mathbf{P}\cdot ii + \mathbf{P}\cdot jj + \mathbf{P}\cdot kk = \mathbf{P}. \end{aligned}\right\} \tag{30–2}$$

Let us refer the unit dyadic to a set of axes $X'Y'Z'$ with unit vectors i', j', k' parallel to the three axes respectively. Then

$$\begin{aligned} \mathbf{I} = a'_{11}i'i' &+ a'_{12}i'j' + a'_{13}i'k' \\ + a'_{21}j'i' &+ a'_{22}j'j' + a'_{23}j'k' \\ + a'_{31}k'i' &+ a'_{32}k'j' + a'_{33}k'k'. \end{aligned}$$

But, since $\mathbf{I}\cdot i' = i'$ from (30–2), we have $a'_{11} = 1$, $a'_{21} = a'_{31} = 0$, etc. So the unit dyadic takes the form

$$\mathbf{I} = i'i' + j'j' + k'k' \tag{30–3}$$

when referred to the axes $X'Y'Z'$.

Next let us choose any three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as consequents of $\mathbf{I}$. We have then

$$\mathbf{I} = \mathbf{ea} + \mathbf{fb} + \mathbf{gc}.$$

Form the scalar product of $\mathbf{I}$ by $\mathbf{b}\times\mathbf{c}$. Then, as $\mathbf{I}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}$,

$$\mathbf{I}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{ea}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{b}\times\mathbf{c},$$

and consequently

$$\mathbf{e} = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}.$$

Similarly

$$\mathbf{f} = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}},$$

$$\mathbf{g} = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}.$$

We see, therefore, that $\mathbf{e}, \mathbf{f}, \mathbf{g}$ are the set of vectors reciprocal to the set $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Hence the antecedents and consequents of the unit dyadic may be taken as any pair of mutually reciprocal sets of vectors. Using the notation of article 7,

$$\mathbf{I} = \mathbf{aa}' + \mathbf{bb}' + \mathbf{cc}' = \mathbf{a}'\mathbf{a} + \mathbf{b}'\mathbf{b} + \mathbf{c}'\mathbf{c}. \tag{30–4}$$

31. Products of Dyadics. — If Ψ and Φ are the two dyadics

$$\Psi = \mathbf{al} + \mathbf{bm} + \mathbf{cn},$$

$$\Phi = \mathbf{ep} + \mathbf{fq} + \mathbf{gr},$$

we define the scalar product by

$$\begin{aligned}\Psi\cdot\Phi &= (\mathbf{al} + \mathbf{bm} + \mathbf{cn})\cdot(\mathbf{ep} + \mathbf{fq} + \mathbf{gr}) \\ &= (\mathbf{l}\cdot\mathbf{e})\mathbf{ap} + (\mathbf{l}\cdot\mathbf{f})\mathbf{aq} + (\mathbf{l}\cdot\mathbf{g})\mathbf{ar} \\ &+ (\mathbf{m}\cdot\mathbf{e})\mathbf{bp} + (\mathbf{m}\cdot\mathbf{f})\mathbf{bq} + (\mathbf{m}\cdot\mathbf{g})\mathbf{br} \\ &+ (\mathbf{n}\cdot\mathbf{e})\mathbf{cp} + (\mathbf{n}\cdot\mathbf{f})\mathbf{cq} + (\mathbf{n}\cdot\mathbf{g})\mathbf{cr}.\end{aligned}$$

Evidently $\Psi\cdot\Phi$ is a dyadic itself. The commutative law does not hold for this product, since clearly $\Phi\cdot\Psi \neq \Psi\cdot\Phi$ in general. However the associative law holds since $(\Psi\cdot\Phi)\cdot\mathbf{X} = \Psi\cdot(\Phi\cdot\mathbf{X})$. If the dyadics are expressed in nonian form, so that

$$\begin{aligned}\Psi &= a_{11}\boldsymbol{ii} + a_{12}\boldsymbol{ij} + a_{13}\boldsymbol{ik} \\ &+ a_{21}\boldsymbol{ji} + a_{22}\boldsymbol{jj} + a_{23}\boldsymbol{jk} \\ &+ a_{31}\boldsymbol{ki} + a_{32}\boldsymbol{kj} + a_{33}\boldsymbol{kk},\end{aligned}$$

$$\begin{aligned}\Phi &= b_{11}\boldsymbol{ii} + b_{12}\boldsymbol{ij} + b_{13}\boldsymbol{ik} \\ &+ b_{21}\boldsymbol{ji} + b_{22}\boldsymbol{jj} + b_{23}\boldsymbol{jk} \\ &+ b_{31}\boldsymbol{ki} + b_{32}\boldsymbol{kj} + b_{33}\boldsymbol{kk},\end{aligned}$$

then

$$\begin{aligned}\Psi\cdot\Phi &= c_{11}\boldsymbol{ii} + c_{12}\boldsymbol{ij} + c_{13}\boldsymbol{ik} \\ &+ c_{21}\boldsymbol{ji} + c_{22}\boldsymbol{jj} + c_{23}\boldsymbol{jk} \\ &+ c_{31}\boldsymbol{ki} + c_{32}\boldsymbol{kj} + c_{33}\boldsymbol{kk},\end{aligned}$$

where

$$c_{ij} = \sum_k a_{ik}b_{kj}.$$

This law of multiplication is identical with that for matrices.

Evidently the dyadic $\mathbf{I}$ acts as unity when multiplied by another dyadic as well as when multiplied by a vector, since

$$\Psi\cdot\mathbf{I} = \mathbf{I}\cdot\Psi = \Psi.$$

32. Reciprocal Dyadics. — Suppose we wish to solve the equation

$$\mathbf{P} = \Psi \cdot \mathbf{Q} \tag{32–1}$$

for the vector $\mathbf{Q}$. As we have seen we can always put the dyadic Ψ in the normal form

$$\Psi = a i_1 i_2 + b j_1 j_2 + c k_1 k_2 \tag{32–2}$$

by a suitable orientation of the two sets of unit vectors i_1, j_1, k_1 and i_2, j_2, k_2. Hence

$$\mathbf{P} = i_1 a Q_{x2} + j_1 b Q_{y2} + k_1 c Q_{z2},$$

where Q_{x2}, Q_{y2}, Q_{z2}, are the components of $\mathbf{Q}$ along i_2, j_2, k_2. So, if P_{x1}, P_{y1}, P_{z1} are the components of $\mathbf{P}$ along i_1, j_1, k_1,

$$P_{x1} = aQ_{x2}, \quad P_{y1} = bQ_{y2}, \quad P_{z1} = cQ_{z2},$$

and

$$\mathbf{Q} = i_2 \frac{1}{a} P_{x1} + j_2 \frac{1}{b} P_{y1} + k_2 \frac{1}{c} P_{z1}$$

$$= \left(\frac{1}{a} i_2 i_1 + \frac{1}{b} j_2 j_1 + \frac{1}{c} k_2 k_1\right) \cdot \mathbf{P}.$$

The dyadic

$$\Psi^{-1} \equiv \frac{1}{a} i_2 i_1 + \frac{1}{b} j_2 j_1 + \frac{1}{c} k_2 k_1 \tag{32–3}$$

is known as the *reciprocal* of Ψ, and the solution of (32–1) for $\mathbf{Q}$ is

$$\mathbf{Q} = \Psi^{-1} \cdot \mathbf{P}. \tag{32–4}$$

Evidently $\Psi \cdot \Psi^{-1} = \Psi^{-1} \cdot \Psi = \mathbf{I}$. In fact, by virtue of this relation, we obtain (32–4) from (32–1) at once by taking the scalar product of both sides of the equation by Ψ^{-1}. We may define reciprocal dyadics, then, as two dyadics whose product is equal to $\mathbf{I}$.

If

$$\Psi = \mathbf{al} + \mathbf{bm} + \mathbf{cn} \tag{32–5}$$

is any dyadic, its reciprocal is

$$\Psi^{-1} = \mathbf{l'a'} + \mathbf{m'b'} + \mathbf{n'c'}, \tag{32–6}$$

where $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are reciprocal to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{l}', \mathbf{m}', \mathbf{n}'$ to $\mathbf{l}, \mathbf{m}, \mathbf{n}$. For, as shown in article 7, $\mathbf{l} \cdot \mathbf{l}' = \mathbf{m} \cdot \mathbf{m}' = \mathbf{n} \cdot \mathbf{n}' = 1$, $\mathbf{l} \cdot \mathbf{m}' = \mathbf{l} \cdot \mathbf{n}' = 0$, etc., and hence

$$\Psi \cdot \Psi^{-1} = \mathbf{aa'} + \mathbf{bb'} + \mathbf{cc'} = \mathbf{I}$$

by (30–4).

If Ψ is given in nonian form, we can write

$$\Psi = \boldsymbol{i}(a_{11}\boldsymbol{i} + a_{12}\boldsymbol{j} + a_{13}\boldsymbol{k}) + \boldsymbol{j}(a_{21}\boldsymbol{i} + a_{22}\boldsymbol{j} + a_{23}\boldsymbol{k}) + \boldsymbol{k}(a_{31}\boldsymbol{i} + a_{32}\boldsymbol{j} + a_{33}\boldsymbol{k}), \quad (32\text{–}7)$$

which has the form (32–5) with $\mathbf{a} = \boldsymbol{i}$, $\mathbf{b} = \boldsymbol{j}$, $\mathbf{c} = \boldsymbol{k}$, $\mathbf{l} = a_{11}\boldsymbol{i} + a_{12}\boldsymbol{j} + a_{13}\boldsymbol{k}$, $\mathbf{m} = a_{21}\boldsymbol{i} + a_{22}\boldsymbol{j} + a_{23}\boldsymbol{k}$, $\mathbf{n} = a_{31}\boldsymbol{i} + a_{32}\boldsymbol{j} + a_{33}\boldsymbol{k}$. Hence $\mathbf{a}' = \boldsymbol{i}$, $\mathbf{b}' = \boldsymbol{j}$, $\mathbf{c}' = \boldsymbol{k}$ and

$$\mathbf{l}' = \frac{\mathbf{m} \times \mathbf{n}}{\mathbf{l}\cdot\mathbf{m} \times \mathbf{n}} = \frac{\boldsymbol{i}A_{11} + \boldsymbol{j}A_{12} + \boldsymbol{k}A_{13}}{|a|},$$

$$\mathbf{m}' = \frac{\mathbf{n} \times \mathbf{l}}{\mathbf{l}\cdot\mathbf{m} \times \mathbf{n}} = \frac{\boldsymbol{i}A_{21} + \boldsymbol{j}A_{22} + \boldsymbol{k}A_{23}}{|a|},$$

$$\mathbf{n}' = \frac{\mathbf{l} \times \mathbf{m}}{\mathbf{l}\cdot\mathbf{m} \times \mathbf{n}} = \frac{\boldsymbol{i}A_{31} + \boldsymbol{j}A_{32} + \boldsymbol{k}A_{33}}{|a|},$$

where $|a|$ is the determinant of the a_{ij}'s and A_{ij} is the cofactor of a_{ij} in the determinant. Hence

$$\Psi^{-1} = \frac{1}{|a|}\{A_{11}\boldsymbol{ii} + A_{21}\boldsymbol{ij} + A_{31}\boldsymbol{ik} + A_{12}\boldsymbol{ji} + A_{22}\boldsymbol{jj} + A_{32}\boldsymbol{jk} + A_{13}\boldsymbol{ki} + A_{23}\boldsymbol{kj} + A_{33}\boldsymbol{kk}\}. \quad (32\text{–}8)$$

Let Ψ be a symmetric dyadic. Then $\mathbf{P}\cdot\Psi = \Psi\cdot\mathbf{P}$ where $\mathbf{P}$ is any vector. Consequently

$$(\Psi\cdot\mathbf{P})\cdot\Psi^{-1} = (\mathbf{P}\cdot\Psi)\cdot\Psi^{-1} = \mathbf{P} = \Psi^{-1}\cdot(\Psi\cdot\mathbf{P}).$$

Hence, as $\Psi\cdot\mathbf{P}$ is an arbitrary vector, Ψ^{-1} is symmetric. We conclude that the reciprocal of a symmetric dyadic is itself symmetric, and similarly that the reciprocal of a skew-symmetric dyadic is skew-symmetric.

Problem 32a. Find the reciprocal of

$$\Psi = 6\boldsymbol{ii} + 0\boldsymbol{ij} + 0\boldsymbol{ik} + 0\boldsymbol{ji} + 34\boldsymbol{jj} + 12\boldsymbol{jk} + 0\boldsymbol{ki} + 12\boldsymbol{kj} + 41\boldsymbol{kk}$$

and check your answer by showing that $\Psi\cdot\Psi^{-1} = \mathbf{I}$.

CHAPTER 2

THE PRINCIPLE OF RELATIVITY

33. **Equivalent Particle-Observers.** — An *event* is a phenomenon occurring at a particular point in space at a particular time. On analysis all physical measurements are found to consist of a record of coincidences of two or more events. In order to specify quantitatively the space interval and the time interval between two separate events by means of a record of coincidences it is customary to employ rigid material measuring rods and isochronous material clocks. The concepts of rigidity and of isochronism, however, require careful definition, for no material rods or clocks possess these properties by inherent right. For instance, a glass rod and a steel rod relatively at rest, which agree in length when compared at one place and time, will not in general be of the same length when compared at another place or time if there has been a change in temperature or in the strength of the electric or magnetic fields in which they lie. Again, earth time and moon time show slight discrepancies, and even the rates of two atomic clocks relatively at rest will not in general maintain their initial ratio if the electromagnetic field in which they are located has changed. In fact, the measurement of distance by a material rod or of time by a material clock suffers from the same uncertainty as the measurement of temperature by an expansion thermometer. Here it was long ago recognized that two thermometers employing different thermometric substances do not give concordant readings at other than the fixed points. The attendant confusion was not resolved until the invention by Kelvin of the thermodynamic scale. Although no laboratory technician would attempt to construct the innumerable reversible heat engines required to realize this ideal scale, he is able by indirect methods to compare with it the scale of the constant pressure or the constant volume gas thermometer which he uses in practice, and thereby to determine whether or not the volume or the pressure of a given gas increases linearly with the temperature. In the measurement of distance and of time we need

an ideal criterion analogous to Kelvin's thermodynamic scale of temperature by which to judge whether the number of placements end to end of a given material measuring rod and the number of oscillations of a given material clock are directly proportional to the space intervals and the time intervals, respectively, between pairs of events; in other words, a criterion of rigidity and isochronism. In the absence of such a criterion the original formulation of the relativity theory was based on undefined concepts of space and time intervals which could not be identified unambiguously with actual observations. Recently Milne[1] has shown how to supply the desired criterion by erecting the space-time structure on the foundation of a constant light-signal velocity. We shall present a modified form of his treatment.

Our fundamental concern is with relative motion, including relative rest as a special case. The concept of motion involves two essential entities: a *moving-element*, and an *observer* or *group of observers* relative to whom the motion of the moving-element takes place. A moving-element is characterized by a point, whether in a material body or not, which can be continuously identified as one and the same. An observer, in order to describe the motion of a moving-element, must possess a means of measuring the distance of the moving-element from himself and a means of measuring the lapse of time. In order to emphasize the fact that a single observer's measurements are confined to the single point occupied by himself, we shall designate such an observer a *particle-observer*. Each observer is supposed to possess a temporal intuition, that is to say, if two events E_1 and E_2 occur at himself, he can judge without ambiguity whether E_2 takes place before E_1, simultaneously with E_1, or after E_1. We shall provide each particle-observer with a device for assigning numbers $\tau_1, \tau_2, \cdots$ to events occurring at himself in such a way that, if event E_2 occurs simultaneously with E_1, the numbers τ_2 and τ_1 assigned to the respective events are the same, whereas, if E_2 occurs after E_1, then $\tau_2 > \tau_1$, and *vice versa*. This device, which may be quite arbitrary in all other characteristics than the one specified, we shall call a *clock*, and we shall name τ the *local time* of the particle-observer under consideration.

Next we shall adopt certain conventions which will enable a particle-observer P to employ light-signals, timed by his clock, in such a way as to describe quantitatively the motion of any moving-element M. Let P dispatch a light-signal to M at time τ_1. On

[1] E. A. Milne, *Relativity, Gravitation and World Structure*, Oxford, 1935.

arrival at M the signal is immediately sent back toward P, whom it reaches at time τ_3. Choosing an arbitrary constant c (a constant whose value, once chosen, remains the same for subsequent repetitions of the experiment) we define the *distance* r_2 of M from P when the signal reaches M by

$$r_2 \equiv \tfrac{1}{2}c(\tau_3 - \tau_1), \tag{33-1}$$

and we define P's value of the *time* at which the signal reaches M by

$$t_2 \equiv \tfrac{1}{2}(\tau_3 + \tau_1). \tag{33-2}$$

Since $\dfrac{r_2}{t_2 - \tau_1} = \dfrac{r_2}{\tau_3 - t_2} = c$, the constant c represents the *velocity* of the light-signal in terms of the conventions adopted for measuring distance and time at a remote point.

It should be noted that both r_2 and t_2 are computed by P from coincidences occurring at himself. The first represents P's estimation of the distance of M and the second P's estimation of the time when the signal reaches M. We shall call t_2 the *extended time* of P at M. If a second particle-observer is located at M, the local time of the second observer when the signal reaches him may be quite different from P's extended time t_2, and his estimation of the distance of P may not agree at all with r_2. The notation used designates local time measured at a particle-observer by the Greek letter τ, the computed time of an event at a distant point being indicated by the Italic letter t.

Evidently each one of two particle-observers P and P' constitutes a moving-element in the experience of the other. Thus P, acting as observer, may describe the motion of P', or P', as observer, may describe the motion of P. We shall designate by letters without primes local times measured by P or quantities computed therefrom, and by corresponding letters with primes local times measured by P' or quantities computed from these times. We attribute to light-signals dispatched from one particle-observer to another the following property: *If two light-signals are sent from one particle-observer to another, the light-signal which is dispatched later from the one will be received later by the other.* This fundamental principle underlies all the theory to be developed. In effect, it is equivalent to limiting our consideration to particle-observers with relative velocities less than the velocity of light.

Now suppose that a light-signal is dispatched from P toward

P' at time τ_1 and is received by the latter at time τ_2'. Let a second light-signal be dispatched from P' toward P at a time τ_1' earlier than τ_2' and be received by P at a time τ_2 later than τ_1, the time τ_1' being so chosen that $\tau_2' - \tau_1' = \tau_2 - \tau_1$. Then we say that τ_1 and τ_1' are *corresponding times*. Evidently this condition can always be fulfilled, for, if $\tau_2' - \tau_1' > \tau_2 - \tau_1$, the light-signal from P' can be replaced by one sent a little later, which will increase both τ_1' and τ_2 by virtue of the principle stated in the last paragraph, making $\tau_2' - \tau_1'$ smaller and $\tau_2 - \tau_1$ larger. The pair of light-signals under discussion is illustrated schematically by the lower solid lines in Fig. 28, the time being plotted vertically and the separation of P and P' horizontally. The curves PP and $P'P'$ are known as the *world-lines* of P and P' respectively.

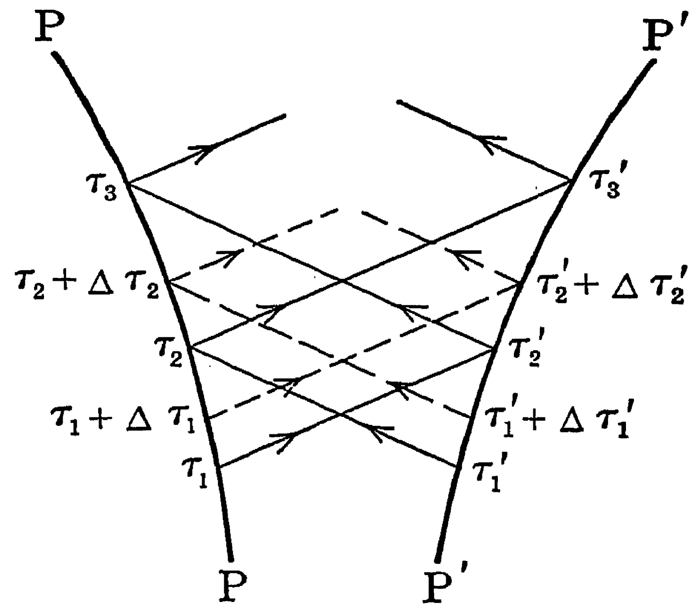

FIG. 28.

The statement that τ_1 and τ_1' are corresponding times does not in general imply that τ_2 and τ_2' are corresponding times also, for, if the signals received by P' and P at the times τ_2' and τ_2 are immediately returned and reach P and P' again at the times τ_3 and τ_3' respectively, the fact that $\tau_2' - \tau_1'$ and $\tau_2 - \tau_1$ are equal does not insure the equality of $\tau_3' - \tau_2'$ and $\tau_3 - \tau_2$. Only in the case of *equivalence*, to be considered next, are τ_2 and τ_2' necessarily corresponding times when τ_1 and τ_1' are. In conformity with our present notation we shall always designate corresponding times by identical subscripts.

In addition to the pair of light-signals dispatched from P and P' at the corresponding times τ_1 and τ_1', consider now another pair dispatched at the corresponding times $\tau_1 + \Delta\tau_1$ and $\tau_1' + \Delta\tau_1'$ respectively, as represented by the broken lines on the figure. Designating the times at which these signals reach P' and P by $\tau_2' + \Delta\tau_2'$ and $\tau_2 + \Delta\tau_2$,

$$(\tau_2' + \Delta\tau_2') - (\tau_1' + \Delta\tau_1') = (\tau_2 + \Delta\tau_2) - (\tau_1 + \Delta\tau_1).$$

But, as $\tau_2' - \tau_1' = \tau_2 - \tau_1$, it follows that $\Delta\tau_2' - \Delta\tau_1' = \Delta\tau_2 - \Delta\tau_1$. *Now, if* $\Delta\tau_1' = \Delta\tau_1$, *and hence* $\Delta\tau_2' = \Delta\tau_2$, *whatever* $\Delta\tau_1$ *may be, we*

say that the clocks of P' and P are equivalent, or that the two particle-observers are equivalent. If, in addition, the clocks of the two particle-observers are *set* so that $\tau_1' = \tau_1$, and therefore all corresponding times are identical, the clocks of the two observers are said to be *synchronous*. In future we shall deal primarily with particle-observers who are equivalent, and, when we are concerned with two particle-observers alone, we shall generally suppose their equivalent clocks to have been synchronized.

It follows from the definition of equivalence that all pairs of corresponding times at two equivalent particle-observers differ by the same amount, and that any pair of times which are the same amount earlier or later than a pair of corresponding times are themselves corresponding. Conversely, if all pairs of corresponding times differ by the same amount, the particle-observers are equivalent. If the clocks of two particle-observers are synchronous, corresponding times are identical. Hence synchronism implies equivalence, although equivalence may exist without synchronism.

Let P and P' (Fig. 28) be equivalent but not necessarily synchronous. If τ_1 and τ_1' are corresponding times, τ_2 and τ_2' are also, since $\tau_2' - \tau_1' = \tau_2 - \tau_1$. Furthermore, the times τ_3 and τ_3' at which the signals dispatched from P' and P at τ_2' and τ_2 are received are corresponding times, for $\tau_3' - \tau_2' = \tau_3 - \tau_2$ since τ_2 and τ_2' are corresponding times. Consequently $\tau_3' - \tau_1' = \tau_3 - \tau_1$. In the case under discussion we may say that the second pair of signals *interlocks* with the first, the signal dispatched from P at the time τ_1 being received by P' at the time τ_2' and immediately returned to P whom it reaches at the time τ_3. Evidently τ_2 is some function of τ_1, which could be obtained empirically by observing the values of τ_2 corresponding to different values of τ_1. Now, if τ_1 becomes τ_2, τ_2 becomes τ_3. So τ_3 must be the *same* function of τ_2 as τ_2 is of τ_1.

If we are given the law of motion of P' relative to P, that is, if we know r_2 as a function of P's extended time t_2 at P', we can express τ_3 as a function of τ_1 by (33–1) and (33–2). Let this functional relation be

$$\tau_3 = F(\tau_1).$$

Since we must have

$$\tau_3 = f(\tau_2), \quad \tau_2 = f(\tau_1), \tag{33–3}$$

it follows that our problem is to find the function f such that

$$f\{f(\tau_1)\} = F(\tau_1). \tag{33–4}$$

Not only are the relations (33–3) *necessary* for equivalence; they are also *sufficient*. For all we need do is to assign the values $\tau_1 + k$, $\tau_2 + k$, $\tau_3 + k$, $\cdots$ to the times τ_1', τ_2', τ_3', $\cdots$ at which the various signals in Fig. 28 are dispatched from P', where k is a constant. Then the clocks of the two particle-observers are equivalent. If $k = 0$ they are synchronous as well. On the other hand, if (33–3) is not satisfied τ_2' is not in general the time at P' corresponding to τ_2 at P when τ_1' and τ_1 correspond, and hence the assignment of the values $\tau_1 + k$, $\tau_2 + k$, $\cdots$ to the times τ_1', τ_2', $\cdots$ fails to make *corresponding* times at the two particle-observers differ by the same amount and therefore fails to make their clocks equivalent. The effect of satisfying (33–3) is to insure that τ_2' corresponds to τ_2 when τ_1' corresponds to τ_1, by requiring τ_2 to become τ_3 when τ_1 becomes τ_2.

Consider two equivalent particle-observers P and P'. From (33–1) the distance of P' from P at time τ_2' is

$$r_2 = \tfrac{1}{2}c(\tau_3 - \tau_1), \tag{33–5}$$

whereas the distance r_2' of P and P' at the corresponding time τ_2 is

$$r_2' = \tfrac{1}{2}c(\tau_3' - \tau_1'). \tag{33–6}$$

But $\tau_3' - \tau_1' = \tau_3 - \tau_1$ as the observers are equivalent. Hence $r_2' = r_2$, that is, *the two distances are the same at corresponding times.* As regards the velocity v_2 of P' relative to P, taken as positive if the two particle-observers are separating and negative if they are approaching, we find from (33–1) and (33–2)

$$v_2 = \frac{dr_2}{dt_2} = c\,\frac{\dfrac{d\tau_3}{d\tau_1} - 1}{\dfrac{d\tau_3}{d\tau_1} + 1}, \tag{33–7}$$

whereas the velocity v_2' of P relative to P' is

$$v_2' = \frac{dr_2'}{dt_2'} = c\,\frac{\dfrac{d\tau_3'}{d\tau_1'} - 1}{\dfrac{d\tau_3'}{d\tau_1'} + 1}. \tag{33–8}$$

As the two particle-observers are equivalent, $d\tau_3' = d\tau_3$ if $d\tau_1' = d\tau_1$. Therefore $v_2' = v_2$, that is, *the two velocities are the same at correspond-*

ing times. Evidently the conclusions reached here hold also for accelerations or for higher derivatives with respect to the time.

As $\frac{d\tau_3}{d\tau_1}$ is necessarily positive, we see from (33–7) that v_2 can never have an absolute magnitude greater than c. For, as $\frac{d\tau_3}{d\tau_1}$ increases from 0 to ∞, v_2 increases monotonically from $-c$ to c. If we solve equation (33–7) for $\frac{d\tau_3}{d\tau_1}$ we get

$$\frac{d\tau_3}{d\tau_1} = \frac{1 + \beta_2}{1 - \beta_2}, \quad \beta_2 \equiv \frac{v_2}{c}, \tag{33–9}$$

a relation we shall find useful later.

From (33–2) it is seen that

$$t_2' - \tau_1' = \tfrac{1}{2}(\tau_3' - \tau_1') = \tfrac{1}{2}(\tau_3 - \tau_1) = t_2 - \tau_1$$

for a pair of equivalent particle-observers P' and P. As the extended times t_2' at P' and t_2 at P differ from the corresponding times τ_1' and τ_1 respectively by the same amount, they have the characteristics of a pair of corresponding times. If P' and P are synchronous, $t_2' = t_2$. In this case, then, r_2', v_2', etc., are the *same* functions of the extended time t_2' of P' as r_2, v_2, etc., are of the extended time t_2 of P.

So far we have confined our attention to the motion of a moving-element relative to a single particle-observer. The next step is to refer motion to an assemblage of equivalent particle-observers. In fact, in a space of more than one dimension, motion cannot be completely defined by reference to one observer alone, for, in addition to motion along the line joining the moving-element to the observer, an angular motion about the observer may take place. We have recourse, then, to a *reference system*, which is defined as a dense assemblage of equivalent particle-observers filling all space, such that each particle-observer is at rest relative to and synchronous with every other. The specification of the motion of a moving-element relative to a reference system constitutes a complete definition of its motion.

In order to show that all the particle-observers of a group are equivalent each to each we proceed as follows. First we show that *any* two particle-observers P' and P'' of the group satisfy the conditions for equivalence with a specified particle-observer P and we provide them

with clocks equivalent to the clock of P. Next we show that P'' satisfies the conditions for equivalence with P', and finally we prove that the *same* clock which makes P'' equivalent to P also makes him equivalent to P'. If the motion of P'' relative to both P and P' is of the same type as that of P' relative to P (i.e., rest, constant velocity, constant acceleration, etc.), and the conditions (33–3) for equivalence of P' with P are satisfied, it is evident that the conditions for equivalence of P'' with both P and P' are also satisfied, and all that remains is to investigate whether the *same* clock at P'' is equivalent to both the clock at P and that at P'.

If two reference systems having the same geometry can be associated with two equivalent particle-observers P and P' respectively, the reference systems are said to be *equivalent*. Furthermore, if we may take as P and P' *any* pair of particle-observers in the two reference systems, the two reference systems are *homogeneously equivalent*. In this case each particle-observer in the one reference system is equivalent to *every* particle-observer in the other.

34. The Principle of Relativity. — The *principle of relativity* asserts that no preferred reference system exists in an effectively empty world such as we are concerned with in the study of electromagnetism. It follows that the laws of physics must be identically the same, and the constants appearing in these laws must have the same values, when they are determined relative to two different reference systems which have the same intrinsic properties. Now the sole function of a reference system is to locate events in space and time. Therefore two reference systems, associated with equivalent particle-observers, which have the same spacial geometry and the same constant light velocity, possess identical intrinsic properties. Such reference systems we have called equivalent. So the laws of nature in general, and the laws of electromagnetism in particular, must be the same when referred to equivalent reference systems. We shall see that this criterion enables us to deduce the rather complex set of field equations perfected by Maxwell, from very simple premises without recourse to experiment. In a very fundamental way we shall be able to explain *why* a current gives rise to a magnetic field, *why* a changing magnetic flux induces an electromotive force, *why* a moving charge is deflected at right-angles to its direction of motion by a magnetic field, and so forth.

The association of a pair of equivalent reference systems with a pair of equivalent particle-observers is necessary not only to insure

the identity of the two reference systems with respect to the underlying constant light-signal velocity, but also to make the distance and time scales of the two systems the same. For the specification of the light-signal velocity determines nothing further than the ratio of the distance scale to the time scale. If we alter these two scales in the same ratio, we change neither the geometry nor the light velocity of a reference system. When, however, two reference systems are associated with equivalent particle-observers, the scale of the one system is exactly specified in terms of that of the other. Only in this event are we justified in asserting that the values of the constants appearing in physical laws, as well as the form of these laws, must be the same relative to the two systems.

As a space of one dimension has no geometry, it is much simpler to treat than a space of three dimensions. Consequently we shall consider it first, confining our attention in the next five articles to equivalent particle-observers and equivalent reference systems in relative motion in a space of one dimension. Later, when we investigate three-dimensional reference systems, we shall limit our discussion to those systems which have Euclidean geometry. For we know that the reference system used in the laboratory is Euclidean within the limits of error of the most precise measurements and that the velocity of light in this system is constant. Our objective is to discover the existence of other equivalent Euclidean reference systems with the same constant light velocity. Then we can formulate laws of electrodynamics which have the same form and contain the same constants relative to each and every one of the group of Euclidean reference systems under consideration.

35. **One-Dimensional Reference System.** — Let P' be a particle-observer at rest relative to a second particle-observer P. Then the distance r_2 of P' from P is equal to the constant r, and (33–1) leads us to the equation

$$\tau_3 = \tau_1 + \frac{2r}{c}. \tag{35–1}$$

This equation can be split into the two equations

$$\tau_3 = \tau_2 + \frac{r}{c}, \quad \tau_2 = \tau_1 + \frac{r}{c}, \tag{35–2}$$

showing that P and P' are equivalent. We shall provide them with

synchronous clocks, so that corresponding times are identical. Then we may write (35–2) in the form

$$\tau_3 = \tau_2' + \frac{r}{c}, \quad \tau_2' = \tau_1 + \frac{r}{c}. \tag{35–3}$$

The solution (35–2) of (35–1) is unique under the conditions that the function $f(\tau)$ appearing in (33–3) is analytic and that $\tau_3 \geqslant \tau_2 \geqslant \tau_1$; conditions demanded by the physical nature of the problem. To show this, put $\tau \equiv \log x$, $r/c \equiv \log n$. Then (35–1) becomes

$$x_3 = n^2 x_1, \tag{35–4}$$

and x_3 and x_2 can be developed as power series in x_2 and x_1, respectively, that is,

$$x_3 = a_0 + a_1 x_2 + a_2 x_2^2 + \cdots,$$

$$x_2 = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots.$$

Since x_3 vanishes with x_1, and x_2 lies between x_1 and x_3, the coefficient a_0 must be identically zero. Eliminating x_2 we find

$$\begin{aligned} x_3 &= a_1(a_1 x_1 + a_2 x_1^2 + \cdots) + a_2(a_1 x_1 + a_2 x_1^2 + \cdots)^2 + \cdots \\ &= a_1^2 x_1 + a_1 a_2 (1 + a_1) x_1^2 + \cdots, \end{aligned}$$

which must be identical with (35–4). Therefore $a_1 = n$, $a_2 = 0, \cdots$, which shows that the only solution is that given by (35–2).

Since P and P' are synchronous, corresponding times τ_c and τ_c' are identical. Furthermore the distance r_2' of P from P' as computed by P' is equal to the distance $r_2 = r$ of P' from P as computed by P. Therefore P is at rest relative to P'.

The extended time t_2 of the event τ_2' (Fig. 28) as calculated by P is

$$t_2 \equiv \tfrac{1}{2}(\tau_3 + \tau_1) = \tau_2' \tag{35–5}$$

from (35–3). Consequently *the extended time of the one particle-observer at the other coincides with the local time of the other.*

Next we shall introduce a third particle-observer P'' at rest relative to P. Since we have shown that particle-observers relatively at rest are equivalent, it follows that P'' is equivalent to P. Let us provide P'' with a clock synchronous with that of P. We shall prove, first, that P'' is at rest relative to P' and therefore is equivalent to P', and, second, that his clock is synchronous with that of P'. Also we shall obtain the addition law for distances measured by particle-observers relatively at rest.

Let us designate by $r_{P'}$ and $r_{P''}$ respectively the distances of P' and P'' from P as computed by P, by r_P' and $r'_{P''}$ respectively the distances of P and P'' from P' as computed by P', and so forth. As shown above $r_P' = r_{P'}$. Now consider the interlocking signals $\tau_1 \longrightarrow \tau_3''$ and $\tau_3'' \longrightarrow \tau_5$ of Fig. 29. Let τ_3 be the time at P corresponding to τ_3'', and τ_3' the time (not necessarily equal to τ_3) at P' corresponding to τ_3''. As we are dealing with a one-dimensional space both the outgoing signal from P to P'' and the returning signal from P'' to

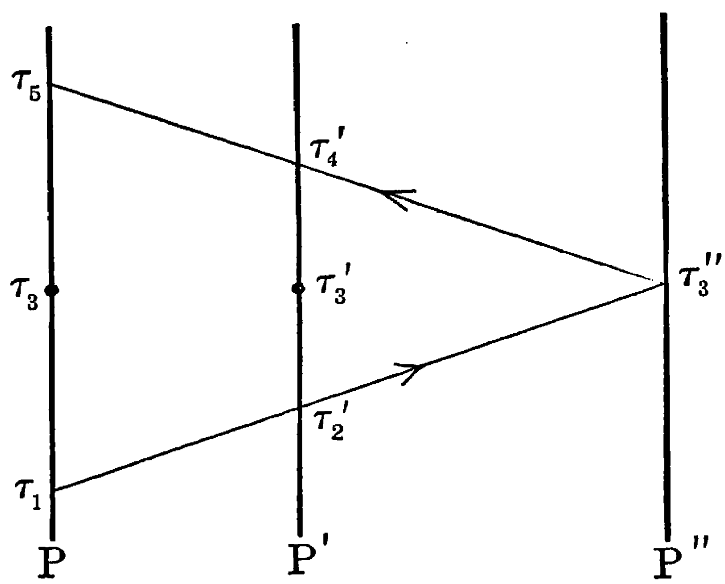

FIG. 29.

P pass through P'. The times of these events we shall designate by τ_2' and τ_4' respectively. Then, as P' is synchronous with P, we have

$$\tau_5 = \tau_4' + \frac{r_{P'}}{c}, \quad \tau_2' = \tau_1 + \frac{r_{P'}}{c}, \tag{35-6}$$

from (35–3). Moreover, as P'' is equivalent to P,

$$\tau_5 = \tau_3 + \frac{r_{P''}}{c}, \quad \tau_3 = \tau_1 + \frac{r_{P''}}{c}, \tag{35-7}$$

from (35–2). Hence

$$r'_{P''} \equiv \tfrac{1}{2}c(\tau_4' - \tau_2') = r_{P''} - r_{P'}.$$

As $r_{P'}$ and $r_{P''}$ are constants, this relation shows that P'' is at rest relative to P' and therefore is equivalent to P'. Since $r''_{P'} = r'_{P''}$, we see that the distance between P' and P'' as measured by either of these observers is equal to the excess of the distance of P'' from P over

that of P' from P as measured by P. Rearranging the terms in the last equation we have the *addition law of distance*:

$$r_{P''} = r_{P'} + r'_{P''}. \tag{35-8}$$

Now, as P'' is equivalent to P', (35–2) gives

$$\tau_4' = \tau_3' + \frac{r'_{P''}}{c}, \quad \tau_3' = \tau_2' + \frac{r'_{P''}}{c}. \tag{35-9}$$

Combining these equations with (35–6), (35–7) and (35–8) we find that $\tau_3' = \tau_3$. But $\tau_3'' = \tau_3$ as P'''s clock is synchronous with P's. Hence $\tau_3'' = \tau_3'$, proving that P'''s clock is also synchronous with that of P'.

We have shown in (35–5) that the local time of an event at P' or P'' is identical with P's extended time of the event. Hence there is no need of distinguishing between the local time of an event at one of the particle-observers and the extended time of the occurrence of that event in the experience of one of the other particle-observers. We may time distant events at P' or P'' by means of the extended time t of P, secure in the knowledge that the local time of the event is the same as P's extended time of the event.

Furthermore we have shown that the distance between P' and P'' as computed by either of them is the same as the excess of the distance of P'' from P over that of P' from P as calculated by P. Consequently we may introduce a distance scale with P as origin and employ only distances as computed by P. Then $r_{P''} - r_{P'}$ is the distance of P'' from P' as measured by any one of the three particle-observers. Distances such as $r_{P''} - r_{P'}$ may be said to constitute the *extended space* of P. All time and space measurements made in the extended time and extended space of P are identical with those made locally by any one of the observers concerned and may be substituted therefor.

It follows from the theorems proved in this article that we can adjoin to *any* particle-observer P a dense linear assemblage of particle-observers P', P'', P''',$\cdots$ at rest relative to P and synchronous with P. Each one of these particle-observers is at rest relative to every other and synchronous with every other. The aggregate of particle-observers, therefore, forms a linear reference system. As all time and space measurements made in the extended time and space of P are identical with those made locally by the particle-observers concerned, we may refer to the extended time and space of P as the time and space of the reference system.

Although a group of particle-observers at rest relative to any given particle-observer may be adjoined to the latter so as to form a linear reference system, it does not follow necessarily that the reference systems adjoined to two equivalent particle-observers in relative motion are homogeneously equivalent. Such is the case when the two equivalent particle-observers have a constant relative velocity, but not when they have a constant relative acceleration, as will be shown later.

It should be noted that the equivalence of two distant particle-observers does not completely determine the time scale of either. If, for instance, the two particle-observers P and P' whose world-lines are portrayed in Fig. 28 are assigned clocks which satisfy the conditions

$$\tau_{n+1} = \tau_n' + k, \quad \tau'_{n+1} = \tau_n + k,$$

where k is a constant and n any real number, these particle-observers are equivalent, whatever monotonically increasing set of numbers are assigned to the times of events occurring in the interval extending from τ_1 to τ_3. Furthermore the two particle-observers are relatively at rest, as is seen by comparing with (35–3), which shows that $k = r/c$. Therefore *any* two particle-observers whose world-lines do not intersect can be furnished clocks which will make them equivalent and relatively at rest, and this may be done in an infinite variety of ways. Nevertheless, however different the rates of different pairs of clocks which accomplish this purpose may be, every pair must make the time intervals $\tau_{n+2} - \tau_n$ the same, for each such time interval is equal to $2r/c$ by (33–1). Therefore the equivalence of the two particle-observers enables them to divide their time scales into equal intervals of magnitude $2r/c$, leaving any further subdivision arbitrary. The interval $2r/c$, however, becomes smaller the nearer together the two particle-observers. So, in the case of a dense assemblage of synchronous particle-observers relatively at rest, such as constitutes a reference system, the time scale of each particle-observer is completely determined.

In one-dimensional space, however, we may associate more than one distinct reference system with a single particle-observer P, since *any* pair of particle-observers whose world-lines do not intersect may be provided with clocks which make them equivalent and relatively at rest. For instance, let P' and P'' be two particle-observers whose world-lines cross each other, but do not intersect that of P. Then we

can provide P and P' with synchronous clocks which make them relatively at rest and build up an associated linear reference system. But we can do the same with P and P'' by means of another pair of clocks. Two such reference systems, associated with a single particle-observer, however, are not in general homogeneously equivalent. In a three-dimensional space, where we are concerned only with reference systems having the same geometry, no such arbitrariness exists.

36. Particle-Observers in Uniform Motion in a Space of One Dimension. — Let P' be a particle-observer moving with a velocity constant in magnitude and direction relative to a second particle-observer P. In the notation of article 33 the equation of motion of P' relative to P is

$$r_2 = v_2(t_2 - t_0), \tag{36–1}$$

where t_0 is the time at which P' passes P. Since r_2 as defined in article 33 is essentially positive, $v_2 = v$, where v is a positive constant, for $t_2 > t_0$, and $v_2 = - v$ for $t_2 < t_0$. This is in agreement with the convention of article 33, according to which the velocity of P' relative to P is positive when P' is receding from P, and negative when P' is approaching P. When P' passes P, v_2 changes sign, although its magnitude remains the same.

Putting $\frac{1}{2}c(\tau_3 - \tau_1)$ for r_2 and $\frac{1}{2}(\tau_3 + \tau_1)$ for t_2, equation (36–1) becomes

$$\frac{\tau_3 - t_0}{\tau_1 - t_0} = \frac{1 + \beta_2}{1 - \beta_2}. \tag{36–2}$$

This equation can be split into the two equations

$$\left.\begin{aligned} \sqrt{1 - \beta_2}(\tau_3 - t_0) &= \sqrt{1 + \beta_2}(\tau_2 - t_0), \\ \sqrt{1 - \beta_2}(\tau_2 - t_0) &= \sqrt{1 + \beta_2}(\tau_1 - t_0), \end{aligned}\right\} \tag{36–3}$$

in which τ_2 is the same function of τ_1 as τ_3 is of τ_2, thus establishing the equivalence of P and P'. This pair of equations may also be written

$$\tau_3 - t_0 = \frac{\tau_2 - t_0}{\sqrt{1 - \beta^2}} + \frac{r_2}{c}, \quad \tau_1 - t_0 = \frac{\tau_2 - t_0}{\sqrt{1 - \beta^2}} - \frac{r_2}{c}, \tag{36–4}$$

where $\beta \equiv v/c$, since $\beta_2{}^2 = \beta^2$ regardless of the sign of v_2.

Adding the two equations (36–4), we find for P's extended time of the event τ_2'

$$t_2 - t_0 \equiv \tfrac{1}{2}\{(\tau_3 - t_0) + (\tau_1 - t_0)\} = \frac{\tau_2 - t_0}{\sqrt{1 - \beta^2}}. \tag{36–5}$$

Since $\tau_2' = \tau_2 + k$, where k is a constant which is zero if the two particle-observers are synchronous, the relation between P's extended time of the event τ_2' and P''s local time of the same event is

$$t_2 - t_0 = \frac{\tau_2' - t_0'}{\sqrt{1 - \beta^2}}, \quad dt_2 = \frac{d\tau_2'}{\sqrt{1 - \beta^2}}, \tag{36–6}$$

where $t_0' = t_0 + k$.

If we plot τ_3 against τ_2, or τ_2 against τ_1, we get the graph shown in Fig. 30, which shows a discontinuity in the slope due to the change in the sign of v_2 but none in the function at the instant of coincidence of P and P'.

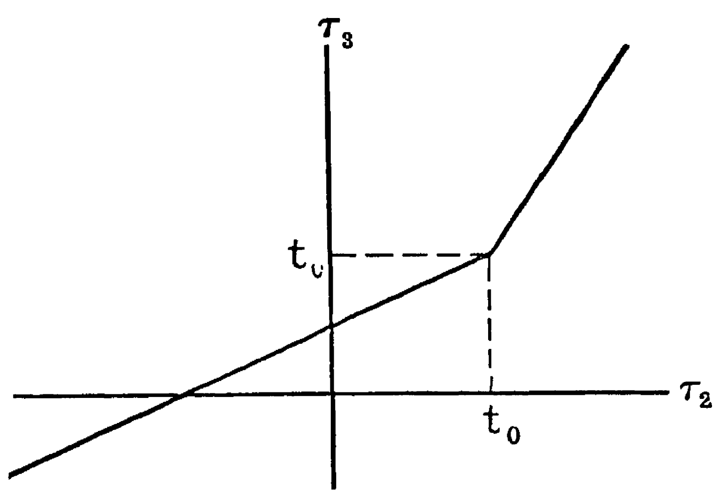

FIG. 30.

It was shown in article 33 that the velocity of P relative to P' as computed by P' and the velocity of P' relative to P as computed by P are the same at corresponding times. Therefore the two velocities have the same constant magnitude.

In addition to the two equivalent particle-observers P and P' moving with constant relative velocity we shall now introduce a third particle-observer P'' moving with constant velocity $v_{P''}$ relative to P and therefore equivalent to P. We shall show that the velocity $v'_{P''}$ of P'' relative to P' is constant and therefore that P'' is equivalent to P' as well as to P. Next we shall show that the same clock which makes P'' equivalent to P makes him equivalent to P'. Finally we shall obtain the addition law of velocity. In order to make our notation consistent throughout, we shall designate here the velocity of P' relative to P by $v_{P'}$ and that of P relative to P' by v_P'. As shown above $v_P' = v_{P'}$.

Consider the interlocking signals $\tau_1 \longrightarrow \tau_3''$ and $\tau_3'' \longrightarrow \tau_5$ of Fig. 31. By (33–9) we have

$$\frac{d\tau_5}{d\tau_1} = \frac{1 + \beta_{P''}}{1 - \beta_{P''}}, \tag{36–7}$$

and

$$\frac{d\tau_4'}{d\tau_2'} = \frac{1 + \beta'_{P''}}{1 - \beta'_{P''}}, \tag{36–8}$$

where $\beta_{P''} \equiv v_{P''}/c$ is constant by hypothesis.

Now as P and P' are equivalent,

$$\frac{d\tau_2'}{d\tau_1} = \frac{d\tau_2}{d\tau_1} = \sqrt{\frac{1 + \beta_{P'}}{1 - \beta_{P'}}}, \quad \frac{d\tau_5}{d\tau_4'} = \frac{d\tau_5}{d\tau_4} = \sqrt{\frac{1 + \beta_{P'}}{1 - \beta_{P'}}},$$

from (36–3). But

$$\frac{d\tau_4'}{d\tau_2'} = \frac{d\tau_4'}{d\tau_5}\frac{d\tau_5}{d\tau_1}\frac{d\tau_1}{d\tau_2'} = \frac{1 + \beta_{P''}}{1 - \beta_{P''}}\frac{1 - \beta_{P'}}{1 + \beta_{P'}}.$$

Comparing with (36–8) we find

$$\frac{1 + \beta'_{P''}}{1 - \beta'_{P''}} = \frac{1 + \beta_{P''}}{1 - \beta_{P''}}\frac{1 - \beta_{P'}}{1 + \beta_{P'}} \qquad (36\text{–}9)$$

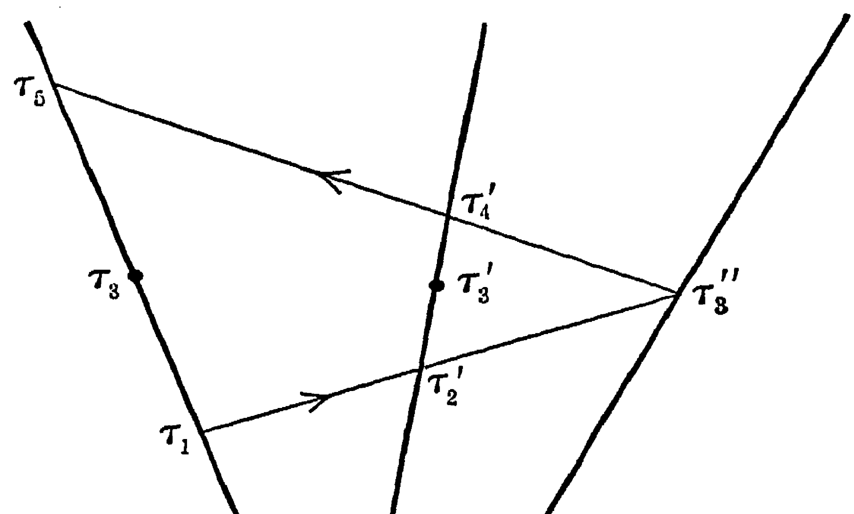

Fig. 31.

which shows that $\beta'_{P''}$ and therefore $v'_{P''}$ is constant. Hence P'' is equivalent to P' as well as to P.

All that remains is to show that the clock furnished to P'' to make him equivalent to P also makes him equivalent to P', that is, in accord with the definition in article 33, to show that $\Delta\tau_3'' = \Delta\tau_3'$, where τ_3' is the time at P' corresponding to τ_3'' at P''. Let τ_3 be the time at P corresponding to τ_3'' at P'', although not necessarily to τ_3' at P'. Applying (36–3) first to P and P'' and then to P' and P'', we have

$$\frac{d\tau_3}{d\tau_1} = \sqrt{\frac{1 + \beta_{P''}}{1 - \beta_{P''}}}$$

and

$$\frac{d\tau_3'}{d\tau_2'} = \sqrt{\frac{1 + \beta'_{P''}}{1 - \beta'_{P''}}}.$$

From the latter

$$\frac{d\tau_3'}{d\tau_1} = \frac{d\tau_3'}{d\tau_2'}\frac{d\tau_2'}{d\tau_1} = \sqrt{\frac{1 + \beta'_{P''}}{1 - \beta'_{P''}}}\sqrt{\frac{1 + \beta_{P'}}{1 - \beta_{P'}}} = \sqrt{\frac{1 + \beta_{P''}}{1 - \beta_{P''}}}$$

with the aid of (36–9). Hence $d\tau_3' = d\tau_3$. But $d\tau_3'' = d\tau_3$ as P'''s clock is equivalent to P's. Consequently $d\tau_3'' = d\tau_3'$ and $\Delta\tau_3'' = \Delta\tau_3'$. It may be remarked that if the clocks of P' and P'' are set to be synchronous with the clock of P they will not be synchronous with each other except in the case where P' and P'' pass P simultaneously.

We may write (36–9) in the more symmetric form

$$\frac{1 - \beta_{P''}}{1 + \beta_{P''}} = \frac{1 - \beta_{P'}}{1 + \beta_{P'}}\frac{1 - \beta'_{P''}}{1 + \beta'_{P''}}, \qquad (36\text{–}10)$$

where each β is positive if the particles concerned are receding and negative if they are approaching. This is the *addition law of velocity* obtained by Einstein [2] in 1905. Solving for $v_{P''}$ it assumes the more usual form

$$v_{P''} = \frac{v_{P'} + v'_{P''}}{1 + \dfrac{v_{P'}v'_{P''}}{c^2}}. \qquad (36\text{–}11)$$

As the particle-observers are equivalent, $v''_P = v_{P''}$, $v_P' = v_{P'}$ and $v''_{P'} = v'_{P''}$.

From (36–10) it is clear that if the positive magnitudes of both $\beta_{P'}$ and $\beta'_{P''}$ are less than unity, that of $\beta_{P''}$ is less than unity. Therefore the addition of two velocities less than c always results in a velocity less than c. Relation (36–10) can be extended immediately to the case of any number of equivalent particle-observers P, P', P'', P''', $\cdots P^{(n)}$ moving with constant relative velocities. The velocity of $P^{(n)}$ relative to P is given by

$$\frac{1 - \beta_{P^{(n)}}}{1 + \beta_{P^{(n)}}} = \frac{1 - \beta_{P'}}{1 + \beta_{P'}}\frac{1 - \beta'_{P''}}{1 + \beta'_{P''}}\frac{1 - \beta''_{P'''}}{1 + \beta''_{P'''}}\cdots\frac{1 - \beta^{(n-1)}_{P^{(n)}}}{1 + \beta^{(n-1)}_{P^{(n)}}}, \qquad (36\text{–}12)$$

where the β's, as before, may be either positive or negative according as the particles concerned are separating or approaching.

As proved in article 35 we can adjoin to each of two equivalent particle-observers P and P' who have a constant relative velocity a dense linear assemblage of synchronous particle-observers relatively

[2] A. Einstein, Ann. d. Phys. 17, p. 891 (1905).

at rest. Each such assemblage constitutes a reference system. Furthermore we have just shown that a third particle-observer P'' moving with constant velocity relative to P has a constant velocity relative to P' and is equivalent to both P and P'. If the velocity $v_{P''}$ of P'' relative to P is made equal to the velocity $v_{P'}$ of P' relative to P, the velocity of P'' relative to P' becomes zero and P'' becomes one of the particle-observers in P''s adjoined reference system. Consequently the reference systems adjoined to P and P' are homogeneously equivalent, that is, each particle-observer of the one is equivalent to every particle-observer of the other. Altogether we can conceive an infinity of such homogeneously equivalent reference systems having velocities ranging by infinitesimal steps from $-c$ to c relative to any selected system. We shall denote the reference systems associated with the particle-observers P, P', P'', $\cdots$ by S, S', S'', $\cdots$ respectively. As each particle-observer in one reference system has the same velocity relative to every particle-observer in any other, we may speak unambiguously of the velocity of one reference system relative to another.

Problem 36a. Show that the solution (36–3) of (36–1) is unique.

Problem 36b. Particle-observers P' and P'' have different constant velocities relative to P. Show that it is impossible to make the three particle-observers synchronous each with each unless they meet simultaneously.

37. Lorentz Space-Time Transformation in One-Dimensional Space. — Consider two synchronous particle-observers P and P' moving with constant relative velocity v. Take P as origin of an associated reference system S and P' as origin of an equivalent associated reference system S', the coordinates x and x' in S and S' respectively being measured positive in the direction of the velocity of S' relative to S. Let Ot and Ot' (Fig. 32) be the world-lines of P and P' respectively. We wish to find the relations between P's and P''s specifications of the position and extended time of the event Q_1.

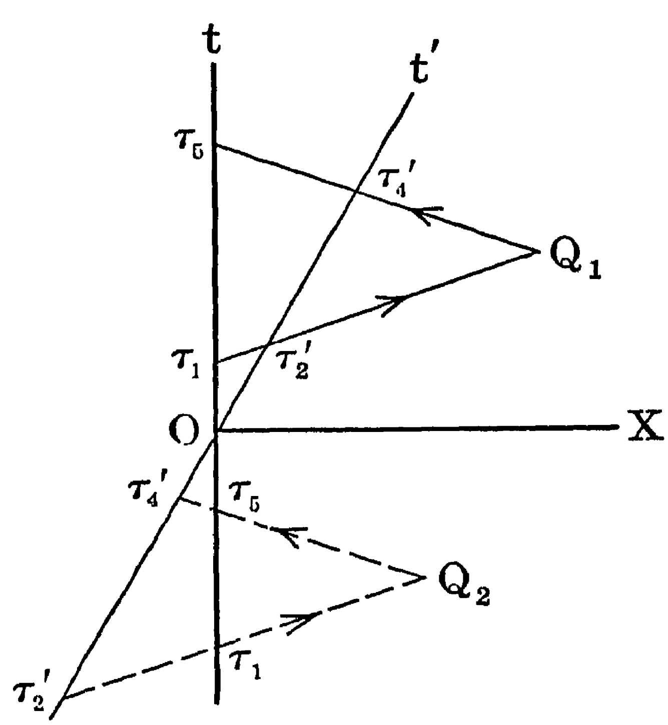

FIG. 32.

Let us take the moment when P' passes P as the zero of their synchronous local times. Then $t_0 = 0$ in (36–3) and, as corresponding times at P and P' are the same, these equations give for the signals indicated in the upper part of the figure

$$\sqrt{1-\beta}\,\tau_5 = \sqrt{1+\beta}\,\tau_4', \quad \sqrt{1-\beta}\,\tau_2' = \sqrt{1+\beta}\,\tau_1, \qquad (37\text{–}1)$$

where $\beta \equiv v/c$. The same equations hold for the signals shown by broken lines in the lower part of the figure, sent and received while P' is approaching P, that is, when the local times are negative.

Now P''s estimation of the distance x' of the event Q_1 is

$$x' \equiv \tfrac{1}{2}c(\tau_4' - \tau_2') = \tfrac{1}{2}c\left\{\sqrt{\frac{1-\beta}{1+\beta}}\,\tau_5 - \sqrt{\frac{1+\beta}{1-\beta}}\,\tau_1\right\}$$

$$= \frac{1}{\sqrt{1-\beta^2}}\left\{\tfrac{1}{2}c(\tau_5 - \tau_1) - \tfrac{1}{2}v(\tau_5 + \tau_1)\right\},$$

and P''s extended time of the event Q_1 is

$$t' \equiv \tfrac{1}{2}(\tau_4' + \tau_2') = \tfrac{1}{2}\left\{\sqrt{\frac{1-\beta}{1+\beta}}\,\tau_5 + \sqrt{\frac{1+\beta}{1-\beta}}\,\tau_1\right\}$$

$$= \frac{1}{\sqrt{1-\beta^2}}\left\{\tfrac{1}{2}(\tau_5 + \tau_1) - \tfrac{1}{2}\beta(\tau_5 - \tau_1)\right\}.$$

Since $x = \tfrac{1}{2}c(\tau_5 - \tau_1)$ and $t = \tfrac{1}{2}(\tau_5 + \tau_1)$ these become

$$x' = \frac{1}{\sqrt{1-\beta^2}}(x - vt), \qquad (37\text{–}2)$$

$$t' = \frac{1}{\sqrt{1-\beta^2}}\left(t - \frac{\beta}{c}x\right). \qquad (37\text{–}3)$$

Similarly, or by solving (37–2) and (37–3) for x and t,

$$x = \frac{1}{\sqrt{1-\beta^2}}(x' + vt'), \qquad (37\text{–}4)$$

$$t = \frac{1}{\sqrt{1-\beta^2}}\left(t' + \frac{\beta}{c}x'\right). \qquad (37\text{–}5)$$

As the extended distances and times measured by P and P' coincide with the distances and times measured locally in their respective reference systems, we can interpret the symbols in these equations in

terms of measurements made by their associated particle-observers at the point and time of occurrence of any event. This will be our general procedure henceforth. The group of equations (37–2) to (37–5) comprise the *Lorentz space-time transformation*.[3] We shall consider some of their implications.

First, consider a rod at rest in system S'. Let x_a' and x_b' be the coordinates of the two ends a and b of the rod relative to S'. Then $x_b' - x_a'$ is the length of the rod as measured by any particle-observer in S'. The length of the rod as measured by any particle-observer in S is naturally defined as the difference $x_b - x_a$ of the coordinates of b and a in S at a given time t. From (37–2), then,

$$x_b - x_a = \sqrt{1 - \beta^2}\,(x_b' - x_a'), \tag{37–6}$$

which shows that S's measurement of the length of the rod is less than S''s in the ratio $\sqrt{1 - \beta^2} : 1$.

We say that a rod remains *rigid* when transferred from one reference system to another if it maintains the same length relative to the second reference system that it had originally relative to the first. Therefore a rigid rod suffers a contraction in its measured length when it is set into uniform motion relative to the observers of a reference system of the type we are discussing. This shortening is known as the *Fitzgerald-Lorentz contraction*.

Next let us rate a clock at rest in S' by means of the mutually synchronous clocks in S with which it comes into coincidence in the course of its motion. As x' remains constant, we need (37–5), which gives

$$t_b - t_a = \frac{1}{\sqrt{1 - \beta^2}}\,(t_b' - t_a') \tag{37–7}$$

for the relation between the time interval $t_b' - t_a'$ on the clock at rest in S' and the time interval $t_b - t_a$ measured by the clocks in S by which it passes. We conclude, therefore, that a clock in uniform motion relative to a reference system of the type which we are discussing runs slow in the ratio $\sqrt{1 - \beta^2} : 1$ as compared with mutually synchronous, equivalent clocks at rest.

From our definition of rigidity it follows that, if a rigidly built clock of S is transferred to S', it will run at the same rate as the clocks of S'. For the simplest type of clock consists of a rigid bar provided

[3] H. A. Lorentz, Proc. Acad. Sci. Amsterdam, 6, p. 809 (1904).

at its two ends with mirrors between which a light-signal passes back and forth. As the bar maintains its original length when transferred from one system to another, and the velocity of light is the same in each, it is evident that a clock of S constructed in this manner will run at the same rate as the clocks of S' when transferred to S'. Hence the same must be true of any type of rigid clock. Consequently, when such a clock is given a uniform motion relative to S, it runs slower as rated by S in the ratio $\sqrt{1 - \beta^2} : 1$ than when at rest in S.

Obviously the effects described are entirely reciprocal, that is, the measured length of a rod in S is less as determined by observers in S' than as determined by observers in S, and a clock in S runs slow as rated by clocks in S'.

From (37–2) and (37–3) we find that

$$x'^2 - c^2t'^2 = x^2 - c^2t^2, \tag{37–8}$$

and also that

$$dx'^2 - c^2dt'^2 = dx^2 - c^2dt^2. \tag{37–9}$$

These quadratic expressions are invariants of the Lorentz transformation. The second measures the *space-time interval* between two nearby events. Although it is an invariant for reference systems moving with constant relative velocity we shall see later that it is not an invariant for reference systems moving with constant relative acceleration.

In Fig. 32 we have shown one line of constant x', namely the world-line Ot' of the particle-observer P' located at the origin ($x' = 0$) of S'. From (37–2) we see that all lines of constant x', that is, world-lines of particles at rest in S', are straight lines parallel to Ot'. A more convenient diagram is obtained by taking x and ct as coordinates in system S, as in Fig. 33. Then lines of constant x', such as the world-line $O\text{-}cT'$ of P', make the angle $\alpha = \tan^{-1}\beta$ with $O\text{-}ct$, and, from (37–3), lines of constant t', such as OX', make the same angle with Ox in the opposite sense. If, therefore, we locate an event Q_1 in the space-time of S by means of the rectangular coordinates x and ct, this event is located in the space-time of S' by the oblique coordinates X' and cT'. Of course we might just as well have used rectangular coordinates to locate events in the space-time of S', in which case it would have been necessary to employ oblique coordinates with an obtuse angle between the axes to locate the same events in the space-time of S. The interval $\overline{Q_1Q_2}$ between two events Q_1 and Q_2,

then, is resolved differently into space and time components according to whether the events are described in terms of the space-time of S or in terms of that of S'.

To find the relation between x' and X', t' and T', we must compare the geometrical transformation

$$x = X' \cos \alpha + cT' \sin \alpha,$$

$$ct = cT' \cos \alpha + X' \sin \alpha,$$

of the coordinates used in Fig. 33 with the Lorentz transformation (37–4) and (37–5). We find

$$x' = X' \sqrt{\frac{1 - \beta^2}{1 + \beta^2}}, \quad t' = T' \sqrt{\frac{1 - \beta^2}{1 + \beta^2}}, \tag{37–10}$$

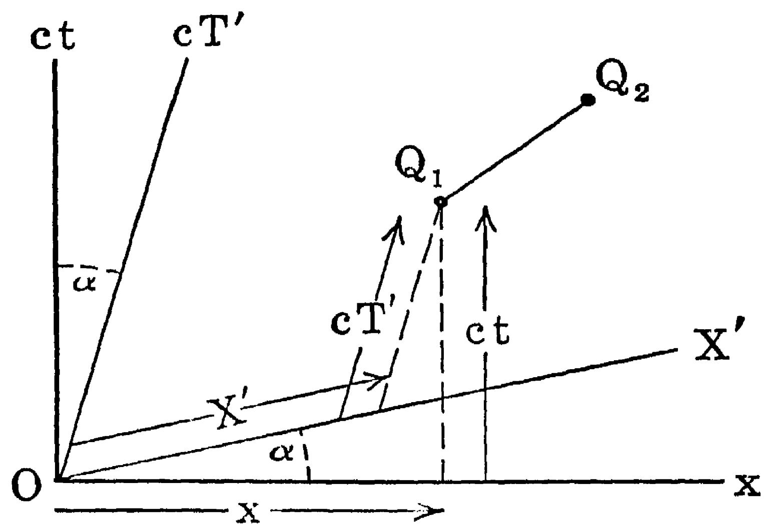

FIG. 33.

showing that the distance and time scales are larger in S' than in S. In the limiting case $\beta = 1$ the scale in S' becomes infinite, that is, any two events which are separated by a finite interval in S become coincident in S'. Thus, to an observer traveling with a velocity approaching that of light from a distant star to the earth, the distance traversed and the time consumed both approach zero.

When S' has a velocity approaching the limiting velocity c relative to S, α approaches $\pi/4$ and the X' and cT' axes in Fig. 33 approach coincidence. Let us designate an observer at the origin O (Fig. 34) of S at the time $t = 0$ as *Here-Now*. If, then, we draw the broken lines through the origin at 45° with the axes, these lines divide space-time into an absolute future and an absolute past, an absolute right

and an absolute left, with respect to the origin *Here-Now*. For, while there exist reference systems for which event Q_1 occurs to the right of, at the same place as, or to the left of O, there is none for which Q_1 does not occur later in time than O. On the other hand, a reference system can be found for which Q_2 occurs earlier than, at the same time as, or later than O, but none for which Q_2 does not occur to the right of O. Similarly Q_3 lies in the absolute past and Q_4 to the absolute left of O.

The space-time diagram we have been discussing was proposed by Minkowski [4] in 1908. A more convenient geometrical representa-

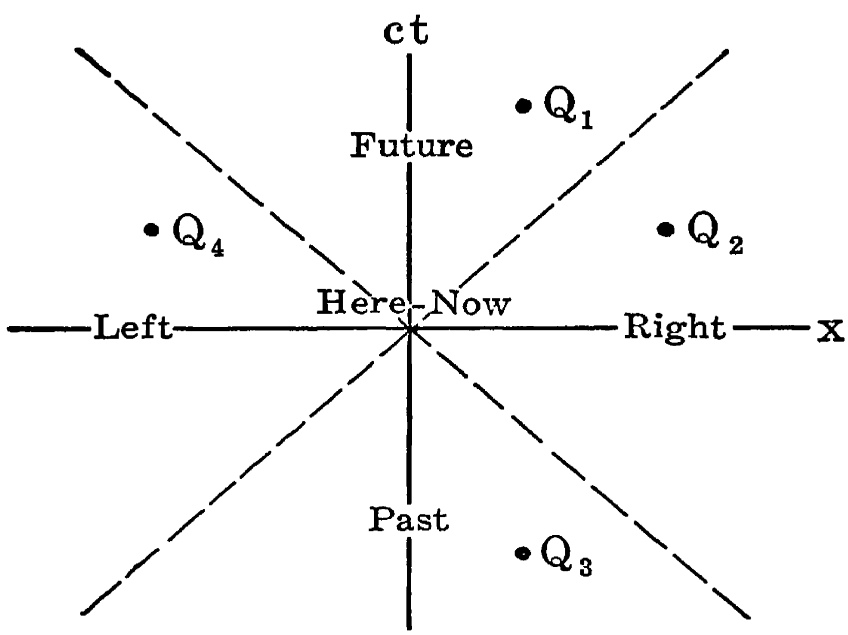

FIG. 34.

tion, introduced by the same author, employs x and the imaginary time $l \equiv \sqrt{-1}ct$ as coordinates. In terms of these variables the Lorentz transformation (37–2) and (37–3) becomes

$$x' = \frac{1}{\sqrt{1-\beta^2}}x + \frac{i\beta}{\sqrt{1-\beta^2}}l, \tag{37–11}$$

$$l' = \frac{1}{\sqrt{1-\beta^2}}l - \frac{i\beta}{\sqrt{1-\beta^2}}x, \tag{37–12}$$

where $i \equiv \sqrt{-1}$. As

$$\left(\frac{1}{\sqrt{1-\beta^2}}\right)^2 + \left(\frac{i\beta}{\sqrt{1-\beta^2}}\right)^2 = 1$$

[4] H. Minkowski, Phys. Zeits. 10, p. 104 (1909).

we may put $\cos\gamma \equiv 1/\sqrt{1-\beta^2}$, $\sin\gamma \equiv -i\beta/\sqrt{1-\beta^2}$. Then these equations become

$$x' = x\cos\gamma - l\sin\gamma,$$

$$l' = l\cos\gamma + x\sin\gamma,$$

exhibiting the transformation in the form of a pure rotation without change of scale in which the axes are turned through the imaginary angle $\tan^{-1}(-i\beta)$. The interval ds between two nearby events is given by the invariant

$$ds^2 \equiv dx'^2 + dl'^2 = dx^2 + dl^2, \qquad (37\text{–}13)$$

which is equivalent to (37–9). The quantity ds is called the *metric* of the xl representative space.

Since the Lorentz transformation consists merely of a rotation of the axes without distortion in the xl space-time, we can employ the methods and theorems of Euclidean vector analysis. This we shall have occasion to do in considerable detail later on.

Problem 37a. Show that the Lorentz transformation can be expressed in the form

$$x' = x\cosh\theta - ct\sinh\theta,$$

$$ct' = ct\cosh\theta - x\sinh\theta,$$

where $\tanh\theta = \beta$. Also show that the radius vector from the origin to any point on the equilaterial hyperbola $\xi^2 - \eta^2 = 1$ gives the scale factor for the system S' (Fig. 33) whose velocity relative to S is given by the tangent of the angle which the radius vector makes with the ξ axis.

38. Transformations for Velocity and Acceleration between Linear Reference Systems with Constant Relative Velocity. — In order to complete our discussion of the motion of an arbitrary moving-element M relative to two equivalent one-dimensional reference systems S and S', the latter of which has a constant velocity v relative to the former, we need to find the relations between the velocities and accelerations of M relative to S and S'. Denote by V and f the velocity and acceleration respectively of M relative to S, and by V' and f' its velocity and acceleration relative to S'. From the Lorentz transformation (37–2) and (37–3) we have

$$\left.\begin{aligned} dx' &= \frac{1}{\sqrt{1-\beta^2}}(dx - vdt), \\ dt' &= \frac{1}{\sqrt{1-\beta^2}}\left(dt - \frac{\beta}{c}dx\right). \end{aligned}\right\} \qquad (38\text{–}1)$$

Hence

$$V' = \frac{dx'}{dt'} = \frac{dx - vdt}{dt - \dfrac{\beta}{c}dx} = \frac{V - v}{1 - \beta\dfrac{V}{c}}, \quad V = \frac{V' + v}{1 + \beta\dfrac{V'}{c}}, \qquad (38\text{–}2)$$

where the second relation can be obtained independently from (37–4) and (37–5) or by solving the first for V. As would be expected, these relations are identical with (36–11). From (38–1) and (38–2)

$$\frac{dt'}{dt} = \frac{1 - \beta\dfrac{V}{c}}{\sqrt{1-\beta^2}} = \frac{\sqrt{1-\beta^2}}{1 + \beta\dfrac{V'}{c}} = \sqrt{\frac{1 - \dfrac{V^2}{c^2}}{1 - \dfrac{V'^2}{c^2}}}, \qquad (38\text{–}3)$$

or

$$\frac{1}{\sqrt{1 - \dfrac{V'^2}{c^2}}}\frac{d}{dt'} = \frac{1}{\sqrt{1 - \dfrac{V^2}{c^2}}}\frac{d}{dt} \qquad (38\text{–}4)$$

is an invariant differential operator. For our purposes, however, the alternative relation

$$\frac{d}{dt'} = \frac{\sqrt{1-\beta^2}}{1 - \beta\dfrac{V}{c}}\frac{d}{dt} \qquad (38\text{–}5)$$

is more convenient. Applying it to (38–2) we find for the acceleration

$$f' = \frac{dV'}{dt'} = \frac{(1-\beta^2)^{3/2}}{\left(1 - \beta\dfrac{V}{c}\right)^3}f, \quad f = \frac{(1-\beta^2)^{3/2}}{\left(1 + \beta\dfrac{V'}{c}\right)^3}f'. \qquad (38\text{–}6)$$

Just as (38–2) can be written in the more symmetrical form

$$\frac{1 - \dfrac{V}{c}}{1 + \dfrac{V}{c}} = \frac{1-\beta}{1+\beta}\,\frac{1 - \dfrac{V'}{c}}{1 + \dfrac{V'}{c}}, \qquad (38\text{–}7)$$

so (38–6) may be written

$$\frac{f}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} = \frac{f'}{\left(1 - \frac{V'^2}{c^2}\right)^{3/2}}. \tag{38–8}$$

The invariant

$$\phi \equiv \frac{f}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} = \frac{d}{dt}\left\{\frac{V}{\sqrt{1 - \frac{V^2}{c^2}}}\right\}, \tag{38–9}$$

which has the same value relative to all equivalent reference systems having constant relative velocities, we shall call the *relativity acceleration*. It is identical with the Galilean acceleration in the reference system in which the moving-element is momentarily at rest.

Constant Acceleration. If the relativity acceleration of a moving-element is constant,

$$V = c\frac{\frac{\phi}{c}(t - t_0)}{\sqrt{1 + \frac{\phi^2}{c^2}(t - t_0)^2}}, \tag{38–10}$$

provided $V = 0$ when $t = t_0$. This can be put in the simpler form

$$\frac{1}{1 - \frac{V^2}{c^2}} = 1 + \frac{\phi^2}{c^2}(t - t_0)^2. \tag{38–11}$$

Integrating (38–10) a second time,

$$x - x_0 = \frac{c^2}{\phi}\left\{\sqrt{1 + \frac{\phi^2}{c^2}(t - t_0)^2} - 1\right\}, \tag{38–12}$$

where x_0 is the smallest value of x, which is attained when $t = t_0$. For small values of $t - t_0$ this reduces to the Galilean expression

$$x - x_0 = \tfrac{1}{2}\phi(t - t_0)^2.$$

It should be noted that, while $x - x_0$ increases without limit with $t - t_0$, V never becomes greater than c.

Energy Equation. If the relativity acceleration ϕ of a moving-element is any function of x, (38–9) gives

$$\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{1}{c^2}\int \phi\, dx = \text{Constant.} \tag{38–13}$$

This is the *energy equation.* The expression $c^2/\sqrt{1 - V^2/c^2}$ represents the *kinetic energy* per unit mass on the relativity theory, and $-\int \phi\, dx$ the *potential energy* per unit mass. The relativity kinetic energy for $V \ll c$ differs from the Newtonian expression by the additive constant c^2, since

$$\frac{c^2}{\sqrt{1-\frac{V^2}{c^2}}} = c^2 + \tfrac{1}{2}V^2 + \cdots.$$

So far as concerns *change* of kinetic energy, however, which is all that is physically measurable, the two are identical for small V. Indeed our selection of $c^2/\sqrt{1 - V^2/c^2}$ to represent the relativity kinetic energy is purely a matter of convenience: we might have appended the constant $-c^2$ so as to make the relativity expression reduce to the Newtonian expression for small V if we had so desired.

For the case of constant ϕ treated above (38–13) gives

$$\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = 1 + \frac{\phi}{c^2}(x - x_0), \tag{38–14}$$

which could have been obtained equally well by eliminating $t - t_0$ between (38–11) and (38–12).

Inverse Square Acceleration. If the relativity acceleration ϕ varies inversely with the square of the distance x from the origin of the observer's reference system, so that $\phi = a/x^2$, (38–13) gives

$$\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = 1 + \frac{a}{c^2}\left(\frac{1}{x_0} - \frac{1}{x}\right), \tag{38–15}$$

provided $V = 0$ when $x = x_0$.

39. Particle-Observers Moving with Constant Relative Acceleration in a Space of One Dimension. — We have in (38–12) the inte-

grated equation of motion of a particle-observer P' moving relative to a second particle-observer P with constant relativity acceleration ϕ. Let us now investigate the possibility of the equivalence of P' and P. In doing so we shall limit ourselves to the case where $x_0 = t_0 = 0$ in (38–12), that is, to the case where P' meets P at time $t = 0$ and then recedes from P in the direction from which he approached, without passing.

In terms of the notation of article 33, (38–12) becomes

$$1 + \frac{\phi}{c^2} r_2 = \sqrt{1 + \frac{\phi^2}{c^2} t_2{}^2}, \tag{39–1}$$

and if we put $\frac{1}{2}c(\tau_3 - \tau_1)$ for r_2 and $\frac{1}{2}(\tau_3 + \tau_1)$ for t_2 as usual we get

$$\frac{1}{\tau_1} - \frac{1}{\tau_3} = \frac{\phi}{c}, \tag{39–2}$$

which may be split into the two equations

$$\frac{1}{\tau_2} - \frac{1}{\tau_3} = \frac{\phi}{2c}, \quad \frac{1}{\tau_1} - \frac{1}{\tau_2} = \frac{\phi}{2c}, \tag{39–3}$$

thus establishing the equivalence of the two particle-observers.

Now consider a third particle-observer P'' who has a constant acceleration $\phi_{P''}$ relative to P, and who meets P at the same time as P' does without passing. Then P'' as well as P' is equivalent to P. We wish to prove that the acceleration $\phi'_{P''}$ of P'' relative to P' is constant and therefore that P'' is equivalent to P' as well as to P. We shall make both P' and P'' synchronous with P by taking $\tau = \tau' = \tau'' = 0$ at the instant of meeting. For uniformity of notation we shall denote the acceleration of P' relative to P by $\phi_{P'}$.

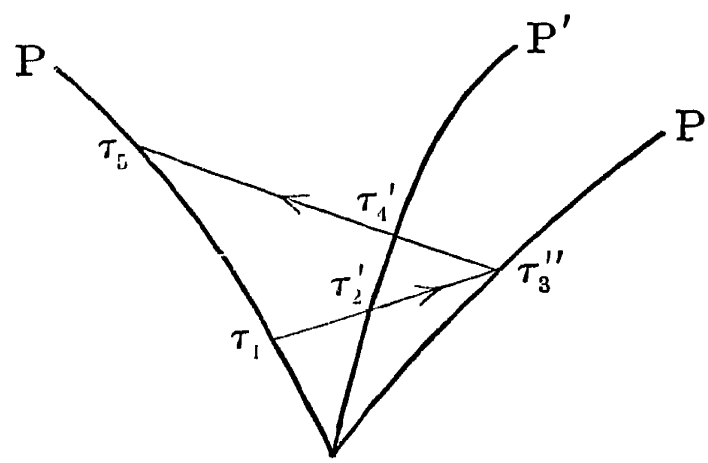

FIG. 35.

Consider the interlocking signals $\tau_1 \rightarrow \tau_3''$ and $\tau_3'' \rightarrow \tau_5$ shown in Fig. 35. From (39–2)

$$\frac{1}{\tau_1} - \frac{1}{\tau_5} = \frac{\phi_{P''}}{c} \tag{39–4}$$

and, from (39–3),

$$\frac{1}{\tau_1} - \frac{1}{\tau_2'} = \frac{\phi_{P'}}{2c}, \quad \frac{1}{\tau_4'} - \frac{1}{\tau_5} = \frac{\phi_{P'}}{2c}, \tag{39–5}$$

since P and P' are synchronous. Combining, we get

$$\frac{1}{\tau_2'} - \frac{1}{\tau_4'} = \frac{\phi_{P''} - \phi_{P'}}{c},$$

which shows that P'' has the constant acceleration $\phi_{P''} - \phi_{P'}$ relative to P'. The *addition law of acceleration* is, then,

$$\phi_{P''} = \phi_{P'} + \phi'_{P''}. \tag{39–6}$$

Since P'' has a constant relativity acceleration relative to P' he is equivalent to P'. All that remains is to show that the clock which makes P'' synchronous with P also makes him synchronous with P', that is, to show that the local time τ_3' at P' corresponding to τ_3'' at P'' is identical with the local time τ_3 at P corresponding to τ_3'' at P''. For then $\tau_3'' = \tau_3'$, since $\tau_3' = \tau_3$ and $\tau_3'' = \tau_3$ on account of the synchronism of both P' and P'' with P.

From (39–3) we have

$$\frac{1}{\tau_1} - \frac{1}{\tau_3} = \frac{\phi_{P''}}{2c},$$

and

$$\frac{1}{\tau_2'} - \frac{1}{\tau_3'} = \frac{\phi'_{P''}}{2c}.$$

Using these two relations to eliminate τ_1 and τ_2' from the first equation of (39–5), we find, with the aid of (39–6), that $\tau_3' = \tau_3$, as required.

Returning to the two equivalent particle-observers P and P' described by equations (39–3), we can adjoin to each a dense linear assembly of synchronous particle-observers constituting a reference system, as proved in article 34. We wish, now, to find the space-time transformation which enables us to pass from the space and time specifications of an event Q (Fig. 36) in the reference system S associated with P to those in the reference system S' associated with P'. We shall take P and P' as origins of distance scales measured positive in the sense of the acceleration ϕ of P' relative to P, and shall take the time at P and P' to be zero when they coincide. This makes

the two particle-observers synchronous. For the signals indicated in the figure, we have, by (39–3),

$$\tau_4' = \tau_4 = \frac{\tau_5}{1 + \dfrac{\phi}{2c}\tau_5}, \quad \tau_2' = \tau_2 = \frac{\tau_1}{1 - \dfrac{\phi}{2c}\tau_1}.$$

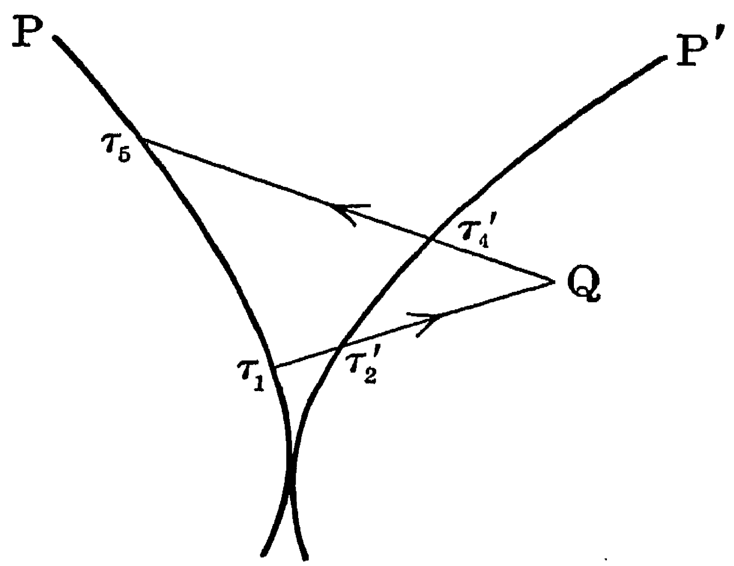

FIG. 36.

So P''s estimation of the distance of the event Q is

$$x' = \tfrac{1}{2}c(\tau_4' - \tau_2') = \frac{\tfrac{1}{2}c(\tau_5 - \tau_1) + \dfrac{\phi}{2}\left(\dfrac{t_5 - \tau_1}{2}\right)^2 - \dfrac{\phi}{2}\left(\dfrac{\tau_5 + \tau_1}{2}\right)^2}{\left\{1 + \dfrac{\phi}{2c}\left(\dfrac{\tau_5 - \tau_1}{2}\right)\right\}^2 - \dfrac{\phi^2}{4c^2}\left(\dfrac{\tau_5 + \tau_1}{2}\right)^2}$$

$$= \frac{x\left(1 + \dfrac{\phi x}{2c^2}\right) - \dfrac{\phi t^2}{2}}{\left(1 + \dfrac{\phi x}{2c^2}\right)^2 - \dfrac{\phi^2 t^2}{4c^2}},$$

and P''s extended time of Q is

$$t' = \tfrac{1}{2}(\tau_4' + \tau_2') = \frac{\tfrac{1}{2}(\tau_5 + \tau_1)}{\left\{1 + \dfrac{\phi}{2c}\left(\dfrac{\tau_5 - \tau_1}{2}\right)\right\}^2 - \dfrac{\phi^2}{4c^2}\left(\dfrac{\tau_5 + \tau_1}{2}\right)^2}$$

$$= \frac{t}{\left(1 + \dfrac{\phi x}{2c^2}\right)^2 - \dfrac{\phi^2 t^2}{4c^2}}.$$

If we put

$$\xi \equiv 1 + \frac{\phi x}{2c^2}, \quad T \equiv \frac{\phi t}{2c},$$

$$\xi' \equiv 1 - \frac{\phi x'}{2c^2}, \quad T' \equiv \frac{\phi t'}{2c},$$

which amounts to taking a new origin in S at $-2c^2/\phi$, and a new origin in S' at $2c^2/\phi$ combined with a change in sense of the axis, these relations take the simpler form

$$\left.\begin{aligned} \xi' &= \frac{\xi}{\xi^2 - T^2}, & \xi &= \frac{\xi'}{\xi'^2 - T'^2}, \\ T' &= \frac{T}{\xi^2 - T^2}, & T &= \frac{T'}{\xi'^2 - T'^2}, \end{aligned}\right\} \tag{39–7}$$

where the equations in the second column are obtained from those in the first by solving for ξ and T.

This transformation gives

$$(\xi^2 - T^2)(\xi'^2 - T'^2) = 1, \tag{39–8}$$

and yields the invariants

$$\frac{T'}{\xi'} = \frac{T}{\xi}, \tag{39–9}$$

and

$$\frac{dx'^2 - c^2dt'^2}{\xi'^2 - T'^2} = \frac{dx^2 - c^2dt^2}{\xi^2 - T^2}. \tag{39–10}$$

Unlike the Lorentz transformation discussed in article 37, the space-time interval $dx^2 - c^2dt^2$ is *not* an invariant for this transformation.

By differentiating (39–7) with ξ' held constant, it is found that the velocity v relative to S of a point fixed in S' is given by

$$\beta \equiv \frac{v}{c} = \frac{2\xi T}{\xi^2 + T^2}. \tag{39–11}$$

When $t = 0$, then, all points in S' are simultaneously at rest in S.

Differentiating (39–11) to find the acceleration of a point in S',

$$f \equiv \frac{dv}{dt} = (1 - \beta^2)^{3/2}\, \xi'\phi. \tag{39–12}$$

A particle-observer in S', therefore, has a constant relativity acceleration $\xi'\phi$ relative to S. In terms of the coordinate measure of S, this is an acceleration

$$\phi_\xi = \frac{\xi\phi}{\xi^2 - T^2} \tag{39-13}$$

for a particle-observer in S' at ξ at time t. At the instant $t = 0$ when all particle-observers in S' are at rest relative to S, this reduces to ϕ/ξ.

It is of interest to note that even when the two reference systems S and S' are relatively at rest space and time measurements do not agree except at the common point occupied by P and P'. For

$$\frac{dx'}{dx} = \frac{dt'}{dt} = \frac{1}{\left(1 + \frac{\phi x}{2c^2}\right)^2} = \left(1 - \frac{\phi x'}{2c^2}\right)^2 \tag{39-14}$$

from (38–7) when $t = t' = 0$. Hence S and S' are not homogeneously equivalent.

Evidently we may conceive of an infinity of reference systems $S, S', S'', \cdots$ formed by adjoining assemblages of particle-observers to the equivalent particle-observers P, P', P'', $\cdots$ which have constant relative accelerations of arbitrary magnitudes and which meet simultaneously without passing.

40. Three-Dimensional Reference System. — In three-dimensional space light-signals dispatched from a particle-observer consist of diverging wave-fronts. When we say that two particle-observers are relatively at rest we mean merely that the distance r_2 between them does not change. Consequently, if P' is a particle-observer at rest relative to another particle-observer P, we prove that the two are equivalent just as in article 35, and, if we suppose them to be provided with synchronous clocks, we show that the extended time of the one at the other coincides with the local time of the other.

We wish now to investigate the possibility of adjoining a reference system to an arbitrary particle-observer P. Let P be located at O (Fig. 37) and let a synchronous particle-observer P_A at rest relative to P be located at A. As corresponding times at the two particle-observers are identical, we need not differentiate between them. Let signals dispatched from P and P_A at time τ_1 be received by the other particle-observer at time τ_2 and immediately returned, each signal reaching the particle-observer with whom it originated at time τ_3. Let

P_B be a third particle-observer at rest relative to P at a point B at the same distance r from P as A. Then the signal dispatched from P at time τ_1 reaches P_B as well as P_A at P's extended time $t_2 = \tau_2$, and a signal immediately returned from P_B will be received by P at the same time τ_3 as the one immediately returned from P_A at A. Now consider a particle-observer P' who is at A when P_A dispatches his first signal (time τ_1), at B when P's first signal reaches him (time $t_2 = \tau_2$) and at some other point C equi-distant from P when P's second signal arrives (time $t_3 = \tau_3$). Provided P' always remains at the same distance r from P, we can furnish him with a clock which is equivalent to and synchronous with that of P, no matter what his motion of rotation about P may be. All that is necessary is that P''s clock shall assign

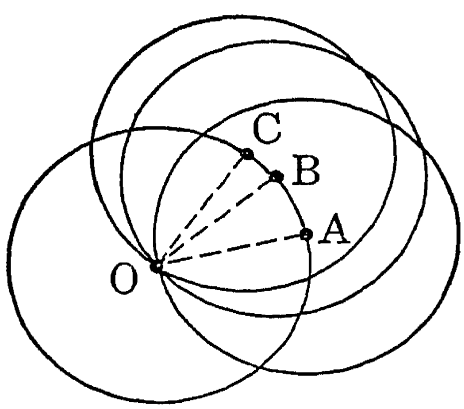

FIG. 37.

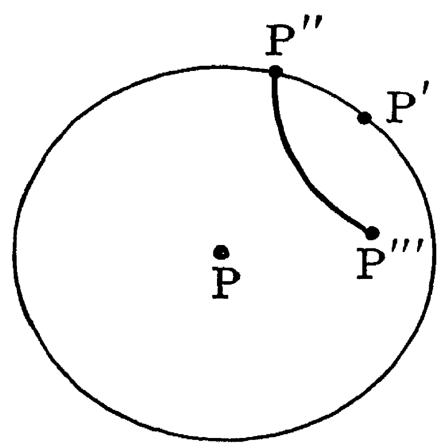

FIG. 38.

the values τ_1, τ_2 and τ_3 respectively to the three events enumerated, and to all such series of events.

Now let P' and P'' (Fig. 38) be two particle-observers at rest at equal distances r from P and synchronous with P. Let P'' also be at rest relative to P' at a distance r' from him. The characteristics of the clocks of P' and P'' are fixed by the condition that they are synchronous with the clock of P. They may not, however, be synchronous with each other. To test this matter, we dispatch signals from each toward the other at the identical times $\tau_1' = \tau_1''$. Suppose the reception of the signal at P'' occurs earlier than that at P'. Then by giving P' and P'' together a rotation in the sense $P'P''$ about P we can delay the reception of the signal at P'' and hasten that at P' until the two times become the same, without disturbing the synchronism of P' and of P'' with P. In this way we make P' and P'' synchronous with each other as well as with P. Then the distance of P' from P'' as measured by P'' is equal to that of P'' from P' as measured by P'.

Next let P''' be a third particle-observer at rest at a fixed distance r from P, a fixed distance r' from P', and a fixed distance r'' from P''. If P''' is synchronous with P he is also synchronous with P' since all particle-observers lying on the intersection of the surface of radius r about P and the surface of radius r' about P' which are synchronous with P are also synchronous with P'. However it may happen that P''' is not synchronous with P''. In this event we can give P'' and P''' together a rotation about P' along the curve $P''P'''$ sufficient to make them synchronous with each other without destroying their synchronism with P and P'.

This is as far as the procedure adopted will take us. As a *reference system* is a dense assemblage of equivalent particle-observers filling all space such that each is at rest relative to and synchronous with every other, it is clear that we cannot associate a reference system with any arbitrary particle-observer. In this respect the three-dimensional case differs from the one-dimensional case. However, with certain particle-observers we may be able to associate reference systems. In such cases it becomes important to investigate the geometry of the reference systems discovered. As we have established a convention for measuring distance, the geometry of a reference system is something to be investigated empirically in any given physical case, by comparing the measured length of the circumference of a circle with the measured length of its diameter, and so forth. The geometry of a reference system is a matter of measurement, not a matter of convention.

In any small region, such as that occupied by the laboratory, we have no difficulty in establishing the existence of one reference system whose geometry is Euclidean within the limits of the most refined measurements which are available to us. We shall be particularly interested in searching for other reference systems which have Euclidean geometry. Such reference systems, as we shall see, fall into two classes: those associated with particle-observers moving with constant relative velocity, and those associated with particle-observers moving with constant relative acceleration. Except for combinations of these two, no other classes exist. A reference system, for instance, cannot be associated with a particle-observer moving with constant speed in a circle relative to an established Euclidean reference system.

41. Euclidean Reference Systems Moving with Constant Relative Velocity. — We shall consider a universe in which there exists a

Euclidean reference system S, that is, a dense assemblage of synchronous particle-observers filling all space which are relatively at rest and have a Euclidean geometry. First we shall show that any particle-observer P' moving with a velocity $\mathbf{v}$ constant in magnitude and direction relative to S is equivalent to every particle-observer in S. Next we shall prove that any two particle-observers P' and P'' having the same constant velocity $\mathbf{v}$ relative to S are relatively at rest and equivalent. Then we shall show that if two particle-observers P'' and P''', having the same velocity $\mathbf{v}$ relative to S as P', are synchronous with P', they are synchronous with each other. These theorems suffice to show that a dense assemblage of particle-observers all moving with the same constant velocity $\mathbf{v}$ relative to S constitute a reference system, which we shall designate by S'. Finally we shall prove that the geometry of S' is Euclidean. Hence, as each particle-observer of S' is equivalent to every particle-observer of S, S and S' are homogeneously equivalent.

Let P be any particle-observer of S and P' a particle-observer who has a constant velocity $\mathbf{v}$ relative to S. If h is the length of the perpendicular in S from P to the path of P', the distance r_2 of P' from P at P's extended time t_2 is given by

$$r_2{}^2 = h^2 + v^2(t_2 - t_0)^2. \tag{41-1}$$

Putting $r_2 \equiv \frac{1}{2}c(\tau_3 - \tau_1)$ and $t_2 \equiv \frac{1}{2}(\tau_3 + \tau_1)$ as usual, this may be written

$$(1 - \beta^2)(\tau_3 - t_0)^2 - 2(1 + \beta^2)(\tau_3 - t_0)(\tau_1 - t_0) + (1 - \beta^2)(\tau_1 - t_0)^2 = \frac{4h^2}{c^2},$$

where $\beta \equiv v/c$. If we put $a^2 \equiv h^2(1 - \beta^2)/v^2$ this becomes

$$\frac{\sqrt{a^2 + (\tau_3 - t_0)^2} + \tau_3 - t_0}{\sqrt{a^2 + (\tau_1 - t_0)^2} + \tau_1 - t_0} = \frac{1 + \beta}{1 - \beta},$$

from which it follows that

$$\left.\begin{aligned} \frac{\sqrt{a^2 + (\tau_3 - t_0)^2} + \tau_3 - t_0}{\sqrt{a^2 + (\tau_2 - t_0)^2} + \tau_2 - t_0} &= \sqrt{\frac{1 + \beta}{1 - \beta}}, \\ \frac{\sqrt{a^2 + (\tau_2 - t_0)^2} + \tau_2 - t_0}{\sqrt{a^2 + (\tau_1 - t_0)^2} + \tau_1 - t_0} &= \sqrt{\frac{1 + \beta}{1 - \beta}}. \end{aligned}\right\} \tag{41-2}$$

These relations can be written more conveniently in the forms

$$\left.\begin{aligned}\tau_3 - t_0 &= \frac{1}{\sqrt{1-\beta^2}}\left\{\tau_2 - t_0 + \beta\sqrt{a^2 + (\tau_2 - t_0)^2}\right\},\\ \tau_2 - t_0 &= \frac{1}{\sqrt{1-\beta^2}}\left\{\tau_1 - t_0 + \beta\sqrt{a^2 + (\tau_1 - t_0)^2}\right\},\end{aligned}\right\} \quad (41\text{–}3)$$

or

$$\left.\begin{aligned}\tau_2 - t_0 &= \frac{1}{\sqrt{1-\beta^2}}\left\{\tau_3 - t_0 - \beta\sqrt{a^2 + (\tau_3 - t_0)^2}\right\},\\ \tau_1 - t_0 &= \frac{1}{\sqrt{1-\beta^2}}\left\{\tau_2 - t_0 - \beta\sqrt{a^2 + (\tau_2 - t_0)^2}\right\}.\end{aligned}\right\} \quad (41\text{–}4)$$

Eliminating a by means of the relation

$$\frac{r_2}{c} \equiv \tfrac{1}{2}(\tau_3 - \tau_1) = \frac{\beta}{\sqrt{1-\beta^2}}\sqrt{a^2 + (\tau_2 - t_0)^2}$$

we get

$$\tau_3 - t_0 = \frac{\tau_2 - t_0}{\sqrt{1-\beta^2}} + \frac{r_2}{c}, \quad \tau_1 - t_0 = \frac{\tau_2 - t_0}{\sqrt{1-\beta^2}} - \frac{r_2}{c}, \qquad (41\text{–}5)$$

which are identical with (36–4) for the one-dimensional case. Consequently both (36–5) and (36–6) are valid for the three-dimensional case.

As P may be *any* particle-observer in S, we have shown that a particle-observer P' moving with constant velocity relative to S is equivalent to every particle-observer in S.

Suppose now that P' is moving along the X axis of S. We shall associate with P' a linear assemblage of particle-observers distributed along this axis, all of which are at rest relative to P' and synchronous with him. As shown in article 36 each of these particle-observers has the same constant velocity relative to S and is equivalent to every particle-observer of S lying on the X axis, and, in view of what we have just proved, equivalent to every particle-observer of S whether lying on the X axis or not. The transformation (37–2) to (37–5) applies, then, to the linear assemblages of particle-observers constituting the coincident X and X' axes of S and S' respectively.

Let Q_t (Fig. 39) be an event occurring at the point x, y, z at the time t and let P_t' be the position of P' in S at this time. If the coordinates of P' relative to S at the time 0 are x_1, 0, 0, then at time t they are $x_1 + vt$, 0, 0. We want to determine the distance and time

of occurrence of the event Q_t relative to P'. To do this we must send a light-signal from P' so as to arrive at Q_t at the time t, and then return it immediately to P'. The signal, then, must leave P' at the

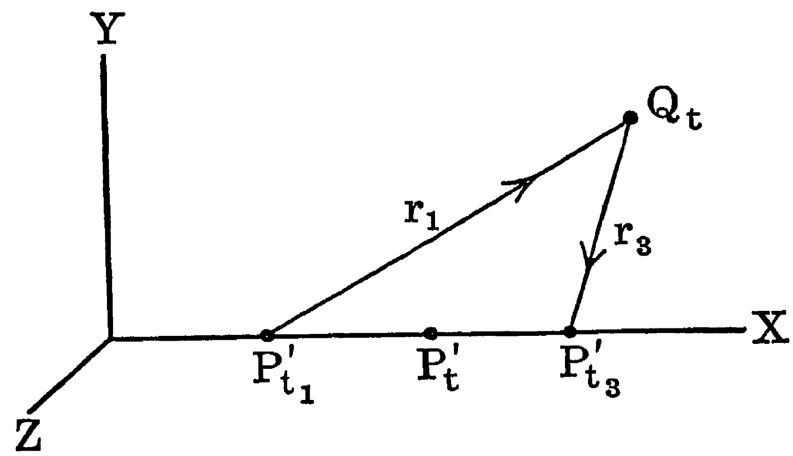

FIG. 39.

time t_1 when P' was at P'_{t_1}, where $t_1 = t - r_1/c$, and must return to P' at the time t_3 when P' will be at P'_{t_3}, where $t_3 = t + r_3/c$. Hence

$$r_1^2 = \left\{x - x_1 - v\left(t - \frac{r_1}{c}\right)\right\}^2 + y^2 + z^2,$$

$$r_3^2 = \left\{x - x_1 - v\left(t + \frac{r_3}{c}\right)\right\}^2 + y^2 + z^2.$$

Solving for r_1 and r_3,

$$r_1 = \frac{\beta(x - x_1 - vt) + \sqrt{(x - x_1 - vt)^2 + (1 - \beta^2)(y^2 + z^2)}}{1 - \beta^2},$$

$$r_3 = \frac{-\beta(x - x_1 - vt) + \sqrt{(x - x_1 - vt)^2 + (1 - \beta^2)(y^2 + z^2)}}{1 - \beta^2},$$

and hence

$$t_1 = t - \frac{r_1}{c}$$

$$= \frac{t - \frac{\beta}{c}(x - x_1) - \frac{1}{c}\sqrt{(x - x_1 - vt)^2 + (1 - \beta^2)(y^2 + z^2)}}{1 - \beta^2},$$

$$t_3 = t + \frac{r_3}{c}$$

$$= \frac{t - \frac{\beta}{c}(x - x_1) + \frac{1}{c}\sqrt{(x - x_1 - vt)^2 + (1 - \beta^2)(y^2 + z^2)}}{1 - \beta^2}.$$

Consequently the local times of P' at P'_{t_1} and P'_{t_3} are respectively

$$\tau_1' = \frac{1}{\sqrt{1-\beta^2}}\left\{t_1 - \frac{\beta}{c}(x_1 + vt_1)\right\}$$

$$= \frac{t - \frac{\beta}{c}x - \frac{1}{c}\sqrt{(x - x_1 - vt)^2 + (1-\beta^2)(y^2+z^2)}}{\sqrt{1-\beta^2}},$$

$$\tau_3' = \frac{1}{\sqrt{1-\beta^2}}\left\{t_3 - \frac{\beta}{c}(x_1 + vt_3)\right\}$$

$$= \frac{t - \frac{\beta}{c}x + \frac{1}{c}\sqrt{(x - x_1 - vt)^2 + (1-\beta^2)(y^2+z^2)}}{\sqrt{1-\beta^2}},$$

and P''s measures of the distance and time of the event Q_t are

$$r' \equiv \tfrac{1}{2}c(\tau_3' - \tau_1') = \frac{\sqrt{(x - x_1 - vt)^2 + (1-\beta^2)(y^2+z^2)}}{\sqrt{1-\beta^2}}, \quad (41\text{–}6)$$

$$t' \equiv \tfrac{1}{2}(\tau_3' + \tau_1') = \frac{t - \frac{\beta}{c}x}{\sqrt{1-\beta^2}}. \quad (41\text{–}7)$$

Now suppose that the event Q_t represents the passage through this point of a particle-observer P'' moving in the X direction relative to S with the same constant velocity $\mathbf{v}$ as P'. If the coordinates of P'' at time 0 are x_2, y_2, z_2, then, at time t, $x = x_2 + vt$, $y = y_2$, $z = z_2$. Hence the distance r' of P'' from P' is

$$r' = \sqrt{\frac{(x_2 - x_1)^2}{1-\beta^2} + y_2^2 + z_2^2}, \quad (41\text{–}8)$$

which is *independent of t*. Consequently P'' is at rest relative to P', and to make him synchronous with P' we must make his local time identical with the extended time of P', in accord with (35–5). Then the local time at P'' is expressed in terms of t and x in S by (41–7).

Next consider a third particle-observer P''' moving in the X direction relative to S with the same constant velocity $\mathbf{v}$ as P' and P''. Evidently P''' is at rest relative to both P' and P''. Designating the

coordinates of P', P'' and P''' in S at time t by the subscripts 1, 2 and 3 respectively, the local times at these three observers are

$$\tau_1' = \frac{t - \frac{\beta}{c} x_1}{\sqrt{1 - \beta^2}}$$

from (37–3), and, if P'' and P''' are synchronized with P',

$$\tau_2' = \frac{t - \frac{\beta}{c} x_2}{\sqrt{1 - \beta^2}}$$

and

$$\tau_3' = \frac{t - \frac{\beta}{c} x_3}{\sqrt{1 - \beta^2}}$$

from (41–7). But the last is just the condition that P''' be synchronous with P''. Hence the three particle-observers are synchronous each with each, and we can adjoin to P' all particle-observers moving with constant velocity $\mathbf{v}$ relative to S to form a reference system of synchronous particle-observers relatively at rest. This reference system will be designated by S' and times and distances measured in it will be indicated by primes. Each particle-observer in S' is equivalent to every particle-observer in S.

That the geometry of S' is Euclidean is seen immediately from (41–8), which is the Pythagorean theorem for S'. For we can construct a Euclidean mesh in S' which projects on S at a given instant t as planes parallel to the YZ, XY and ZX coordinate planes. Distances measured in the Y and Z directions are the same whether determined in S or in S', while the distance in S between two points fixed in S' having the same Y and Z coordinates is less than that determined in S' in the ratio $\sqrt{1 - \beta^2} : 1$.

Since the two reference systems S and S' have the same geometry and since each particle-observer of the one is equivalent to every particle-observer of the other, S and S' are homogeneously equivalent. Every particle-observer in S' has the same constant velocity v relative to S, and conversely every particle-observer in S has the velocity v in the opposite sense relative to S'. We may speak of v, therefore, as the velocity of either reference system relative to the other.

42. Lorentz Space-Time Transformation in Three-Dimensional Space. — Consider two equivalent Euclidean reference systems S and S' with a constant relative velocity. In S we shall embed a set of rectangular axes with origin at P and X axis in the direction of the velocity $\mathbf{v}$ of S' relative to S. Next we shall take the locus of all points in S' which move along the X axis of S as the X' axis of a set of rectangular axes in S', orienting the X' axis of S' in the same sense as the X axis of S. The Y' axis and the Z' axis of S' we shall take as the loci of those particles in S' which lie respectively on the Y axis and on the Z axis of S at the time $t = 0$ in S. Then the origin P' of $X'Y'Z'$ coincides with the origin P of XYZ at time $t = 0$. We shall take the time t' of this event at P' to be zero, thus synchronizing the clocks of P and P'.

Since P' here is taken as the particle-observer who passed the origin of S at the time $t = 0$, the distance x_1 in (41–6) is zero. The space transformation between S' and S is obtained from this equation by making first $y = z = 0$, then $z = x - vt = 0$, etc. The extended time in P''s experience of an event occurring at x, y, z at time t is given immediately by (41–7). In all the space-time transformation is

$$\left.\begin{aligned} x' &= \frac{1}{\sqrt{1-\beta^2}}(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \frac{1}{\sqrt{1-\beta^2}}\left(t - \frac{\beta}{c}x\right). \end{aligned}\right\} \qquad (42\text{–}1)$$

Equations (42–1) specify the transformation from S to S'. Solving for x, y, z, t we get the transformation from S' to S. Altogether, if we put $k \equiv 1/\sqrt{1-\beta^2}$,

$$\left.\begin{aligned} x' &= k(x - vt), & x &= k(x' + vt'), \\ y' &= y, & y &= y', \\ z' &= z, & z &= z', \\ t' &= k\left(t - \frac{\beta}{c}x\right), & t &= k\left(t' + \frac{\beta}{c}x'\right), \end{aligned}\right\} \qquad (42\text{–}2)$$

where the transformation from S' to S differs from that from S to S'

only in the sign of terms containing the first power of the relative velocity. Comparing with the one-dimensional case treated in article 37, we see that the only new element is the addition of the Y and Z axes in S and of the Y' and Z' axes in S'. While a rigid rod moving in the direction of its length with velocity $\mathbf{v}$ experiences the shortening calculated in article 37, the relations between y' and y, and z' and z, show that no change takes place in the length of a rod which is given a transverse velocity.

From either column of (42–2) we find that

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2 \qquad (42\text{–}3)$$

and

$$dx'^2 + dy'^2 + dz'^2 - c^2dt'^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \qquad (42\text{–}4)$$

are invariants of the Lorentz transformation. The second is the three-dimensional space-time interval between two nearby events.

Minkowski's two graphical representations of the Lorentz transformation can be used equally well in the case of a three-dimensional space by adjoining to the x and ct or l coordinates the perpendicularly measured y and z coordinates. Thus the diagram showing the world-line of a particle moving in three-dimensional space requires four dimensions for its complete portrayal. In the case where $x, y, z, l \equiv \sqrt{-1}ct$ are chosen as coordinates, the Lorentz transformation is equivalent to a simple rotation of the axes in the xl plane, the coordinates transforming according to the equations

$$\left.\begin{aligned} x' &= kx + i\beta kl, & x &= kx' - i\beta kl', \\ y' &= y, & y &= y', \\ z' &= z, & z &= z', \\ l' &= kl - i\beta kx, & l &= kl' + i\beta kx', \end{aligned}\right\} \qquad (42\text{–}5)$$

in accord with (37–11) and (37–12), where $i \equiv \sqrt{-1}$.

Problem 42a. Let P' and Q' be two particle-observers in S', the coordinates of Q' relative to P' in S at any time t being dx, dy, dz. Light-signals dispatched simultaneously from P' and Q' at time t reach Q' and P' respectively at times $t + dt_{Q'}$ and $t + dt_{P'}$. Show that

$$dt_{Q'} - dt_{P'} = \frac{2v}{c^2 - v^2}\,dx.$$

Problem 42b. In the Michelson-Gale [5] experiment the two parts of a bifurcated beam of light are sent in opposite senses around the periphery

[5] Michelson and Gale, Astrophys. Jour. 61, p. 140 (1925).

of an area A on the surface of the earth. Show that, to a first approximation, the difference in phase on reunion is given by

$$\frac{\Delta\lambda}{\lambda} = 4\frac{\Omega}{\lambda c} A \sin\gamma,$$

where λ is the wave-length, Ω the angular velocity of the earth, and γ the latitude.

Hint. Use the result of 42*a* and Stokes' theorem.

Problem 42c. Write the Lorentz transformation in vector form.

Ans.
$$\mathbf{r}' = \mathbf{r} + (k-1)\frac{\mathbf{v}\cdot\mathbf{r}}{v^2}\mathbf{v} - kt\,\mathbf{v},$$
$$t' = k\left(t - \frac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right).$$

43. Transformations for Velocity and Acceleration between Euclidean Reference Systems with Constant Relative Velocity. — If we differentiate the equations in the first column of (42–2) we get, just as in article 38, for the relations between the components of the velocities $\mathbf{V}$ and $\mathbf{V}'$ of an arbitrary moving-element M relative to the inertial systems S and S' respectively:

$$\left.\begin{aligned}
V_x' &= \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \dfrac{\beta}{c}dx} = \frac{V_x - v}{1 - \beta\dfrac{V_x}{c}},\\
V_y' &= \frac{dy'}{dt'} = \frac{dy}{k\left(dt - \dfrac{\beta}{c}dx\right)} = \frac{V_y}{k\left(1 - \beta\dfrac{V_x}{c}\right)},\\
V_z' &= \frac{dz'}{dt'} = \frac{dz}{k\left(dt - \dfrac{\beta}{c}dx\right)} = \frac{V_z}{k\left(1 - \beta\dfrac{V_x}{c}\right)};\\
V_x &= \frac{V_x' + v}{1 + \beta\dfrac{V_x'}{c}},\\
V_y &= \frac{V_y'}{k\left(1 + \beta\dfrac{V_x'}{c}\right)},\\
V_z &= \frac{V_z'}{k\left(1 + \beta\dfrac{V_x'}{c}\right)}.
\end{aligned}\right\} \quad (43\text{–}1)$$

It is evident from these equations that a particle-observer P'' who has a constant velocity relative to S has a constant velocity relative to S' also. We can, therefore, adjoin to P'' a dense assemblage of synchronous particle-observers relatively at rest to form a Euclidean reference system S'' which is homogeneously equivalent to both S and S'. Altogether we can introduce a triple infinity of such homogeneously equivalent reference systems having velocities in all directions ranging by infinitesimal steps from $-c$ to c relative to any selected Euclidean system. Such a group of equivalent Euclidean reference systems with constant relative velocities less than c will be called a *Lorentz group*. Macroscopic material particles, when free from external influences, are known to lie permanently in the reference systems of a particular Lorentz group. The systems of this group are named *inertial systems*. The principle of relativity requires that the laws of physics shall have the same form, and the constants contained in them the same values, relative to all inertial systems. Henceforth we shall be concerned primarily with the Lorentz group constituting the inertial systems.

Squaring and adding the equations in the second column of (43–1) we obtain the transformation

$$1 - \frac{V^2}{c^2} = \frac{(1 - \beta^2)\left(1 - \dfrac{V'^2}{c^2}\right)}{\left(1 + \boldsymbol{\beta} \cdot \dfrac{\mathbf{V}'}{c}\right)^2} \tag{43–2}$$

for the square of the velocity, where $\boldsymbol{\beta} \equiv \mathbf{v}/c$. This relation shows us that the velocity of light is the *only* velocity which has the same magnitude, whatever its direction may be, relative to all inertial systems; a proposition of which we shall make important use later. Furthermore it shows us that if $v^2 < c^2$ and $V'^2 < c^2$, then $V^2 < c^2$ whatever the direction of $\mathbf{V}'$ may be. Hence a moving-element having a velocity less than c relative to one inertial system has a velocity less than c relative to every inertial system.

From (42–2) and (43–2) we find

$$\frac{dt'}{dt} = \frac{1 - \boldsymbol{\beta}\cdot\dfrac{\mathbf{V}}{c}}{\sqrt{1 - \beta^2}} = \frac{\sqrt{1 - \beta^2}}{1 + \boldsymbol{\beta}\cdot\dfrac{\mathbf{V}'}{c}} = \sqrt{\frac{1 - \dfrac{V^2}{c^2}}{1 - \dfrac{V'^2}{c^2}}}, \tag{43–3}$$

which shows that

$$\frac{1}{\sqrt{1-\dfrac{V'^2}{c^2}}}\frac{d}{dt'}=\frac{1}{\sqrt{1-\dfrac{V^2}{c^2}}}\frac{d}{dt} \tag{43-4}$$

is an invariant differential operator just as in the one-dimensional case treated in article 38.

Availing ourselves of (43–3) to calculate from (43–1) the relations between the components of the accelerations **f** and **f**′ of an arbitrary moving element M relative to S and S' respectively, we find

$$\left.\begin{aligned}
f_x' &= \frac{dV_x'}{dt'} = \frac{f_x}{k^3\left(1-\beta\dfrac{V_x}{c}\right)^3},\\
f_y' &= \frac{dV_y'}{dt'} = \frac{f_y-\dfrac{\beta}{c}(f_yV_x-f_xV_y)}{k^2\left(1-\beta\dfrac{V_x}{c}\right)^3},\\
f_z' &= \frac{dV_z'}{dt'} = \frac{f_z-\dfrac{\beta}{c}(f_zV_x-f_xV_z)}{k^2\left(1-\beta\dfrac{V_x}{c}\right)^3};\\
f_x &= \frac{f_x'}{k^3\left(1+\beta\dfrac{V_x'}{c}\right)^3},\\
f_y &= \frac{f_y'+\dfrac{\beta}{c}(f_y'V_x'-f_x'V_y')}{k^2\left(1+\beta\dfrac{V_x'}{c}\right)^3},\\
f_z &= \frac{f_z'+\dfrac{\beta}{c}(f_z'V_x'-f_x'V_z')}{k^2\left(1+\beta\dfrac{V_x'}{c}\right)^3}.
\end{aligned}\right\} \tag{43-5}$$

Squaring and combining these relations we obtain the invariant

$$\frac{f'^2}{\left(1 - \frac{V'^2}{c^2}\right)^2} + \frac{\left(\mathbf{f}' \cdot \frac{\mathbf{V}'}{c}\right)^2}{\left(1 - \frac{V'^2}{c^2}\right)^3} = \frac{f^2}{\left(1 - \frac{V^2}{c^2}\right)^2} + \frac{\left(\mathbf{f} \cdot \frac{\mathbf{V}}{c}\right)^2}{\left(1 - \frac{V^2}{c^2}\right)^3} \cdot \quad (43\text{–}6)$$

If the moving-element M is momentarily at rest in S', $\mathbf{V}' = 0$ and the relations between $\mathbf{f}$ and $\mathbf{f}'$ given in (43–5) simplify to

$$f_x' = k^3 f_x, \quad f_y' = k^2 f_y, \quad f_z' = k^2 f_z. \quad (43\text{–}7)$$

Under the same circumstances it is easily shown by differentiating once more with respect to the time that

$$\left.\begin{aligned} \dot{f}_x' &= k^4 \dot{f}_x + 3k^6 \beta \frac{f_x^2}{c}, \\ \dot{f}_y' &= k^3 \dot{f}_y + 3k^5 \beta \frac{f_x f_y}{c}, \\ \dot{f}_z' &= k^3 \dot{f}_z + 3k^5 \beta \frac{f_x f_z}{c}. \end{aligned}\right\} \quad (43\text{–}8)$$

We saw from (43–2) that the only velocity which has the same magnitude relative to all inertial systems is the velocity of light c. Consider a particle moving with this constant speed. For such a particle we shall show now that the only acceleration which has the same magnitude, irrespective of direction, relative to all inertial systems is the acceleration zero. It follows, then, that the only motion possible with a velocity of the same magnitude, and an acceleration of the same magnitude, relative to all inertial systems, is motion in a straight line with the velocity of light.

Designating the velocity of the particle under consideration by $\mathbf{c}$, we must have $\mathbf{f} \cdot \mathbf{c} = 0$ relative to S since the speed remains constant. Incidentally, as the speed relative to any other inertial system S' is equal to the constant c by (43–2), $\mathbf{f}' \cdot \mathbf{c}' = 0$ as well. Now if we square and add the three equations in the first column of (43–5) after putting $\mathbf{V} = \mathbf{c}$, we find, with the aid of the condition $\mathbf{f} \cdot \mathbf{c} = 0$,

$$f'^2 = \frac{(1 - \beta^2)^2}{\left(1 - \boldsymbol{\beta} \cdot \frac{\mathbf{c}}{c}\right)^4} f^2. \quad (43\text{–}9)$$

This relation shows that the only values of f'^2 and f^2 which are the same for all $\boldsymbol{\beta}$'s are the values zero.

Problem 43a. Find the transformation between S and S' of the components of the time rate of increase of acceleration and verify (43–8).

Problem 43b. Write the velocity and acceleration transformations in vector form.

Ans.

$$\mathbf{V}' = \frac{1}{k\left(1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}\right)}\left\{\mathbf{V} + (k-1)\frac{\mathbf{v}\cdot\mathbf{V}}{v^2}\mathbf{v} - k\mathbf{v}\right\},$$

$$\mathbf{f}' = \frac{1}{k^2\left(1 - \frac{\mathbf{v}\cdot\mathbf{V}}{c^2}\right)^3}\left\{\mathbf{f} + \left(\frac{1}{k} - 1\right)\frac{\mathbf{v}\cdot\mathbf{f}}{v^2}\mathbf{v} - \frac{1}{c^2}\mathbf{v}\times(\mathbf{f}\times\mathbf{V})\right\}.$$

44. Relativity Precession. — In the last article we considered a moving-element which has a varying velocity relative to an inertial system S. Such a moving-element can be imagined to pass by infinitesimal steps from a state of momentary rest in an inertial system S' having a velocity $\mathbf{v}$ relative to S equal to that of the moving-element at the time t, to an inertial system S'' having a velocity $\mathbf{v} + d\mathbf{v}$ relative to S equal to that of the moving-element at the time $t + dt$, and so forth. Let us apply this point of view to a vector associated with a moving particle in such a way that it partakes of the motion. As usual we can represent the moving vector geometrically by an arrow $\overline{OP}$ whose origin O is at all times coincident with the moving particle to which it is attached, the direction of the arrow being the same as that of the vector and the length of the arrow being proportional to the magnitude of the vector. Since the vector is associated with a particle, however, we must choose a scale of proportionality which makes the length of the representative arrow very small compared with the radius of curvature of the path of the particle to which it is attached. Then the distance of the terminus P of the arrow from its origin O can be considered as a differential and we can neglect squares and higher powers of the length of the arrow in our analysis.

We want to find the relation between the time rate of change of the vector relative to S and its time rate of change relative to the inertial system S' in which the moving particle to which it is attached is momentarily at rest. We shall find that if the vector has no rotation at any time relative to the inertial system in which its origin is momentarily at rest, it will not in general maintain its original direction rela-

tive to S. For instance, if the origin of the vector describes a closed curve, the orientation of the vector after a complete revolution will differ from that before, the vector having rotated relative to S in the opposite sense to the revolution of its origin. This rotation is known as *relativity precession.*

Let the coordinates of the origin O and of the terminus P of the representative arrow relative to S' at the instant t' at which O is at rest in this inertial system be x', y', z' and $x' + dx', y' + dy', z' + dz'$ respectively. If t is the time in S when O is at rest in S', the time in S of the event $P_{t'}$ is $t + k\dfrac{\beta}{c}\,dx'$ and the coordinates in S of $P_{t'}$ relative to O_t are $k\,dx', dy', dz'$. The components of the arrow in S, however, are the coordinates of P at time t relative to O at time t, that is, of P_t relative to O_t. So, if the velocity of P relative to S is designated by $\mathbf{V}$, the components of the arrow relative to S are

$$dx = k\,dx' - V_x k\frac{\beta}{c}\,dx',$$

$$dy = dy' - V_y k\frac{\beta}{c}\,dx',$$

$$dz = dz' - V_z k\frac{\beta}{c}\,dx'.$$

Denote the vector as viewed by observers in S and S' by $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ respectively. Then, as the components of $\boldsymbol{\rho}$ are to the components of $\boldsymbol{\rho}'$ in the ratios dx/dx', dy/dy', dz/dz', the equations above can be combined to give the vector relation

$$\boldsymbol{\rho} = \boldsymbol{\rho}' + (k - 1)\frac{\boldsymbol{\rho}'\cdot\mathbf{v}}{v^2}\mathbf{v} - k\frac{\boldsymbol{\rho}'\cdot\mathbf{v}}{c^2}\mathbf{V}, \qquad (44\text{–}1)$$

where $\mathbf{v}$ is the velocity of S' relative to S. As $\mathbf{V}$ differs from $\mathbf{v}$ only by a quantity of the order of dx, dy, dz, it follows that

$$\boldsymbol{\rho}\cdot\mathbf{v} = \frac{1}{k}\boldsymbol{\rho}'\cdot\mathbf{v}. \qquad (44\text{–}2)$$

For the case under consideration the invariant differential operator (43–4) becomes

$$k\frac{d}{dt} = \frac{d}{dt'}.$$

Operating with this on (44–1) we obtain, with the aid of (44–2), the relation

$$\frac{d\boldsymbol{\rho}}{dt} = \frac{1}{k}\left\{\frac{d\boldsymbol{\rho}'}{dt'} + \left(\frac{1}{k} - 1\right)\frac{d\boldsymbol{\rho}'}{dt'} \cdot \mathbf{v}\frac{\mathbf{v}}{v^2}\right\} - k^2\frac{\boldsymbol{\rho}\cdot\mathbf{v}}{c^2}\mathbf{f} \qquad (44\text{–}3)$$

between the time rate of change of the vector in S' and its time rate of change in S. While the first term on the right vanishes if the vector remains invariable in magnitude and direction relative to the inertial system in which it is momentarily at rest, the second term still persists unless the acceleration $\mathbf{f}$ of the vector relative to S vanishes or the vector is perpendicular to the direction of its velocity $\mathbf{v}$ relative to S.

If we limit our further discussion to the case where the vector $\boldsymbol{\rho}'$ remains unchanged in magnitude and direction relative to the inertial system in which it is momentarily at rest, the angular velocity $\boldsymbol{\omega}$ of $\boldsymbol{\rho}$ relative to S is tiven by

$$\rho^2\boldsymbol{\omega} = \boldsymbol{\rho} \times \frac{d\boldsymbol{\rho}}{dt}$$

or

$$\boldsymbol{\omega} = -k^2\frac{\boldsymbol{\rho}\cdot\mathbf{v}}{\rho^2c^2}\boldsymbol{\rho} \times \mathbf{f}. \qquad (44\text{–}4)$$

Suppose now that the vector $\boldsymbol{\rho}$ describes a plane orbit. As the component of $\boldsymbol{\rho}$ perpendicular to the plane of the orbit suffers no rotation due to the presence of the factor $\boldsymbol{\rho}\cdot\mathbf{v}$ in (44–3), we need consider only the component $\mathbf{r}$ (Fig. 40) of the vector in the plane of the orbit. Let ψ be the angle which the normal to the path makes with some fixed direction in S and α the angle which $\mathbf{r}$ makes with the tangent to the path. If, then, we resolve the acceleration into its tangential and normal components,

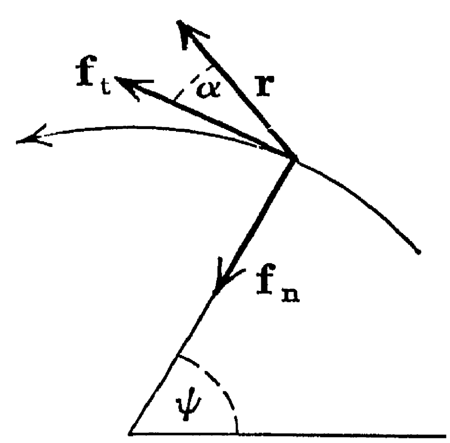

FIG. 40.

$$|\mathbf{r} \times \mathbf{f}| = |\mathbf{r} \times \mathbf{f}_t| + |\mathbf{r} \times \mathbf{f}_n|$$
$$= r\frac{dv}{dt}\sin\alpha + rv\frac{d\psi}{dt}\cos\alpha.$$

Hence

$$\omega = \frac{d\psi}{dt} - \frac{d\alpha}{dt} = -k^2\beta\sin\alpha\cos\alpha\frac{d\beta}{dt} - k^2\beta^2\cos^2\alpha\frac{d\psi}{dt}$$

or

$$\frac{d\psi}{dt} = \frac{1 - \beta^2}{1 - \beta^2 \sin^2 \alpha} \frac{d\alpha}{dt} - \frac{\beta \sin \alpha \cos \alpha}{1 - \beta^2 \sin^2 \alpha} \frac{d\beta}{dt},$$

which gives the integral

$$\tan^{-1} \left\{ \sqrt{1 - \beta^2} \tan \alpha \right\} = \int \frac{d\psi}{\sqrt{1 - \beta^2}}. \qquad (44\text{–}5)$$

From this equation it appears that α increases by 2π and the vector $\boldsymbol{\rho}$ returns to its original orientation relative to the velocity $\mathbf{v}$ when ψ increases by the amount specified by the equation

$$\int \frac{d\psi}{\sqrt{1 - \beta^2}} = 2\pi. \qquad (44\text{–}6)$$

If the orbit is periodic and β^2 constant, as would be the case for a circular orbit of a particle subject to a central force, the vector $\boldsymbol{\rho}$ will return to the same orientation relative to $\mathbf{v}$ when the particle to which this vector is attached has revolved through an angle $2\pi\sqrt{1 - \beta^2}$. If $\beta \ll 1$, this means that the vector $\boldsymbol{\rho}$ precesses about the normal to the plane of the orbit in the *opposite* sense to the revolution of the particle through an angle approximately equal to $\pi\beta^2$ per revolution. Thus the axis of rotation of the earth or of the spinning electron in an atom experiences this precession in addition to that caused by any impressed torque. In the case of the earth β is so small that the relativity precession is beyond the limits of observation, but in the case of the electron it is by no means negligible and plays an important part in modern theories of atomic structure.

Relativity precession was discovered by Thomas, and the reader who is interested in further details should consult the papers of Thomas [6] and Page.[7]

45. Euclidean Reference Systems Moving with Constant Relative Acceleration. — Starting with a single Euclidean reference system with constant light velocity we have found a triple infinity of other equivalent Euclidean reference systems with the same constant light velocity, such that each system has a constant velocity relative to the first or to any other of the group. The question then arises: does this Lorentz group exhaust all possibilities, or do other categories of equivalent Euclidean reference systems exist which have other types

[6] L. H. Thomas, Phil. Mag. 3, p. 1 (1927).
[7] L. Page, Phys. Rev. 33, p. 572 (1929).

of motion than constant velocity relative to the fundamental system? The answer to this question has been given by Page,[8] who has shown that a reference system may be adjoined to a particle-observer moving with a constant relativity acceleration ϕ relative to the fundamental system, and that the geometry of the system so constructed is Euclidean. With every such relatively accelerated reference system may be associated in turn a Lorentz group of homogeneously equivalent Euclidean reference systems, the light velocity in all the reference systems under consideration being, of course, the same constant c. Finally, Engstrom and Zorn [9] and Robertson [10] have shown that no further categories of Euclidean reference systems with constant light velocity exist, whether equivalent to the original reference system or not. Hence, with a given particle-observer there can be associated at most a single Euclidean reference system. In this respect the three-dimensional case differs from the one-dimensional case (art. 35) in which no geometry exists.

The principle of relativity requires that the laws of physics shall have the same form, and the constants contained in them the same values, relative to all equivalent Euclidean reference systems, including those moving with constant relative acceleration as well as those moving with constant relative velocity. While, however, we are able to transfer a macroscopic particle from one field-free inertial system to another by the application of a force for a suitable time, we know no means of transferring such a particle from an inertial system to an equivalent Euclidean system moving with a constant relative acceleration. It may be, however, that such transfers occur in the microscopic world, and that the imperfectly comprehended motions of electrons, neutrons and protons in the atom are to be explained in such a manner.

So far as the formulation of the laws of electrodynamics are concerned, we may limit our consideration to the single Lorentz group of equivalent Euclidean reference systems with constant light velocity comprising the inertial systems. The principle of relativity, applied to this Lorentz group alone, we shall denominate the *restricted principle of relativity*. It tells us that the laws of physics must be the same relative to all inertial systems.

While we are not warranted at the present time in devoting to

[8] L. Page, Phys. Rev. **49**, p. 254 (1936).
[9] Engstrom and Zorn, Phys. Rev. **49**, p. 701 (1936).
[10] H. P. Robertson, Phys. Rev. **49**, p. 755 (1936).

relatively accelerated equivalent Euclidean reference systems the space necessary to give a detailed description of their kinematical relationships, it may be of interest to write down without proof the space-time transformation. Adjoining Y and Z rectangular axes to the X axis of the one-dimensional case treated in article 39, and putting

$$\xi \equiv 1 + \frac{\phi x}{2c^2}, \quad \eta \equiv \frac{\phi y}{2c^2}, \quad \zeta \equiv \frac{\phi z}{2c^2}, \quad T \equiv \frac{\phi t}{2c},$$

$$\xi' \equiv 1 - \frac{\phi x'}{2c^2}, \quad \eta' \equiv \frac{\phi y'}{2c^2}, \quad \zeta' \equiv \frac{\phi z'}{2c^2}, \quad T' \equiv \frac{\phi t'}{2c},$$

and $\rho^2 \equiv \xi^2 + \eta^2 + \zeta^2$, the desired transformation is

$$\left.\begin{aligned}
\xi' &= \frac{\xi}{\rho^2 - T^2}, & \xi &= \frac{\xi'}{\rho'^2 - T'^2}, \\
\eta' &= \frac{\eta}{\rho^2 - T^2}, & \eta &= \frac{\eta'}{\rho'^2 - T'^2}, \\
\zeta' &= -\frac{\zeta}{\rho^2 - T^2}, & \zeta &= -\frac{\zeta'}{\rho'^2 - T'^2}, \\
T' &= \frac{T}{\rho^2 - T^2}, & T &= \frac{T'}{\rho'^2 - T'^2},
\end{aligned}\right\} \tag{45-1}$$

where the negative sign in the relations connecting ζ' and ζ is due to the fact that these axes have been taken in opposite senses so as to make each set of axes right-handed.

The reader desiring a more complete discussion of relatively accelerated reference systems may consult the papers of Page [11] and Page and Adams.[12]

Problem 45a. Show that if a ray of light follows a straight track in one of two relatively accelerated reference systems, its path is a straight line in the other.

[11] L. Page, Phys. Rev. **49**, p. 254 (1936).
[12] Page and Adams, Phys. Rev. **49**, p. 466 (1936).

CHAPTER 3

THE ELECTROMAGNETIC FIELD

46. Emission Theory of Electromagnetism. — Faraday has shown how an electric field can be represented graphically by continuous lines of force originating on positive charges and terminating on negative charges. We shall endow this geometrical representation [1] with sufficient physical substantiality to allow us to picture lines of electric force as filaments which may move relative to the observer's inertial system in company with the charges to which they belong. Since, however, different charges may move with different velocities and in different directions, kinematical properties can be attributed to lines of force only if we treat the field of each element of charge as a separate entity. Hence our point of view differs from Faraday's in that we shall ascribe to each element of charge its own diverging or converging group of lines of force, instead of representing the resultant electric field by lines of force. Further we shall suppose that the lines of force associated with an element of charge may penetrate the fields of other charges, or in fact other charges themselves, without being in any way influenced by the latter. When a number of elementary charges are present, then, their independent fields coexist, and the resultant field will be considered, not as an entity in itself, but as a combination of the overlapping elementary fields of the individual elements of charge.

In order to make the discontinuous representation of a field by lines of force effectively continuous, we shall imagine the number of lines proceeding from even the smallest element of charge to be very large. We may, then, group adjacent lines of force into bundles or tubes containing equal numbers of lines, the number of lines to a tube being determined in such a way as to make the total number of tubes diverging from a positive charge or converging on a negative charge proportional to the magnitude of the charge. If the factor of proportionality is taken as unity, as will be our procedure, the number of

[1] L. Page, Am. Jour. Sci., 38, p. 169 (1914).

tubes of force in the field of an element of charge is identical with the number of units of electricity in the charge, and we are led automatically to the Heaviside-Lorentz units * of electric and magnetic quantities. Now, if a bounded surface is placed in the field of an elementary charge, some of the tubes associated with the charge may pass in whole through the surface, and others only in part. In the latter case the fraction of a tube passing through the surface is the ratio of the number of lines which intersect the surface to the total number of lines in the tube.

At a given instant an elementary electric field is completely described at every point by the direction and the density of the lines of force. These two characteristics may be specified quantitatively by a vector function **E** of the coordinates which has at every point the direction of the lines of force at that point and a magnitude equal to the number of tubes of force per unit cross-section. The function **E** is called the *electric intensity* of the elementary field. At a point where a number of elementary fields overlap, we define the *resultant electric intensity* as the vector sum of the electric intensities of the individual fields. Evidently at a distance from a charged body large compared with its greatest linear dimension the individual fields due to all the elements of charge on the body have so nearly the same direction that we can treat the resultant field as if it were a single elementary field.

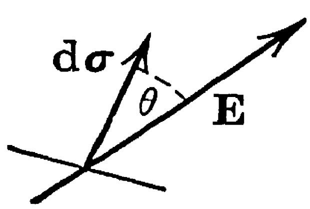

FIG. 41.

Let dN be the number of tubes of electric force of an elementary field passing through an element of surface $d\sigma$ (Fig. 41) whose normal makes an angle θ with the electric intensity. As the projection of $d\sigma$ on a plane at right angles to the lines of force is $d\sigma \cos\theta$, the number of tubes per unit cross-section and therefore the magnitude of the electric intensity is

$$E = \frac{dN}{d\sigma \cos\theta}. \qquad (46\text{–}1)$$

Consequently the component of **E** along the normal to the surface is equal to the number of tubes per unit area passing through it. If, therefore, a number of elementary fields $\mathbf{E}_1$, $\mathbf{E}_2$, $\cdots$, making angles θ_1, θ_2, $\cdots$ with the normal to the surface, overlap in the neighborhood

*A table of equivalents in Heaviside-Lorentz units, electromagnetic units and electrostatic units is given on p. xii.

of $d\sigma$ to produce the resultant field $\mathbf{E}$ making an angle θ with the normal,

$$E \cos \theta = E_1 \cos \theta_1 + E_2 \cos \theta_2 + \cdots.$$

But

$$E_1 = \frac{dN_1}{d\sigma \cos \theta_1}, \quad E_2 = \frac{dN_2}{d\sigma \cos \theta_2}, \quad \cdots.$$

Therefore

$$E = \frac{dN_1 + dN_2 + \cdots}{d\sigma \cos \theta},$$

showing that (46–1) holds also for the resultant of a number of elementary fields provided we interpret dN as the total number of tubes of force passing through the surface $d\sigma$.

Each line of force can be considered as the locus of a dense linear aggregate of moving-elements, or points. We wish to find the velocity law of these moving-elements. Now the principle of relativity tells us that both the velocity and the acceleration of a constituent moving-element in an elementary electric field must have the same magnitudes relative to all inertial systems. As shown in article 43 this requires that the moving-elements shall move in straight lines with the velocity c of light. Furthermore, the fact that the lines of force originate on the charge whose field they represent, requires that lines of force in the immediate neighborhood of the charge can have no finite component of linear velocity at right angles to themselves relative to the inertial system in which the charge is momentarily at rest. It follows that the direction of motion relative to this inertial system of each moving-element in the immediate neighborhood of the charge must be tangent to the line of force of which it forms a part. Hence the moving-elements in the immediate neighborhood of the charge must move radially either toward or away from the charge, and the charge must be considered either as a sink or as a source of moving-elements. Either alternative, or any combination of the two, leads to Maxwell's equations, the first corresponding to advanced fields and potentials and the second to retarded fields and potentials. However, the teleological implications of the first alternative are generally considered sufficient to warrant its exclusion from physical reality, for, if the moving-elements constituting an elementary field should move toward the charge responsible for the field, the future history of the charge would be completely determined by the present nature of the distant portions of its own field, and it would be impos-

sible for the charge to be influenced in any inconsistent manner by the fields of other charges. Hence we are forced to adopt the second alternative and to consider an element of charge of either sign as a source of moving-elements projected from it in straight lines with velocity c. As lines of force diverge from positive charges and converge on negative charges, the angle between a line of force and the direction of motion of the moving-elements which carry it is never greater than $\pi/2$ in the field of an element of positive charge, and never less than $\pi/2$ in the field of an element of negative charge. Although we adopt the second of the two possible alternatives considered, many of the formulas we shall develop require only an occasional change in sign to make them fit the first alternative, which has been discussed in some detail in a paper by Page.[2]

We picture an element of charge, then, as a group of emittors each of which shoots forth a stream of moving-elements with the velocity of light, in much the same way as a machine gun fires a stream of bullets. The locus, at any instant, of the moving-elements which have been projected from a given emittor, constitutes a line of force. Now we shall adopt two assumptions suggested by symmetry. First we assume that, relative to the inertial system in which the element of charge is momentarily at rest, the emittors are distributed uniformly in angle, and second, that the emittors have no rotation relative to this inertial system.

To summarize, the emission theory starts with the elementary electric field as the fundamental entity in its description of electromagnetic phenomena. The velocity law of the moving-elements constituting this field has been indicated uniquely by the requirements of the principle of relativity. In addition we have made two fundamental assumptions regarding the distribution and angular motion of the emittors from which the lines of force spring forth, which are strongly suggested, if not demanded, by symmetry. In the succeeding articles of this chapter we shall prove that the four field equations formulated by Maxwell to express in analytical form the experimental discoveries of Coulomb, Ampère, and Faraday are purely kinematical relations deducible in whole from the characteristics specified in the present article for an elementary field.

47. Transformation of the Electric and the Magnetic Intensity.—Our first objective is to express the electric intensity $\mathbf{E}'$ of an elementary electric field as measured in an inertial system S' in terms of

[2] L. Page, Phys. Rev. 24, p. 296 (1924).

the electric intensity **E** of the same field as measured in another inertial system S. Let P (Fig. 42) be the position of a moving-element of a line of force at time t in S and t' in S', and let Q be the position of an adjacent moving-element of the *same* line of force at time t in S. At time t' in S' the latter moving-element will be at a slightly different point Q'. Then $\overline{PQ}$ is an element of the line of force in S at time t, and $\overline{PQ'}$ the same element of the line of force in S' at time t'.

Orient the axes fixed in the two inertial systems so that the X and X' axes are parallel to the velocity **v** of S' relative to S. We shall designate the coordinates in S of Q relative to P by dx, dy, dz, and of

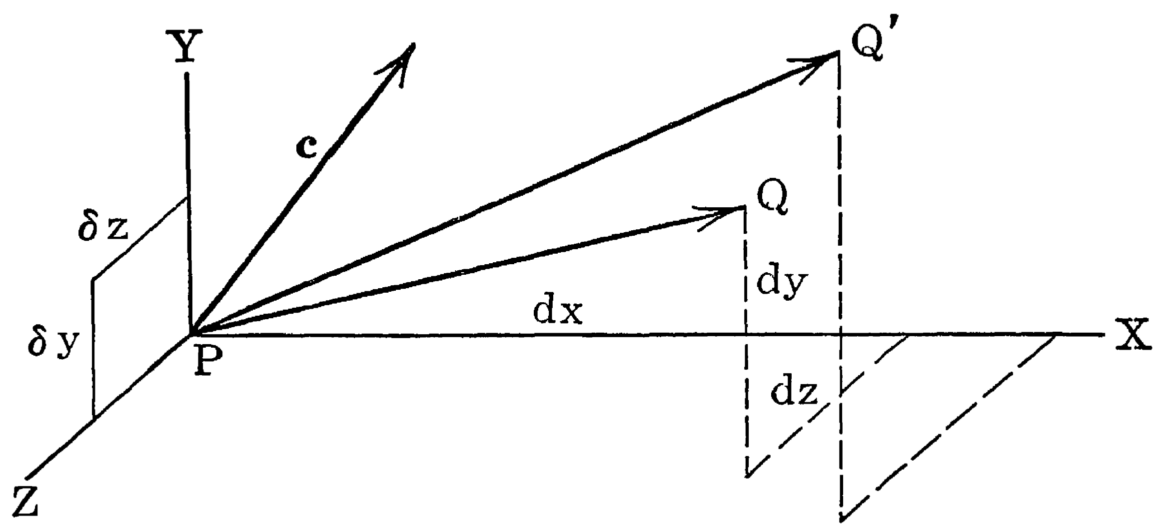

FIG. 42.

Q' relative to P by dx_1, dy_1, dz_1. Also we shall understand by dx', dy', dz' the coordinates of Q' relative to P as measured in S'. The kinematical part of our analysis consists of two steps. First we express dx_1, dy_1, dz_1 in terms of dx, dy, dz, and then we utilize the Lorentz transformation between dx', dy', dz' and dx_1, dy_1, dz_1 to express dx', dy', dz' in terms of dx, dy, dz.

If the velocity of the moving-element at Q is **c**, and the time in S at which this moving-element reaches Q' is $t + dt$,

$$\begin{aligned} dx_1 &= dx + c_x dt, \\ dy_1 &= dy + c_y dt, \\ dz_1 &= dz + c_z dt. \end{aligned}$$

As the time in S' is the same at Q' as at P, that is, $dt' = 0$, it follows from (42–2) that

$$dt = \frac{\beta}{c} dx_1.$$

Hence

$$dx_1 = \frac{dx}{1 - \beta\frac{c_x}{c}},$$

$$dy_1 = dy + \frac{\beta\frac{c_y}{c}\,dx}{1 - \beta\frac{c_x}{c}},$$

$$dz_1 = dz + \frac{\beta\frac{c_z}{c}\,dx}{1 - \beta\frac{c_x}{c}}.$$

To obtain from these expressions the coordinates dx', dy', dz' of Q' relative to P as measured in S' we must use the last three equations in the second column of (42–2), making $dt' = 0$, since P and Q' are reached at the same time according to the standard of simultaneity of S'. In this way we find

$$\left.\begin{aligned} dx' &= \frac{dx}{k\left(1 - \beta\frac{c_x}{c}\right)}, \\ dy' &= dy + \frac{\beta\frac{c_y}{c}\,dx}{1 - \beta\frac{c_x}{c}}, \\ dz' &= dz + \frac{\beta\frac{c_z}{c}\,dx}{1 - \beta\frac{c_x}{c}}. \end{aligned}\right\} \qquad (47\text{–}1)$$

As the direction cosines of the line of force through P are proportional to dx', dy', dz' as measured in S', and proportional to dx, dy, dz as measured in S, these three equations specify the direction cosines of the line of force in S' in terms of its direction cosines in S. Since

the electric intensity at P is parallel to the line of force through P,

$$\frac{E_x'}{dx'} = \frac{E_y'}{dy'} = \frac{E_z'}{dz'}$$

and

$$\frac{E_x}{dx} = \frac{E_y}{dy} = \frac{E_z}{dz}.$$

If, then, we determine one component of the electric intensity in S' in terms of the corresponding component in S, the transformation for the remaining components can be obtained at once from (47–1). The component for which the transformation is most easily obtained is that parallel to the X axis, since an elementary rectangle at P parallel to the YZ plane has the same area in S' as in S. Let δN be the number of tubes of force passing through the rectangle of edges δy, δz (Fig. 42). Then, as $\delta y' = \delta y$, $\delta z' = \delta z$, and the component of the electric intensity normal to any element of surface is equal to the number of tubes of force per unit area passing through the surface,

$$E_x' = \frac{\delta N}{\delta y' \delta z'} = \frac{\delta N}{\delta y \delta z} = E_x.$$

Consequently, using (47–1),

$$\begin{aligned} E_y' &= E_x' \frac{dy'}{dx'} = kE_x' \left\{ \frac{dy}{dx} \left(1 - \beta \frac{c_x}{c} \right) + \beta \frac{c_y}{c} \right\} \\ &= k \left\{ E_y - \beta \left(\frac{c_x}{c} E_y - \frac{c_y}{c} E_x \right) \right\} \\ &= k \left\{ E_y - \frac{\beta}{c} \left| \mathbf{c} \times \mathbf{E} \right|_z \right\}, \end{aligned}$$

and similarly

$$E_z' = k \left\{ E_z + \frac{\beta}{c} \left| \mathbf{c} \times \mathbf{E} \right|_y \right\}.$$

As the vector $(1/c)\mathbf{c} \times \mathbf{E}$ appears in these and many other expressions to be derived later, it is convenient to represent it by a single symbol and to attach a name to it. We shall write $\mathbf{H} \equiv (1/c)\mathbf{c} \times \mathbf{E}$ and call this quantity the *magnetic intensity* of the elementary electric field. When we have deduced the complete set of electromagnetic equations we shall be able to identify $\mathbf{H}$ unambiguously with the magnetic intensity as defined experimentally by Ampère and Maxwell.

In terms of **H** the transformation for the components of **E** becomes

$$\left.\begin{aligned} E_x' &= E_x, \\ E_y' &= k\{E_y - \beta H_z\}, \\ E_z' &= k\{E_z + \beta H_y\}. \end{aligned}\right\} \tag{47–2}$$

Next we shall find the transformation for the components of **H**. To obtain it we need the transformation for the components of **c** as well as that for the components of **E**. The former, given by (43–1) with c_x, c_y, c_z replacing V_x, V_y, V_z respectively, is

$$\left.\begin{aligned} c_x' &= \frac{c_x - v}{1 - \beta\dfrac{c_x}{c}}, \\ c_y' &= \frac{c_y}{k\left(1 - \beta\dfrac{c_x}{c}\right)}, \\ c_z' &= \frac{c_z}{k\left(1 - \beta\dfrac{c_x}{c}\right)}. \end{aligned}\right\} \tag{47–3}$$

The derivation is facilitated by noting that

$$\mathbf{c}\cdot\mathbf{H} \equiv c_xH_x + c_yH_y + c_zH_z = 0.$$

Then, as $\mathbf{H}' \equiv (1/c)\mathbf{c}' \times \mathbf{E}'$,

$$\begin{aligned} H_x' &= \frac{c_y'}{c}E_z' - \frac{c_z'}{c}E_y' \\ &= \frac{\dfrac{c_y}{c}(E_z + \beta H_y) - \dfrac{c_z}{c}(E_y - \beta H_z)}{1 - \beta\dfrac{c_x}{c}} \\ &= \frac{H_x - \beta\dfrac{c_x}{c}H_x}{1 - \beta\dfrac{c_x}{c}} = H_x, \end{aligned}$$

$$H_y' = \frac{c_z'}{c} E_x' - \frac{c_x'}{c} E_z'$$

$$= k \frac{(1 - \beta^2) \frac{c_z}{c} E_x - \left(\frac{c_x}{c} - \beta\right)(E_z + \beta H_y)}{1 - \beta \frac{c_x}{c}}$$

$$= k \frac{H_y - \beta \frac{c_x}{c} H_y + \beta\left(E_z - \beta \frac{c_x}{c} E_z\right)}{1 - \beta \frac{c_x}{c}} = k\{H_y + \beta E_z\},$$

and similarly with H_z'. Collecting results:

$$\left.\begin{aligned} H_x' &= H_x, \\ H_y' &= k\{H_y + \beta E_z\}, \\ H_z' &= k\{H_z - \beta E_y\}, \end{aligned}\right\} \qquad (47\text{–}4)$$

a set of equations which differs from (47–2) only in the interchange of the components of **E** and **H** and in the change in sign of the terms in β.

The inverse transformations are obviously

$$\left.\begin{aligned} E_x &= E_x', \\ E_y &= k\{E_y' + \beta H_z'\}, \\ E_z &= k\{E_z' - \beta H_y'\}, \end{aligned}\right\} \qquad (47\text{–}5)$$

and

$$\left.\begin{aligned} H_x &= H_x', \\ H_y &= k\{H_y' - \beta E_z'\}, \\ H_z &= k\{H_z' + \beta E_y'\}, \end{aligned}\right\} \qquad (47\text{–}6)$$

since the velocity of S relative to S' is the negative of that of S' relative to S.

Just as we defined the resultant electric intensity in a region where a number of elementary fields overlap as the vector sum of the electric intensities of the individual fields, so we shall define the *resultant magnetic intensity* as the vector sum of the magnetic intensities due to the separate elementary fields. Then, as (47–2), (47–4), (47–5) and (47–6) are linear in the components of **E** and **H**, and do not contain

the components of **c** explicitly, these transformations apply as well to the resultant **E** and **H** as to the electric and magnetic intensities of a single elementary field.

It follows from (47–2) and (47–4) that

$$\mathbf{E}'\cdot\mathbf{H}' = \mathbf{E}\cdot\mathbf{H} \tag{47–7}$$

and

$$E'^2 - H'^2 = E^2 - H^2 \tag{47–8}$$

are invariants of the Lorentz transformation.

It appears from the transformation equations (47–2) and (47–4) that the resolution of an electromagnetic field into electric and magnetic parts depends upon the state of motion of the observer. A field which is entirely magnetic to an observer in one inertial system may be in part magnetic and in part electric to an observer in another. Only the expressions (47–7) and (47–8) have the same values for observers in all inertial systems. From the first of these we see that if **E** and **H** are at right angles in one inertial system they are at right angles in all, and from the second we see that if the electric field is more intense in one inertial system than in another, the magnetic field also is more intense in the first than in the second.

These considerations suggest the following question: can we eliminate either the electric part or the magnetic part of any electromagnetic field by giving the observer a suitable velocity? Suppose that we have an electric field **E** and a magnetic field **H** in inertial system S, and wish to find an inertial system S' in which the electromagnetic field is entirely magnetic. Since $E_x' = E_x$ in (47–2), we see first that S' must move with a velocity **v** at right angles to **E**. Then (47–2) shows that all three components of **E**′ vanish if

$$\beta = \frac{E_y}{H_z} = -\frac{E_z}{H_y},$$

a pair of equations which can be satisfied for $\beta < 1$ only if

$$E_yH_y + E_zH_z = 0,$$

and

$$E^2 < (H_y^2 + H_z^2).$$

Consequently, if **E** is perpendicular to **H** and less than **H** in magnitude, **E**′ is zero in the inertial system moving relative to S with the velocity

$$\mathbf{v} = c\,\frac{\mathbf{E}\times\mathbf{H}}{H^2} \tag{47–9}$$

or, in fact, in any inertial system moving at right angles to **E** at an angle with **E** × **H** whose cosine is greater than E/H, with a velocity v equal to c times the quotient of E by the component of **H** perpendicular to **v**. In the inertial system S' moving with the velocity (47–9) relative to S, **H**′ is in the same direction as **H** and has the magnitude $\sqrt{H^2 - E^2}$.

Similarly it follows from (47–4) that, if **H** is perpendicular to **E** and less than **E** in magnitude, **H**′ is zero in the inertial system moving relative to S with the velocity

$$\mathbf{v} = c\frac{\mathbf{E} \times \mathbf{H}}{E^2} \tag{47–10}$$

or in any inertial system moving at right angles to **H** at an angle with **E** × **H** whose cosine is greater than H/E, with a velocity v equal to c times the quotient of H by the component of **E** perpendicular to **v**. In the inertial system S' moving with the velocity (47–10) relative to S, **E**′ is in the same direction as **E** and has the magnitude $\sqrt{E^2 - H^2}$.

In the case of crossed (i.e., perpendicular) electric and magnetic fields, then, it is always possible to find an inertial system S' in which one of the vectors **E**′ and **H**′ vanishes, with the single exception of the case where $E = H$. This case is of especial interest since it is that of a plane electromagnetic wave. Here (47–9) and (47–10) are satisfied simultaneously, but only with a value of β equal to unity. As we transfer our observations from S to inertial systems moving in the direction of propagation of the wave (i.e., that of the vector **E** × **H**) with higher and higher velocities, the equal magnitudes of the perpendicular vectors **E**′ and **H**′ become smaller and smaller, approaching zero as v approaches c. At the same time the length of the wave grows larger and the frequency smaller. Just the reverse changes occur if we transfer our observations to inertial systems moving in the direction opposite to that of propagation.

A simple example of the dependence of the resolution of an electromagnetic field into electric and magnetic parts on the state of motion of the observer is provided by the infinite parallel plate condenser. Let the plates be at rest in S parallel to the ZX coordinate plane with the Y coordinate of the negatively charged plate greater than that of the positively charged plate. If, then, the charge per unit area is ρ_σ, the electromagnetic field between the plates is $E_x = 0$, $E_y = \rho_\sigma$, $E_z = 0$, $H_x = 0$, $H_y = 0$, $H_z = 0$ to an observer in S. To

an observer moving with velocity v in the X direction in the region between the plates, $E_x' = 0$, $E_y' = k\rho_\sigma$, $E_z' = 0$, $H_x' = 0$, $H_y' = 0$, $H_z' = -k\beta\rho_\sigma$. The latter observer attributes the increased intensity of the electric field to the Fitzgerald-Lorentz contraction, and the presence of the magnetic field to the currents produced by the motion of the charged plates relative to his inertial system.

Again, consider the uniform magnetic field $H_x = 0$, $H_y = H$, $H_z = 0$ between the large parallel plane pole-pieces of a magnet at rest in S. Relative to S', $E_x' = 0$, $E_y' = 0$, $E_z' = k\beta H$, $H_x' = 0$, $H_y' = kH_y$, $H_z' = 0$. The electric field in S' gives rise to an electromotive force along a straight wire at rest in this inertial system with its length parallel to the Z axis. This electromotive force

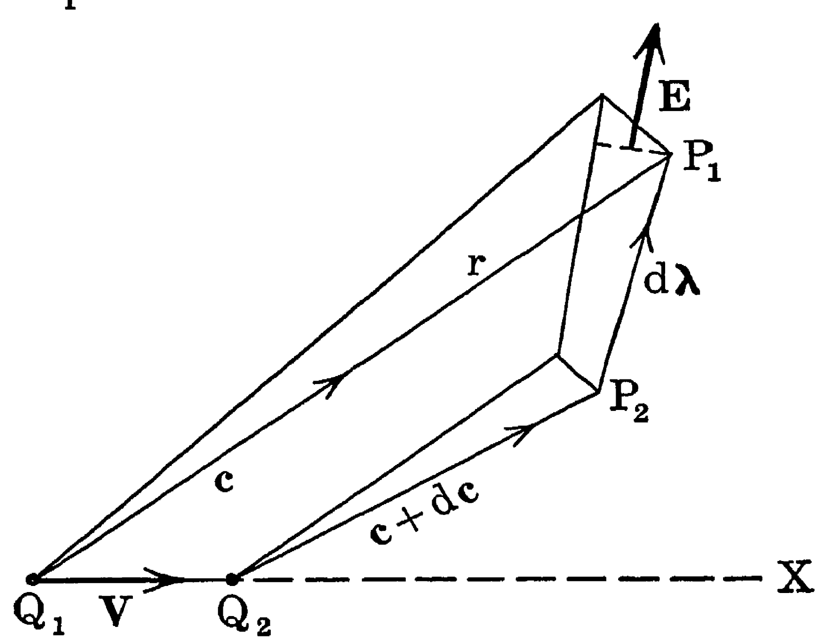

FIG. 43.

is evident to an observer in S through the work which it can perform on the free electrons in the conductor. As there is no electric field in S, the observer in this system attributes the electromotive force to the motion of the wire relative to him through a region in which a magnetic field is present. An electromotive force so produced is called a *motional electromotive force* since it exists for the observer in S only in a body in motion relative to his inertial system.

48. Field of a Point Charge. — Consider an elementary or point charge e which has a velocity **V** and acceleration **f** relative to the observer's inertial system S. Let the charge be located at Q_1 (Fig. 43) at time t_0 and at Q_2 at time $t_0 + dt_0$. Then $\overline{Q_1Q_2} = Vdt_0$. We wish to find the electric intensity **E** at a point P_1 distant r from Q_1 at the

time $t = t_0 + r/c$ at which a moving-element emitted from e at the time t_0 reaches P_1. Let P_2 be the point occupied at the time $t_0 + r/c$ by a moving-element projected from the same emittor at the time $t_0 + dt_0$. Then the line of force at P_1 at time $t_0 + r/c$ has the direction of the vector $d\boldsymbol{\lambda}$ drawn from P_2 to P_1.

If $\mathbf{c}$ is the velocity of the first moving-element and $\mathbf{c} + d\mathbf{c}$ that of the second,

$$d\boldsymbol{\lambda} = \mathbf{c}\frac{r}{c} - \mathbf{V}dt_0 - (\mathbf{c} + d\mathbf{c})\left(\frac{r}{c} - dt_0\right)$$

$$= (\mathbf{c} - \mathbf{V})dt_0 - \frac{r}{c}\,d\mathbf{c}, \tag{48–1}$$

and

$$\mathbf{c}\cdot d\boldsymbol{\lambda} = c^2\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)dt_0, \tag{48–2}$$

since $d\mathbf{c}$ is perpendicular to $\mathbf{c}$.

Let n be the number of tubes of force passing through a unit area at P_1 perpendicular to the radius vector r. Then

$$\mathbf{E} = n\,\frac{c\,d\boldsymbol{\lambda}}{\mathbf{c}\cdot d\boldsymbol{\lambda}}$$

$$= \frac{n}{c\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\left\{\mathbf{c} - \mathbf{V} - \frac{r}{c}\frac{d\mathbf{c}}{dt_0}\right\}. \tag{48–3}$$

It remains to evaluate $d\mathbf{c}/dt_0$ and n. The first is obtained from the condition that the emittors have no angular motion relative to the inertial system S' in which the charge is momentarily at rest at the time t_0, and the second from the condition that the emittors are distributed uniformly in angle relative to S'.

To find $d\mathbf{c}/dt_0$ we can make use of (44–3), understanding by $\boldsymbol{\rho}'$ a vector of constant magnitude having the direction of the emittor under consideration, and making $d\boldsymbol{\rho}'/dt_0' = 0$. In fact, as the direction of motion of a moving-element leaving the charge is the same as that of the emittor from which it comes, relative to the inertial system in which the charge is momentarily at rest, we can put the velocity $\mathbf{c}'$ of the moving-element relative to S' for $\boldsymbol{\rho}'$. Then we can find $\boldsymbol{\rho}$ from (44–1) and put its time derivative in (44–3) to determine $d\mathbf{c}/dt_0$.

However it is somewhat simpler to obtain $d\mathbf{c}'/dt_0'$ first and then to evaluate $d\mathbf{c}/dt_0$ by means of the transformations already derived. If the charge is accelerated, it passes from rest in S' at time t_0' to rest in an inertial system S'' moving with velocity $\mathbf{f}'dt_0'$ relative to S' at time $t_0' + dt_0'$. Consequently, if $\boldsymbol{\rho}'$ is a vector in S' having the direction of the emittor and a magnitude equal to the velocity of light,

$$\frac{d\boldsymbol{\rho}'}{dt_0'} = \frac{\mathbf{c}'' - \mathbf{c}'}{dt_0'},$$

where $\mathbf{c}'$ is the velocity relative to S' of the moving-element emitted at time t_0' and $\mathbf{c}''$ that relative to S'' of the moving-element emitted at time $t_0' + dt_0'$. Neglecting differentials of the second order, we have from (47–3)

$$\mathbf{c}'' = \frac{\mathbf{c}' + d\mathbf{c}' - \mathbf{f}'dt_0'}{1 - \dfrac{\mathbf{c}'\cdot\mathbf{f}'}{c^2}\, dt_0'} = \mathbf{c}' + d\mathbf{c}' + \frac{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'}{c^2}\, dt_0',$$

where $\mathbf{c}' + d\mathbf{c}'$ is the velocity relative to S' of the moving-element emitted at time $t_0' + dt_0'$. Therefore

$$\frac{d\boldsymbol{\rho}'}{dt_0'} = \frac{d\mathbf{c}'}{dt_0'} + \frac{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'}{c^2}.$$

The reader should note that $d\boldsymbol{\rho}'/dt_0'$ is not in general equal to $d\mathbf{c}'/dt_0'$ even though we have made $\boldsymbol{\rho}' = \mathbf{c}'$ at the instant t_0' when the charge is momentarily at rest in S'.

As the emittors have no motion of rotation,

$$\frac{d\mathbf{c}'}{dt_0'} = -\frac{(\mathbf{f}' \times \mathbf{c}') \times \mathbf{c}'}{c^2} = \mathbf{f}' - \frac{\mathbf{c}'\cdot\mathbf{f}'}{c^2}\,\mathbf{c}'. \tag{48–4}$$

Now we must transform this equation to system S. As we are differentiating at the charge, we make use of (43–4) with $V' = 0$ and $V = v$, getting $d/dt_0' = k\, d/dt_0$. The transformation for the components of $\mathbf{c}'$ are given by (47–3) and those for the components of $\mathbf{f}'$ by (43–7). Carrying through the algebra separately for each rectangular component of the vector equation (48–4) we obtain the three relations

$$\frac{dc_x}{dt_0} = \frac{k^2}{c^2}\left\{(c^2 - \mathbf{c}\cdot\mathbf{v})f_x - \mathbf{c}\cdot\mathbf{f}(c_x - v)\right\},$$

$$\frac{dc_y}{dt_0} = \frac{k^2}{c^2}\{(c^2 - \mathbf{c}\cdot\mathbf{v})f_y - \mathbf{c}\cdot\mathbf{f}c_y\},$$

$$\frac{dc_z}{dt_0} = \frac{k^2}{c^2}\{(c^2 - \mathbf{c}\cdot\mathbf{v})f_z - \mathbf{c}\cdot\mathbf{f}c_z\}.$$

We may now replace $\mathbf{v}$ in these equations by the equal velocity $\mathbf{V}$ of the charge, and combine the resulting equations into the single vector equation

$$\frac{d\mathbf{c}}{dt_0} = -\frac{\{\mathbf{f} \times (\mathbf{c} - \mathbf{V})\} \times \mathbf{c}}{c^2\left(1 - \frac{V^2}{c^2}\right)}. \tag{48-5}$$

The calculation of n is all that remains. Let α and α' be the angles which an emittor makes with $\mathbf{v}$ in S and S' respectively. Then $\rho \sin \alpha = \rho' \sin \alpha'$ and $\rho \cos \alpha = \sqrt{1 - \beta^2}\, \rho' \cos \alpha'$, giving $\tan \alpha' = \sqrt{1 - \beta^2} \tan \alpha$. But, if θ is the angle which $\mathbf{c}$ makes with $\mathbf{v}$,

$$\tan \alpha = \frac{c \sin \theta}{c \cos \theta - v} = \frac{\sin \theta}{\cos \theta - \beta}.$$

Therefore

$$\tan \alpha' = \frac{\sqrt{1 - \beta^2} \sin \theta}{\cos \theta - \beta}$$

and

$$\sin \alpha' d\alpha' = \frac{1 - \beta^2}{(1 - \beta \cos \theta)^2} \sin \theta d\theta.$$

Now, if dN is the number of tubes diverging from the charge inside the conical angle between α' and $\alpha' + d\alpha'$,

$$dN = \frac{e}{2} \sin \alpha' d\alpha' = \frac{e}{2} \frac{(1 - \beta^2)}{\left(1 - \frac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right)^2} \sin \theta d\theta,$$

since the emittors are distributed uniformly in angle as viewed from S'. The area subtending the conical angle under consideration at a distance r from the charge is

$$d\sigma = 2\pi r^2 \sin \theta d\theta.$$

Consequently the number of tubes of force passing through a unit area at P_1 perpendicular to the radius vector is

$$n = \frac{dN}{d\sigma} = \frac{e}{4\pi r^2} \frac{\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^2}, \tag{48–6}$$

where we have replaced $\mathbf{v}$ by its equal $\mathbf{V}$.

Substituting (48–5) and (48–6) in (48–3) we obtain the expression

$$\mathbf{E} = \left[\frac{e}{4\pi r^2 c\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\left\{\left(1 - \frac{V^2}{c^2}\right)(\mathbf{c} - \mathbf{V}) + \frac{r}{c^3}\{\mathbf{f}\times(\mathbf{c}-\mathbf{V})\}\times\mathbf{c}\right\}\right] \tag{48–7}$$

for the electric intensity at the field-point P_1 at time t in terms of the coordinates, velocity and acceleration of the charge at the earlier time t_0. Such an expression is said to be *retarded*, which we indicate by enclosing it in heavy square brackets. We shall use this notation consistently henceforth, understanding that all coordinates, velocity components and acceleration components of a charge are to be evaluated for the time $t - [r]/c$ when they are contained within square brackets. The position of the charge at this earlier time is called its *effective position* for the determination of the field at the field-point P_1 at the time t.

Since $\mathbf{H}$ stands for the vector function $(1/c)\mathbf{c} \times \mathbf{E}$ by definition, the retarded expression for the magnetic intensity due to a point charge is

$$\mathbf{H} = \left[\frac{e}{4\pi r^2 c\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\left\{-\left(1 - \frac{V^2}{c^2}\right)\frac{\mathbf{c}\times\mathbf{V}}{c} + \frac{r}{c^4}\mathbf{c}\times(\{\mathbf{f}\times(\mathbf{c}-\mathbf{V})\}\times\mathbf{c})\right\}\right]. \tag{48–8}$$

At small distances from the charge the parts of these expressions varying inversely with the square of the distance r predominate, whereas at great distances these parts are negligible compared with the parts which vary inversely with the first power of r. The latter parts of $\mathbf{E}$ and $\mathbf{H}$ constitute the *radiation field* of the charge.

49. Differentiation of Retarded Quantities. — The expressions (48–7) and (48–8) for the electric and magnetic intensities of the field of a point charge can be expressed as derivatives with respect to the

coordinates and the time of two potentials, one a scalar function and the other a vector function. In order to find them we must first develop formulas for the derivatives of a retarded quantity.

Let P (Fig. 44) be the field-point with coordinates x, y, z at which we wish to find the field at time t of a charge e whose effective position for this point and time is Q. The coordinates of Q we designate by

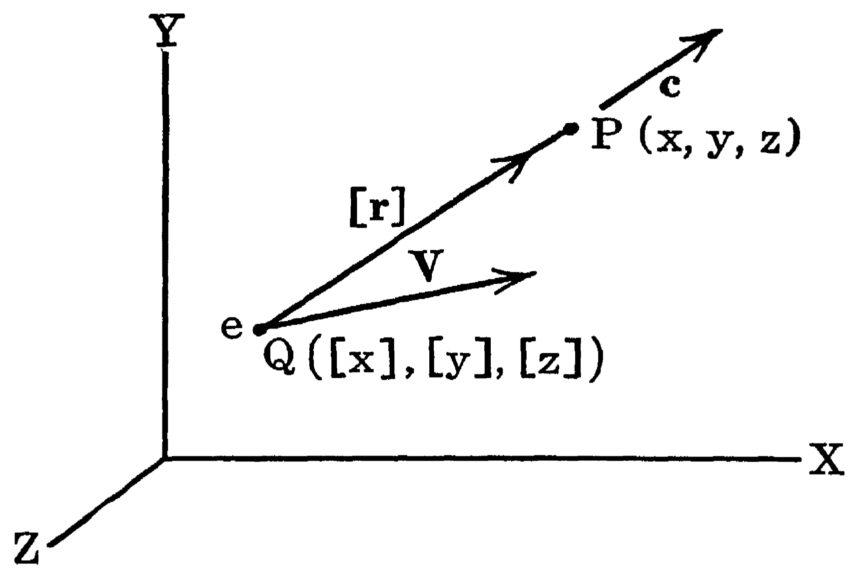

FIG. 44.

$[x]$, $[y]$, $[z]$ and the time at which it is occupied by the charge by $[t]$. Let $[\mathbf{r}]$ represent the vector $\overline{QP}$. Then

$$t - [t] = \frac{[r]}{c} = \frac{\sqrt{(x - [x])^2 + (y - [y])^2 + (z - [z])^2}}{c}, \quad (49\text{–}1)$$

where, if the charge is in motion relative to the axes XYZ, the coordinates $[x]$, $[y]$, $[z]$ and also the components $[V_x]$, $[V_y]$, $[V_z]$ of the velocity of the charge are in general functions of $[t]$.

Consider a quantity $\psi(x, y, z, [t])$, where $[t]$ occurs explicitly or implicitly in $[x], \cdots, [V_x], \cdots$, etc. As $[t]$ may be considered to be a function of x, y, z, t we have two types of partial derivatives of ψ to consider. For the moment we shall distinguish them by denoting the derivative with respect to x when y, z, $[t]$ are held constant by $\frac{\partial\psi}{\partial x}$, and the derivative with respect to x when the x contained implicitly in $[t]$ as well as the x appearing explicitly is varied, y, z, t being kept constant, by $\frac{d\psi}{dx}\cdot$ By $\frac{\partial\psi}{\partial[t]}$ we shall mean the derivative

with respect to $[t]$ when explicit x, y, z are held constant, and by $\dfrac{d\psi}{dt}$ the derivative with respect to the t contained implicitly in $[t]$, both explicit and implicit x, y, z being kept constant. Then

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial[t]}\frac{d[t]}{dt},$$

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial[t]}\frac{d[t]}{dx},$$

and similarly for derivatives with respect to y and z.

Differentiating (49–1) with respect to t we find that

$$1 - \frac{d[t]}{dt} = -\left\{\frac{(x-[x])[V_x]+(y-[y])[V_y]+(z-[z])[V_z]}{[r]c}\right\}\frac{d[t]}{dt}$$

$$= -\left[\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right]\frac{d[t]}{dt}$$

since $\mathbf{c}$ has the same direction as $[\mathbf{r}]$, and therefore

$$\frac{d\psi}{dt} = \frac{1}{\left[1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right]}\frac{\partial\psi}{\partial[t]}. \tag{49–2}$$

In like manner

$$-\frac{d[t]}{dx} =$$

$$\left\{\frac{(x-[x])\left(1-\dfrac{d[x]}{dx}\right)+(y-[y])\left(-\dfrac{d[y]}{dx}\right)+(z-[z])\left(-\dfrac{d[z]}{dx}\right)}{[r]c}\right\}.$$

But

$$\frac{d[x]}{dx} = [V_x]\frac{d[t]}{dx}, \text{ etc.}$$

Hence

$$-\frac{d[t]}{dx} = \frac{c_x}{c^2} - \left[\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right]\frac{d[t]}{dx}, \text{ etc.,}$$

and

$$\frac{d\psi}{dx} = \frac{\partial\psi}{dx} - \frac{\dfrac{c_x}{c^2}}{\left[1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right]}\frac{\partial\psi}{\partial[t]}$$

$$= \frac{\partial\psi}{\partial x} - \frac{c_x}{c^2}\frac{d\psi}{dt}. \tag{49–3}$$

Similar expressions hold for the derivatives with respect to y and z.

50. Scalar and Vector Potentials. — In this article we shall show that the retarded expressions (48–7) and (48–8) for the electric and magnetic intensities in the field of a point charge can be represented by derivatives of a scalar and a vector potential.

As

$$\left[\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right] = \frac{(x-[x])[V_x] + (y-[y])[V_y] + (z-[z])[V_z]}{[r]c}$$

appears in both (48–7) and (48–8) we start by writing down its derivatives. Evidently

$$\frac{\partial}{\partial[t]}\left[\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right] = \left[-\frac{V^2}{rc} + \frac{\overline{\mathbf{c}\cdot\mathbf{V}}^2}{rc^3} + \frac{\mathbf{c}\cdot\mathbf{f}}{c^2}\right],$$

$$\frac{\partial}{\partial x}\left[\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right] = \left[-\frac{\mathbf{c}\cdot\mathbf{V}}{rc^2}\frac{c_x}{c} + \frac{1}{r}\frac{V_x}{c}\right].$$

Also, from (49–1),

$$\frac{\partial[r]}{\partial[t]} = \left[-\frac{\mathbf{c}\cdot\mathbf{V}}{c}\right],$$

$$\frac{\partial[r]}{\partial x} = \frac{c_x}{c}.$$

Consequently (49–2) gives

$$\frac{1}{c}\frac{d}{dt}\left[\frac{1}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right] = \left[\frac{c\left\{\left(1-\dfrac{V^2}{c^2}\right) - \left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right) + \dfrac{r}{c^3}\,\mathbf{c}\cdot\mathbf{f}\right\}}{r^2c\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\right] \tag{50–1}$$

and (49–3) leads to

$$\frac{d}{dx}\left[\frac{1}{r\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]=\left[\frac{c_x\left\{-\left(1-\frac{V^2}{c^2}\right)-\frac{r}{c^3}\mathbf{c}\cdot\mathbf{f}\right\}+V_x\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}{r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\right]. \tag{50–2}$$

Hence

$$-\frac{d}{dx}\left[\frac{e}{4\pi r\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]-\frac{1}{c}\frac{d}{dt}\left[\frac{eV_x}{4\pi rc\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]$$

$$=\left[\frac{e}{4\pi r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\left\{\left(1-\frac{V^2}{c^2}\right)(c_x-V_x)\right.\right.$$

$$\left.\left.+\frac{r}{c^3}\left\{\mathbf{c}\cdot\mathbf{f}(c_x-V_x)-(c^2-\mathbf{c}\cdot\mathbf{V})f_x\right\}\right\}\right].$$

We recognize this as the x-component of the vector function (48–7). Now if we consider the quantities which are differentiated to be given as functions of x, y, z, t we can replace $\frac{d}{dt}$ by $\frac{\partial}{\partial t}$, $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, etc., and the electric intensity due to the point charge under consideration may be expressed in the form

$$\mathbf{E}=-\nabla\left[\frac{e}{4\pi r\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]-\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{e\mathbf{V}}{4\pi rc\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]. \tag{50–3}$$

Similarly

$$\frac{d}{dy}\left[\frac{eV_z}{4\pi rc\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]-\frac{d}{dz}\left[\frac{eV_y}{4\pi rc\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]$$

$$=\left[\frac{e}{4\pi r^2c\left(1-\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\left\{-\left(1-\frac{V^2}{c^2}\right)\left(\frac{c_yV_z-c_zV_y}{c}\right)\right.\right.$$

$$\left.\left.-\frac{r}{c^4}\left\{\mathbf{c}\cdot\mathbf{f}(c_yV_z-c_zV_y)+(c^2-\mathbf{c}\cdot\mathbf{V})(c_yf_z-c_zf_y)\right\}\right\}\right]$$

which we recognize as the x-component of (48–8). Therefore

$$\mathbf{H} = \nabla \times \left[\frac{e\mathbf{V}}{4\pi rc\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)} \right]. \tag{50–4}$$

Since $\mathbf{E}$ and $\mathbf{H}$ for the resultant of a number of elementary fields are the vector sums of the electric and magnetic intensities respectively of the individual overlapping fields, it follows that

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{50–5}$$

$$\mathbf{H} = \nabla \times \mathbf{A}, \tag{50–6}$$

for any field, where the *scalar potential* Φ and the *vector potential* $\mathbf{A}$ are given by the retarded expressions

$$\Phi = \sum_i \left[\frac{e_i}{4\pi r_i\left(1 - \frac{\mathbf{c}_i\cdot\mathbf{V}_i}{c^2}\right)} \right], \tag{50–7}$$

$$\mathbf{A} = \sum_i \left[\frac{e_i\mathbf{V}_i}{4\pi r_i c\left(1 - \frac{\mathbf{c}_i\cdot\mathbf{V}_i}{c^2}\right)} \right]. \tag{50–8}$$

The potentials for a continuous distribution of charge may be put in a simpler form. Suppose we wish to find these potentials at a field-point P (Fig. 45) at a time t due to a continuous distribution of charge of density $\rho(x, y, z, t)$ per unit volume. Consider a fixed volume element $d\tau$ at Q in the shape of a rectangular parallelopiped of altitude $\overline{AB} = dr$ at a distance r from P. Let the charge density at Q at time $t - r/c$ be denoted by $[\rho]$. If the charge in the neighborhood of Q has the velocity $\mathbf{V}$, the charge $[de]$ which lies in $ABCD$ when in its effective position for calculation of the field at P at time t is not

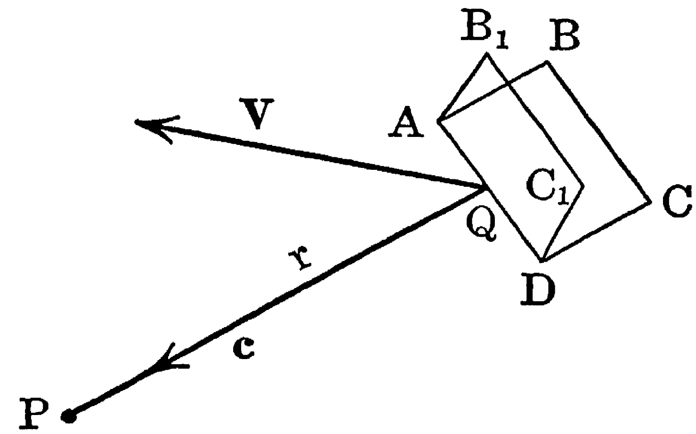

FIG. 45.

that resident in $ABCD$ at the time $t - r/c$, but that located in the smaller parallelopiped AB_1C_1D of altitude dr_1, where

$$dr_1 = dr - \left[\frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right] dr = \left[1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right] dr,$$

since the charge adjacent to the base BC of $ABCD$ must reach this position at the time $t - (r + dr)/c$ to be effective in giving rise to a field at P at time t. Therefore

$$[de] = [\rho]\left[1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right] d\tau$$

and (50–7) and (50–8) become

$$\Phi = \frac{1}{4\pi}\int_\tau \frac{[\rho]}{r}\, d\tau, \tag{50–9}$$

$$\mathbf{A} = \frac{1}{4\pi c}\int_\tau \frac{[\rho\mathbf{V}]}{r}\, d\tau, \tag{50–10}$$

where r need not be enclosed in brackets since it represents now a distance in the observer's inertial system which is not a function of the time.

Since the effective position of a charge for calculation of the field at P at time t is that occupied at the time $t - [r]/c$, the positions and velocities of the charges appearing in (50–7) and (50–8), or the charge density $[\rho]$ and current density $[\rho\mathbf{V}]$ appearing in (50–9) and (50–10), are those that would be *seen* by an observer at P at the time t.

In the case where the charge density and current density relative to the observer's inertial system are not functions of the time, the values of these quantities in any fixed volume element $d\tau$ are the same at time $t - r/c$ as at time t, and the retarded expressions (50–9) and (50–10) for the potentials may be replaced by the simultaneous expressions

$$\Phi = \frac{1}{4\pi}\int_\tau \frac{\rho}{r}\, d\tau, \tag{50–11}$$

$$\mathbf{A} = \frac{1}{4\pi c}\int_\tau \frac{\rho\mathbf{V}}{r}\, d\tau. \tag{50–12}$$

These formulas may be used to calculate the fields not only of charges permanently at rest in the observer's inertial system, but also to compute the electric and magnetic fields produced by a charged

sphere rotating with constant angular velocity about a fixed diameter the charge density of which is a function only of the distance from its center, the magnetic field due to constant currents flowing in any network of fixed conductors, and, in general, the fields due to any steady flow of charge.

Finally, we notice from (49–3) that

$$\frac{d}{dx}\left[\frac{\dfrac{V_x}{c}}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]=\left[\frac{V_x}{c}\right]\frac{d}{dx}\left[\frac{1}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]-\left[\frac{\dfrac{c_x f_x}{c^3}}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^2}\right].$$

Making use of (50–2), then,

$$\frac{d}{dx}\left[\frac{\dfrac{V_x}{c}}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]+\frac{d}{dy}\left[\frac{\dfrac{V_y}{c}}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]+\frac{d}{dz}\left[\frac{\dfrac{V_z}{c}}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right]$$

$$=\left[\frac{\dfrac{\mathbf{c}\cdot\mathbf{V}}{c}\left\{-\left(1-\dfrac{V^2}{c^2}\right)-\dfrac{r}{c^3}\mathbf{c}\cdot\mathbf{f}\right\}+\dfrac{V^2}{c}\left\{1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right\}-\dfrac{r}{c^2}\mathbf{c}\cdot\mathbf{f}\left\{1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right\}}{r^2c\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\right]$$

$$=-\frac{1}{c}\frac{d}{dt}\left[\frac{1}{r\left(1-\dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\right].$$

Consequently the scalar and vector potentials given by (50–7) and (50–8), considered as functions of x, y, z, t, satisfy the differential equation

$$\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0, \tag{50–13}$$

no matter whether they refer to a single elementary field or to the resultant of a number of overlapping fields.

51. Differential Equations of the Electromagnetic Field. — Obviously any vector field $\mathbf{U}(x, y, z)$ can be represented graphically by lines of force drawn so as to have everywhere the direction of the vector $\mathbf{U}$ and in such density that the number of tubes of force (a tube being defined as a bundle of a specified number of lines) per unit cross-section is equal to the magnitude of $\mathbf{U}$. In special cases the

lines of force may be everywhere continuous, but in general the lines of force required for the graphical representation of an arbitrary vector function of the coordinates are discontinuous, some lines ending where the field becomes weaker or beginning where it becomes stronger, or some of the lines reversing their sense at a point where the field suddenly changes direction.

The reasoning, employed in article 46 to show that the moving-elements of which the lines of force of an elementary electric field are the locus must move in straight lines with the velocity of light, can be applied as well to any elementary vector field of physical significance, and serves to tell us how the portion of the field represented by continuous lines of force changes with the time. If, however, there are discontinuities in the lines of force, as at an element of charge in an electric field, we must take account also of the motion of these discontinuities in computing the entire time rate of change of the field. We shall now derive the differential equations of an elementary vector field $\mathbf{U}(x, y, z, t)$, the lines of force of which are moving with velocity $\mathbf{c}$, and apply the results to the electric fields in which we are interested.

First consider any closed surface σ. As we proved in detail for the electric field discussed in article 46 the component of $\mathbf{U}$ along the outward-drawn normal to a surface element $d\boldsymbol{\sigma}$ at any point is equal to the number of tubes of force per unit area passing through it from the negative to the positive side. Therefore the flux $\mathbf{U}\cdot d\boldsymbol{\sigma}$ of the vector field through the surface element $d\boldsymbol{\sigma}$ is equal to the number dN of tubes of force passing through it. Integrating over the entire surface σ,

$$\int_\sigma \mathbf{U}\cdot d\boldsymbol{\sigma} = N, \tag{51–1}$$

where N is the excess of the number of tubes of force emerging from the region enclosed by the surface σ over the number entering. Converting the surface integral into a volume integral by Gauss' theorem, we have

$$\int_\tau \nabla\cdot\mathbf{U}d\tau = N,$$

where τ is the volume enclosed by the surface σ. Evidently, if the lines of force are continuous throughout the region τ, as many enter as leave this region and $N = 0$. Only when some of the lines originate

or terminate inside the region is N different from zero. Let ρ be the number of tubes of force which originate (ρ positive) or terminate (ρ negative) per unit volume. Then $N = \int_\tau \rho d\tau$ and the equation above becomes

$$\int_\tau \nabla \cdot \mathbf{U} d\tau = \int_\tau \rho d\tau.$$

As this equation holds for any volume τ, we get the *divergence equation*

$$\nabla \cdot \mathbf{U} = \rho \tag{51-2}$$

of the vector field $\mathbf{U}$.

Next consider the bounded surface σ (Fig. 46) fixed in the observer's inertial system. The number N of tubes of force passing through it is given by the integral (51–1) taken over the surface. Differentiating with respect to the time,

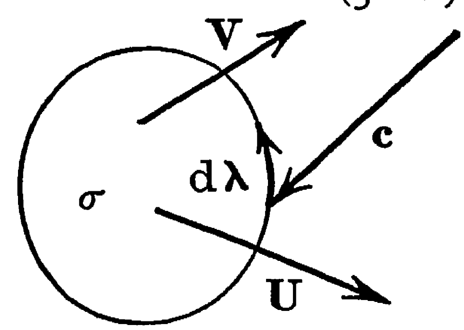

FIG. 46.

$$\frac{dN}{dt} = \int_\sigma \frac{\partial \mathbf{U}}{\partial t} \cdot d\boldsymbol{\sigma}. \tag{51-3}$$

But the time rate of increase of the number of tubes of force through the surface is equal to the number crossing the periphery from without to within per unit time plus the number of origins of discontinuous tubes passing through the surface per unit time from the positive to the negative side. The first is $\oint \mathbf{U} \cdot d\boldsymbol{\lambda} \times \mathbf{c}$ integrated around the periphery of the surface, and, if $\mathbf{V}$ is the velocity of the origins of tubes, the second is $-\int_\sigma \rho \mathbf{V} \cdot d\boldsymbol{\sigma}$ integrated over the surface. Therefore

$$\begin{aligned} \frac{dN}{dt} &= \oint \mathbf{c} \times \mathbf{U} \cdot d\boldsymbol{\lambda} - \int_\sigma \rho \mathbf{V} \cdot d\boldsymbol{\sigma} \\ &= \int_\sigma \nabla \times (\mathbf{c} \times \mathbf{U}) \cdot d\boldsymbol{\sigma} - \int_\sigma \rho \mathbf{V} \cdot d\boldsymbol{\sigma}. \end{aligned} \tag{51-4}$$

Equating (51–4) to (51–3) we have

$$\int_\sigma \nabla \times (\mathbf{c} \times \mathbf{U}) \cdot d\boldsymbol{\sigma} = \int_\sigma \left(\frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{V} \right) \cdot d\boldsymbol{\sigma}.$$

As this equation holds for any surface σ, we obtain the *circuital equation*

$$\nabla \times \left\{\frac{1}{c}\mathbf{c} \times \mathbf{U}\right\} = \frac{1}{c}\left\{\frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{V}\right\} \tag{51–5}$$

of the vector field $\mathbf{U}$.

Now we are ready to write down the differential equations of the electromagnetic field. If we put $\mathbf{E}$ for $\mathbf{U}$ in (51–2) and (51–5) the quantity ρ, since it has been defined as the number of ends of tubes per unit volume, represents the electric charge per unit volume, and $\rho\mathbf{V}$ the electric current per unit cross-section. Moreover the product $(1/c)\mathbf{c} \times \mathbf{E}$ has been designated by the symbol $\mathbf{H}$ and named the magnetic intensity. Therefore the two equations under consideration become

$$\left.\begin{aligned} \nabla \cdot \mathbf{E} &= \rho, \\ \nabla \times \mathbf{H} &= \frac{1}{c}(\dot{\mathbf{E}} + \rho\mathbf{V}), \end{aligned}\right\} \tag{51–6}$$

where the dot over a letter indicates the partial derivative with respect to the time. It may be appropriate to emphasize that these equations are purely kinematical relations governing the moving lines of force of the vector field $\mathbf{E}$ as viewed by an observer in the inertial system S, and that their derivation does not require the use of even the Lorentz transformation.

We have shown that $\mathbf{E}$ and $\mathbf{H}$ can be expressed as derivatives of a scalar potential Φ and a vector potential $\mathbf{A}$ by (50–5) and (50–6). From these equations we obtain at once the pair of differential equations

$$\left.\begin{aligned} \nabla \cdot \mathbf{H} &= \nabla \cdot \nabla \times \mathbf{A} = 0, \\ \nabla \times \mathbf{E} &= -\nabla \times \nabla\Phi - \frac{1}{c}\nabla \times \frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c}\dot{\mathbf{H}}. \end{aligned}\right\} \tag{51–7}$$

These, together with (51–6), constitute the complete set of differential equations of the electromagnetic field. Collected they are:

$$\left.\begin{aligned} \nabla \cdot \mathbf{E} &= \rho, \quad (a) & \nabla \cdot \mathbf{H} &= 0, \quad (b) \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\dot{\mathbf{H}}, \quad (c) & \nabla \times \mathbf{H} &= \frac{1}{c}(\dot{\mathbf{E}} + \rho\mathbf{V}). \quad (d) \end{aligned}\right\} \tag{51–8}$$

As these equations are identical with those formulated by Maxwell to describe in analytical form the experimental discoveries of Cou-

lomb, Ampère and Faraday, we can now identify unambiguously the vector $\mathbf{H} \equiv (1/c)\mathbf{c} \times \mathbf{E}$ with the magnetic intensity as measured experimentally. It may be recalled that the entire group of equations has been deduced by applying the relativity principle to the simple geometrical representation of the electric field of an element of charge relative to the inertial system in which it is momentarily at rest, by means of uniformly diverging lines of force the emittors of which have no angular velocity about the charge. As the differential equations are linear in $\mathbf{E}$, $\mathbf{H}$, ρ, and $\rho\mathbf{V}$ and do not contain the components of $\mathbf{c}$ explicitly, they apply as well to the resultant of a number of overlapping elementary fields as to each individual field.

By means of Gauss' and Stokes' theorems we can put the field equations in the integral form:

$$\left.\begin{array}{llll} \displaystyle\int_\sigma \mathbf{E}\cdot d\sigma = \int_\tau \rho d\tau, & (a) & \displaystyle\int_\sigma \mathbf{H}\cdot d\sigma = 0, & (b) \\ \displaystyle\oint \mathbf{E}\cdot d\lambda = -\frac{1}{c}\int_\sigma \dot{\mathbf{H}}\cdot d\sigma, & (c) & \displaystyle\oint \mathbf{H}\cdot d\lambda = \frac{1}{c}\int_\sigma (\dot{\mathbf{E}}+\rho\mathbf{V})\cdot d\sigma, & (d) \end{array}\right\} \quad (51\text{–}9)$$

where (a) and (b), in which the surface integrals are taken over the closed surface σ bounding the volume τ, are Gauss' laws for electric and magnetic fields, (c) is Faraday's law for the induced electromotive force around a closed circuit, and (d) is Ampère's law for the magnetomotive force around a closed circuit as corrected by Maxwell to include the so-called displacement current.

The differential equations satisfied by the scalar and vector potentials are easily obtained from (50–5), (50–6), (50–13) and (51–8). From (50–5) and (51–8a) we have at once

$$\nabla\cdot\nabla\Phi + \frac{1}{c}\nabla\cdot\dot{\mathbf{A}} = -\rho,$$

and from (50–5), (50–6) and (51–8d)

$$\nabla\cdot\nabla\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\dot{\Phi}\right) = -\frac{1}{c}\rho\mathbf{V}.$$

Making use of (50–13) these become

$$\left.\begin{array}{l} \nabla\cdot\nabla\Phi - \dfrac{1}{c^2}\ddot{\Phi} = -\rho, \\ \nabla\cdot\nabla\mathbf{A} - \dfrac{1}{c^2}\ddot{\mathbf{A}} = -\dfrac{1}{c}\rho\mathbf{V}, \end{array}\right\} \quad (51\text{–}10)$$

giving Φ in terms of the charge density ρ, and $\mathbf{A}$ in terms of the current density $\rho\mathbf{V}$. In empty space, where ρ and $\rho\mathbf{V}$ vanish, both Φ and $\mathbf{A}$ satisfy a wave equation with velocity of propagation c. If, furthermore, the field is static, Φ and $\mathbf{A}$ each satisfy Laplace's equation, that is,

$$\left.\begin{aligned} \nabla\cdot\nabla\Phi &= 0, \\ \nabla\cdot\nabla\mathbf{A} &= 0. \end{aligned}\right\} \qquad (51\text{–}11)$$

Problem 51a. Show that the equation of continuity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0$$

is a consequence of (51–8).

52. Boundary Conditions. — We can obtain from the field equations (51–9) the boundary conditions at a surface MN (Fig. 47) of discontinuity separating two regions τ_1 and τ_2. Let ρ_σ be the charge per unit area on the surface and $\mathbf{V}$ the velocity with which it is moving, and let $\mathbf{n}_1$ be a unit vector normal to the surface and $\mathbf{t}_1$ a unit vector tangent to the surface. If we apply (*a*) and (*b*) to the small pill-box $ABCD$ with bases of area $\Delta\sigma$ parallel to the surface, we get

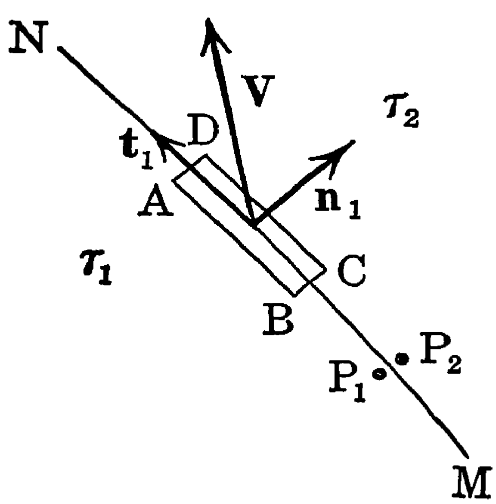

Fig. 47.

$$\mathbf{E}_1\cdot\Delta\boldsymbol{\sigma}_1 + \mathbf{E}_2\cdot\Delta\boldsymbol{\sigma}_2 = \rho_\sigma\Delta\sigma,$$
$$\mathbf{H}_1\cdot\Delta\boldsymbol{\sigma}_1 + \mathbf{H}_2\cdot\Delta\boldsymbol{\sigma}_2 = 0,$$

where the subscripts 1 and 2 refer to the regions τ_1 and τ_2 respectively, $\Delta\boldsymbol{\sigma}_1 = -\mathbf{n}_1\Delta\sigma$ and $\Delta\boldsymbol{\sigma}_2 = \mathbf{n}_1\Delta\sigma$ being the vectors representing the bases AB and CD, respectively, of the pill-box. Therefore

$$\left.\begin{aligned} \mathbf{E}_2\cdot\mathbf{n}_1 &= \mathbf{E}_1\cdot\mathbf{n}_1 + \rho_\sigma, \\ \mathbf{H}_2\cdot\mathbf{n}_1 &= \mathbf{H}_1\cdot\mathbf{n}_1, \end{aligned}\right\} \qquad (52\text{–}1)$$

are the boundary conditions for the normal components of $\mathbf{E}$ and $\mathbf{H}$.

To find the boundary conditions for the tangential components, let $ABCD$ represent a small rectangle, the long sides $\overline{AB}$ and $\overline{CD}$, of length $\Delta\lambda$, being parallel to $\mathbf{t}_1$. Applying (*c*) and (*d*) to this rectangular path

$$\mathbf{E}_1\cdot\Delta\boldsymbol{\lambda}_1 + \mathbf{E}_2\cdot\Delta\boldsymbol{\lambda}_2 = 0,$$

$$\mathbf{H}_1\cdot\Delta\boldsymbol{\lambda}_1 + \mathbf{H}_2\cdot\Delta\boldsymbol{\lambda}_2 = \frac{1}{c}\,\rho_\sigma\mathbf{V}\cdot(\mathbf{n}_1 \times \mathbf{t}_1)\Delta\lambda,$$

where $\Delta\lambda_1 = -\mathbf{t}_1\Delta\lambda$ and $\Delta\lambda_2 = \mathbf{t}_1\Delta\lambda$ are the vector lengths of $\overline{AB}$ and $\overline{CD}$, respectively. Hence

$$\left.\begin{aligned} \mathbf{E}_2\cdot\mathbf{t}_1 &= \mathbf{E}_1\cdot\mathbf{t}_1, \\ \mathbf{H}_2\cdot\mathbf{t}_1 &= \mathbf{H}_1\cdot\mathbf{t}_1 + \frac{1}{c}\,\rho_\sigma\mathbf{V}\times\mathbf{n}_1\cdot\mathbf{t}_1, \end{aligned}\right\} \tag{52–2}$$

are the boundary conditions for the tangential components of **E** and **H**, the second equation stating that the difference between the components of $\mathbf{H}_2$ and $\mathbf{H}_1$ along $\mathbf{t}_1$ is equal to the component of the current density $\rho_\sigma\mathbf{V}$ along a tangent to the surface at right angles to $\mathbf{t}_1$ in the sense of the unit vector $\mathbf{n}_1 \times \mathbf{t}_1$.

The scalar potential Φ and the vector potential **A** are continuous across a charged surface. To prove this, consider two opposite points P_1 and P_2 at equal small distances either side of the surface, and treat the portion σ_1 of the surface in the neighborhood of P_1 and P_2 separately from the remaining portion σ_2. Evidently the contributions to the potentials due to the charge and current on σ_2 are continuous as we pass from P_1 to P_2. The small portion σ_1 of the surface may be considered to be plane, and, as all parts of it are very close to P_1 and P_2, we may neglect retardation and use the simultaneous expressions (50–11) and (50–12) to calculate Φ and **A** at P_1 and P_2. But the form of these expressions shows that the potentials are the same at the two points, establishing the proposition we undertook to prove.

From the continuity of the potentials the equality of the tangential components of **E** expressed in (52–2) follows immediately by virtue of (50–5), and the equality of the normal components of **H** expressed in (52–1) follows as a result of (50–6). To show the latter in detail, let the X axis be normal to the surface at the point under consideration, so that $\mathbf{n}_1 = \boldsymbol{i}$. Then

$$H_n = \boldsymbol{i}\cdot\nabla\times\mathbf{A} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

As both derivatives are along the surface, the continuity of A_z and A_y demands the continuity of H_n. The second of the two boundary conditions (52–1) and the first of (52–2), then, can be replaced by the condition that the potentials are continuous.

53. Generalization of the Field Equations. — It is of interest to investigate the form which the equations of the electromagnetic field would take in a world containing magnetic as well as electric charges. Let us denote by $\mathbf{E}_E$ the electric intensity and by $\mathbf{c}_E$ the velocity of

the moving-elements of the elementary electric fields produced by electric charge of volume density ρ_E moving with velocity $\mathbf{V}_E$, and let us put $\mathbf{H}_E \equiv (1/c)\mathbf{c}_E \times \mathbf{E}_E$. Similarly we shall denote by $\mathbf{H}_H$ the magnetic intensity and by $\mathbf{c}_H$ the velocity of the moving-elements of the elementary magnetic fields produced by magnetic charge of volume density ρ_H moving with velocity $\mathbf{V}_H$, and we shall put $\mathbf{E}_H \equiv -(1/c)\mathbf{c}_H \times \mathbf{H}_H$. Then, corresponding to the transformations (47–2) and (47–4), which take the form

$$
\begin{aligned}
E'_{Ex} &= E_{Ex},\\
E'_{Ey} &= k\{E_{Ey} - \beta H_{Ez}\},\\
E'_{Ez} &= k\{E_{Ez} + \beta H_{Ey}\};\\
H'_{Ex} &= H_{Ex},\\
H'_{Ey} &= k\{H_{Ey} + \beta E_{Ez}\},\\
H'_{Ez} &= k\{H_{Ez} - \beta E_{Ey}\};
\end{aligned}
$$

in our present notation, we have also

$$
\begin{aligned}
H'_{Hx} &= H_{Hx},\\
H'_{Hy} &= k\{H_{Hy} + \beta E_{Hz}\},\\
H'_{Hz} &= k\{H_{Hz} - \beta E_{Hy}\};\\
E'_{Hx} &= E_{Hx},\\
E'_{Hy} &= k\{E_{Hy} - \beta H_{Hz}\},\\
E'_{Hz} &= k\{E_{Hz} + \beta H_{Hy}\}.
\end{aligned}
$$

Putting $\mathbf{E} \equiv \mathbf{E}_E + \mathbf{E}_H$ and $\mathbf{H} \equiv \mathbf{H}_H + \mathbf{H}_E$ we obtain by addition transformations for the resultant of any number of overlapping elementary fields due in part to electric charges and in part to magnetic charges. These are

$$
\left.
\begin{aligned}
E_x' &= E_x,\\
E_y' &= k\{E_y - \beta H_z\},\\
E_z' &= k\{E_z + \beta H_y\};
\end{aligned}
\right\} \qquad (53\text{–}1)
$$

and

$$
\left.
\begin{aligned}
H_x' &= H_x,\\
H_y' &= k\{H_y + \beta E_z\},\\
H_z' &= k\{H_z - \beta E_y\};
\end{aligned}
\right\} \qquad (53\text{–}2)
$$

which are identical in form with (47–2) and (47–4).

The field equations (51–8) for the resultant field due to electric charges are

$$\left.\begin{aligned} &\nabla\cdot\mathbf{E}_E = \rho_E, \quad (a) \qquad \nabla\cdot\mathbf{H}_E = 0, \qquad (b) \\ &\nabla\times\mathbf{E}_E = -\frac{1}{c}\dot{\mathbf{H}}_E, \ (c) \quad \nabla\times\mathbf{H}_E = \frac{1}{c}(\dot{\mathbf{E}}_E + \rho_E\mathbf{V}_E), \ (d) \end{aligned}\right\} \tag{53–3}$$

in our present notation. For the resultant field due to magnetic charges we have in place of (50–5) and (50–6)

$$\mathbf{H}_H = -\nabla\Phi_H - \frac{1}{c}\frac{\partial\mathbf{A}_H}{\partial t}, \tag{53–4}$$

$$\mathbf{E}_H = -\nabla\times\mathbf{A}_H, \tag{53–5}$$

the negative sign in the second being due to the fact that the definition of $\mathbf{E}_H$ does not correspond in sign with that of $\mathbf{H}_E$. These equations, together with (51–2) and (51–5), give us the following field equations for the resultant field due to magnetic charges:

$$\left.\begin{aligned} &\nabla\cdot\mathbf{H}_H = \rho_H, \quad (a) \qquad \nabla\cdot\mathbf{E}_H = 0, \qquad (b) \\ &\nabla\times\mathbf{H}_H = \frac{1}{c}\dot{\mathbf{E}}_H, \ (c) \quad \nabla\times\mathbf{E}_H = -\frac{1}{c}(\dot{\mathbf{H}}_H + \rho_H\mathbf{V}_H). \ (d) \end{aligned}\right\} \tag{53–6}$$

To find the differential equations satisfied by Φ_H and $\mathbf{A}_H$ we substitute (53–4) and (53–5) in (a) and (d) above, getting

$$\nabla\cdot\nabla\Phi_H + \frac{1}{c}\nabla\cdot\dot{\mathbf{A}}_H = -\rho_H,$$

$$\nabla\cdot\nabla\mathbf{A}_H - \frac{1}{c^2}\ddot{\mathbf{A}}_H - \nabla\left(\nabla\cdot\mathbf{A}_H + \frac{1}{c}\dot{\Phi}_H\right) = -\frac{1}{c}\rho_H\mathbf{V}_H.$$

But, as in (50–13) for the field due to electric charges,

$$\nabla\cdot\mathbf{A}_H + \frac{1}{c}\frac{\partial\Phi_H}{\partial t} = 0. \tag{53–7}$$

Therefore

$$\left.\begin{aligned} \nabla\cdot\nabla\Phi_H - \frac{1}{c^2}\ddot{\Phi}_H &= -\rho_H, \\ \nabla\cdot\nabla\mathbf{A}_H - \frac{1}{c^2}\ddot{\mathbf{A}}_H &= -\frac{1}{c}\rho_H\mathbf{V}_H, \end{aligned}\right\} \tag{53–8}$$

showing that in empty space Φ_H and $\mathbf{A}_H$, like Φ_E and $\mathbf{A}_E$, satisfy a wave equation with velocity of propagation c which reduces to Laplace's equation for a static field.

Adding the field equations (53–3) and (53–6), pair by pair, we obtain for the resultant of any number of overlapping elementary fields due in part to electric and in part to magnetic charges:

$$\left.\begin{array}{llll} \nabla\cdot\mathbf{E} = \rho_E, & (a) & \nabla\cdot\mathbf{H} = \rho_H, & (b) \\ \nabla\times\mathbf{E} = -\dfrac{1}{c}(\dot{\mathbf{H}} + \rho_H\mathbf{V}_H), & (c) & \nabla\times\mathbf{H} = \dfrac{1}{c}(\dot{\mathbf{E}} + \rho_E\mathbf{V}_E). & (d) \end{array}\right\} \quad (53\text{–}9)$$

In this generalized form, the field equations are completely symmetrical as regards electric and magnetic quantities.

CHAPTER 4

THE ELEMENTARY CHARGE AND THE FORCE EQUATION

54. Fields of Point Charges Moving with Constant Velocity and with Constant Acceleration. — In the last chapter we have deduced in closed form retarded expressions for the electric and magnetic intensities, relative to the observer's inertial system S, of the field of an element of charge moving with an arbitrary velocity and an arbitrary acceleration. These expressions, (48–7) and (48–8), specify **E** and **H** in terms of the distance $[r]$ of the effective position of the charge, and the velocity $[\mathbf{V}]$ and the acceleration $[\mathbf{f}]$ in the effective position. Often, however, we wish to know **E** and **H** at a point $P(x, y, z)$ at a time t in terms of the distance, velocity and acceleration of the charge at the same time t. Expressions of this character for **E** and **H**, or for the potentials Φ and **A**, we shall call *simultaneous*, as distinguished from the retarded expressions obtained earlier. Evidently simultaneous expressions for the complete field of an element of charge involve a knowledge of its entire past history. We can obtain simultaneous expressions for **E** and **H** in closed form for three types of motion: (*a*) rest, (*b*) motion with constant velocity, (*c*) motion with constant acceleration. For the general case of motion with an arbitrary velocity and an arbitrary acceleration we must content ourselves with infinite series.

(*a*) *Rest.* The field of a point charge permanently at rest in the observer's inertial system is extremely simple. As the effective position coincides with the simultaneous position, and as $[\mathbf{V}] = [\mathbf{f}] = 0$ and the radius vector has the same direction as **c**, equations (48–7) and (48–8) give

$$\mathbf{E} = \frac{e}{4\pi r^2}\frac{\mathbf{r}}{r}, \quad \mathbf{H} = 0, \tag{54–1}$$

where $\mathbf{r} = [\mathbf{r}]$ is the vector distance from the charge of the point P at which the field is to be determined.

(*b*) *Constant Velocity.* If a point charge moves with constant velocity $\mathbf{V}$ relative to the observer's inertial system, $[\mathbf{V}] = \mathbf{V}$ and $[\mathbf{f}] = 0$. Hence

$$\mathbf{E} = \left[\frac{e}{4\pi r^2 c\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\left(1 - \frac{V^2}{c^2}\right)(\mathbf{c} - \mathbf{V})\right]$$

from (48–7). As the distance of the simultaneous position O of the charge e from its effective position Q (Fig. 48) is $V[r]/c$,

$$\left[r\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)\right] = [r] - \frac{V}{c}[r]\cos\alpha = r\sqrt{1 - \sin^2(\theta - \alpha)},$$

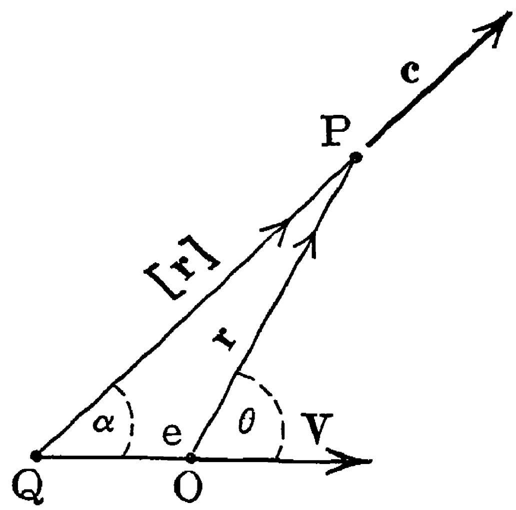

Fig. 48.

where θ and α are the angles which $\mathbf{r}$ and $[\mathbf{r}]$, respectively, make with $\mathbf{V}$. But $[r]\sin(\theta - \alpha) = (V[r]/c)\sin\theta$. Hence

$$\left[r\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)\right] = r\sqrt{1 - \frac{V^2}{c^2}\sin^2\theta}.$$

Furthermore, as the triangle formed by the vectors $\mathbf{c}$, $\mathbf{V}$ and $\mathbf{c} - \mathbf{V}$ is similar to the triangle PQO,

$$\frac{\mathbf{c} - \mathbf{V}}{c} = \frac{\mathbf{r}}{[r]}. \tag{54–2}$$

Consequently

$$\mathbf{E} = \frac{e}{4\pi r^2}\frac{\left(1 - \dfrac{V^2}{c^2}\right)}{\left(1 - \dfrac{V^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\mathbf{r}}{r}. \tag{54–3}$$

The magnetic intensity is most easily obtained from the definition $\mathbf{H} \equiv (1/c)\,\mathbf{c} \times \mathbf{E}$. As $\mathbf{c} \times \mathbf{r} = \mathbf{V} \times \mathbf{r}$ from (54–2),

$$\mathbf{H} = \frac{e}{4\pi r^2}\frac{\left(1 - \dfrac{V^2}{c^2}\right)}{\left(1 - \dfrac{V^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\mathbf{V} \times \mathbf{r}}{cr}. \tag{54–4}$$

We see from (54–3) that the lines of electric force are straight lines

radiating outwards from the simultaneous position O of the charge, and that they are most dense in the equatorial plane $\theta = \pi/2$ and least dense along the polar axis $\theta = 0$. As V approaches c the disparity in density increases, the lines of force crowding more and more toward the equatorial plane. If we draw lines of magnetic force in the direction of $\mathbf{H}$ it is apparent from (54–4) that they will be circles in planes at right angles to $\mathbf{V}$ with their centers on the line through the charge in the direction of the velocity. Like the electric field, the magnetic field, for a given r, is most intense in the equatorial plane, the disparity in intensity increasing with increase in V.

Although it is often convenient to represent the magnetic intensity of an elementary field by lines of magnetic force in the direction of $\mathbf{H}$ with a density proportional to the magnitude of $\mathbf{H}$, it must be noted that these lines do not possess the fundamental characteristics of lines of electric force and cannot be represented in general as the loci of moving-elements traveling with velocity $\mathbf{c}$. For while the number of lines of electric force in an elementary field is and remains the same relative to all inertial systems, no matter how the motion of the charge with which they are associated may change, the number of lines of magnetic force is different to observers in different inertial systems and in any one inertial system the number increases with increase in velocity of the charge. On account of the field equation $\nabla \cdot \mathbf{H} = 0$, however, the lines of magnetic force always form closed curves.

(*c*) *Constant Acceleration*. The integrated equation of motion of a point charge moving with constant relativity acceleration ϕ along the X axis is given by (38–12). If we take the origin O of coordinates at the point where the charge comes to rest in the observer's inertial system S, and the origin of time at the instant of this event, $x_0 = t_0 = 0$, and (38–12) simplifies to

$$x = \frac{c^2}{\phi}\left\{\sqrt{1 + \frac{\phi^2 t^2}{c^2}} - 1\right\}. \tag{54–5}$$

Under the same initial conditions, the velocity relation (38–10) reduces to

$$\frac{V}{c} = \frac{\dfrac{\phi t}{c}}{\sqrt{1 + \dfrac{\phi^2 t^2}{c^2}}}, \quad 1 - \frac{V^2}{c^2} = \frac{1}{1 + \dfrac{\phi^2 t^2}{c^2}}. \tag{54–6}$$

We shall compute simultaneous expressions for **E** and **H** at P (Fig. 49) only for the instant when the charge is at rest at the origin O. The distance $[x]$ of the effective position Q of the charge from O and the velocity $[V]$ of the charge when in this position are obtained by making $t = -[r]/c$ in (54–5) and (54–6) respectively.

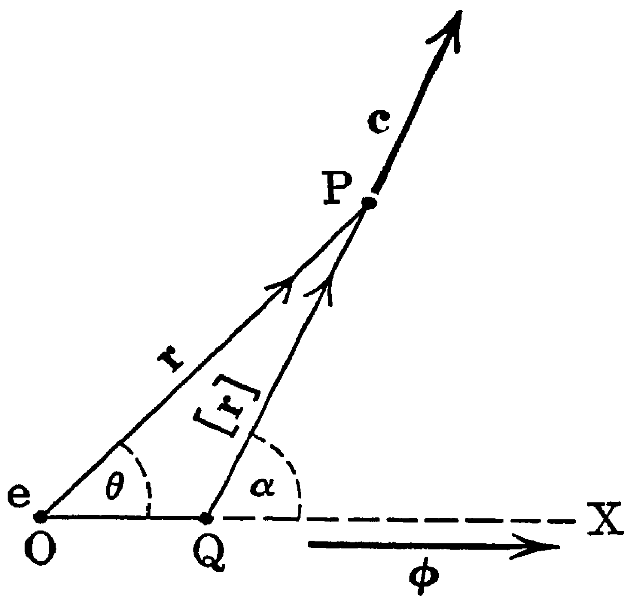

FIG. 49.

If we take the cross product of **c** by the factor inside the braces in (48–7) we get

$$\left[\left(1 - \frac{V^2}{c^2}\right)\mathbf{v} + \frac{r\mathbf{f}}{c}\right] \times \mathbf{c}.$$

But

$$\left[\left(1 - \frac{V^2}{c^2}\right)\mathbf{v}\right] = -\left[\left(1 - \frac{V^2}{c^2}\right)^{3/2} \frac{r\boldsymbol{\phi}}{c}\right]$$

from (54–6) and

$$\left[\frac{r\mathbf{f}}{c}\right] = \left[\left(1 - \frac{V^2}{c^2}\right)^{3/2} \frac{r\boldsymbol{\phi}}{c}\right]$$

from (38–9). Consequently $\mathbf{c} \times \mathbf{E} = 0$ and **E** at P is directed along the line $\overline{QP}$ drawn from the effective position of the charge. As $\mathbf{H} \equiv (1/c)\mathbf{c} \times \mathbf{E}$ there is no magnetic field at the instant considered.

It follows that

$$\mathbf{E} = \frac{1}{c^2}\mathbf{c}\cdot\mathbf{E}\mathbf{c} = \left[\frac{e\left(1 - \frac{V^2}{c^2}\right)\mathbf{c}}{4\pi r^2 c\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^2}\right]. \tag{54–7}$$

From the figure and (54–5) with $t = -[r]/c$ we have

$$r\cos\theta - [r]\cos\alpha = \frac{c^2}{\phi}\left\{\sqrt{1 + \frac{\phi^2[r]^2}{c^4}} - 1\right\},$$

where θ and α are the angles which $\mathbf{r}$ and $[\mathbf{r}]$, respectively, make with $\boldsymbol{\phi}$, or

$$\sqrt{1 + \frac{\phi^2[r]^2}{c^4} + \frac{\phi[r]}{c^2}\cos\alpha} = 1 + \frac{\phi r}{c^2}\cos\theta.$$

Now, putting $t = -[r]/c$ in (54–6), we find that

$$\left[\frac{\sqrt{1 - \dfrac{V^2}{c^2}}}{1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}}\right] = \frac{1}{\sqrt{1 + \dfrac{\phi^2[r]^2}{c^4} + \dfrac{\phi[r]}{c^2}\cos\alpha}} = \frac{1}{1 + \dfrac{\phi r}{c^2}\cos\theta}.$$

It remains to express $[r]$ in terms of r and θ. From the figure and (54–5) again,

$$[r]^2 = r^2 - 2r\frac{c^2}{\phi}\left\{\sqrt{1 + \frac{\phi^2[r]^2}{c^4}} - 1\right\}\cos\theta + \frac{c^4}{\phi^2}\left\{\sqrt{1 + \frac{\phi^2[r]^2}{c^4}} - 1\right\}^2$$

which gives

$$[r] = r\frac{\sqrt{1 + \dfrac{\phi r}{c^2}\cos\theta + \dfrac{\phi^2 r^2}{4c^4}}}{1 + \dfrac{\phi r}{c^2}\cos\theta}.$$

Therefore

$$\mathbf{E} = \frac{e}{4\pi r^2\left(1 + \dfrac{\phi r}{c^2}\cos\theta + \dfrac{\phi^2 r^2}{4c^4}\right)}\frac{\mathbf{c}}{c}, \quad \mathbf{H} = 0. \qquad (54\text{–}8)$$

From the figure we have

$$\left.\begin{aligned} \sin(\alpha - \theta) &= \frac{\overline{OQ}\sin\theta}{[r]} = \frac{\dfrac{\phi r}{2c^2}\sin\theta}{\sqrt{1 + \dfrac{\phi r}{c^2}\cos\theta + \dfrac{\phi^2 r^2}{4c^4}}}, \\ \cos(\alpha - \theta) &= \frac{r - \overline{OQ}\cos\theta}{[r]} = \frac{1 + \dfrac{\phi r}{2c^2}\cos\theta}{\sqrt{1 + \dfrac{\phi r}{c^2}\cos\theta + \dfrac{\phi^2 r^2}{4c^4}}}. \end{aligned}\right\} \qquad (54\text{–}9)$$

Consequently the radial and transverse components of **E** are

$$\left.\begin{aligned} E_r &= \frac{e}{4\pi r^2}\frac{1+\dfrac{\phi r}{2c^2}\cos\theta}{\left(1+\dfrac{\phi r}{c^2}\cos\theta+\dfrac{\phi^2 r^2}{4c^4}\right)^{3/2}}, \\ E_\theta &= \frac{e}{4\pi r^2}\frac{\dfrac{\phi r}{2c^2}\sin\theta}{\left(1+\dfrac{\phi r}{c^2}\cos\theta+\dfrac{\phi^2 r^2}{4c^4}\right)^{3/2}}. \end{aligned}\right\} \quad (54\text{–}10)$$

We can also express **E** as the sum of a component along the radius vector **r** and a component in the direction of $\boldsymbol{\phi}$. Then, as $\sin\alpha/\sin\theta = r/[r]$,

$$\mathbf{E} = \frac{e}{4\pi r^2}\frac{1}{\left(1+\dfrac{\phi r}{c^2}\cos\theta+\dfrac{\phi^2 r^2}{4c^4}\right)^{3/2}}\left\{\left(1+\frac{\phi r}{c^2}\cos\theta\right)\frac{\mathbf{r}}{r}-\frac{r}{2c^2}\boldsymbol{\phi}\right\}. \quad (54\text{–}11)$$

This expression for **E** shows us that there is a component of the electric intensity opposite to the acceleration $\boldsymbol{\phi}$ of the charge e in addition to the component along the radius vector. The former, which falls off inversely with the first power of r for $r \ll c^2/\phi$, would give rise to an electrical force on a neighboring charge of the same sign in the opposite direction to the acceleration of e. We shall see later that this force contributes to the mutual mass reaction of a closely packed group of charged particles.

Next we shall investigate the geometry of the field. Evidently the lines of electric force are curves in planes through the X axis. As the line of force through P has the direction of $[\mathbf{r}]$, its differential equation is

$$r\frac{d\theta}{dr} = \tan(\alpha-\theta) = \frac{\dfrac{\phi r}{2c^2}\sin\theta}{1+\dfrac{\phi r}{2c^2}\cos\theta}$$

from (54–9). The solution of this equation is

$$\cot\theta + \frac{\phi r}{2c^2}\csc\theta = \frac{\phi b}{c^2},$$

where b is the constant of integration, or, in more significant form,

$$\left(r\cos\theta + \frac{c^2}{\phi}\right)^2 + (r\sin\theta - b)^2 = \frac{c^4}{\phi^2} + b^2. \qquad (54\text{–}12)$$

The lines of electric force, then, are circles through the instantaneous position O of the charge with centers lying on the plane $x = -c^2/\phi$.

As the lines of force at the distance [r] from the effective position Q of the charge are directed along the radius vectors drawn from Q, a sphere of radius $[r]$ with Q as center will be intersected orthogonally by all the lines of force diverging from the charge. The family of spheres so constructed may be called *level surfaces*. Although the field under discussion cannot be described in terms of a scalar potential alone, the level surfaces correspond in many of their properties to equipotential surfaces in electrostatics.

A section of the field cut by a plane through the X axis is shown in Fig. 50, the lines of force being represented by solid lines and the level

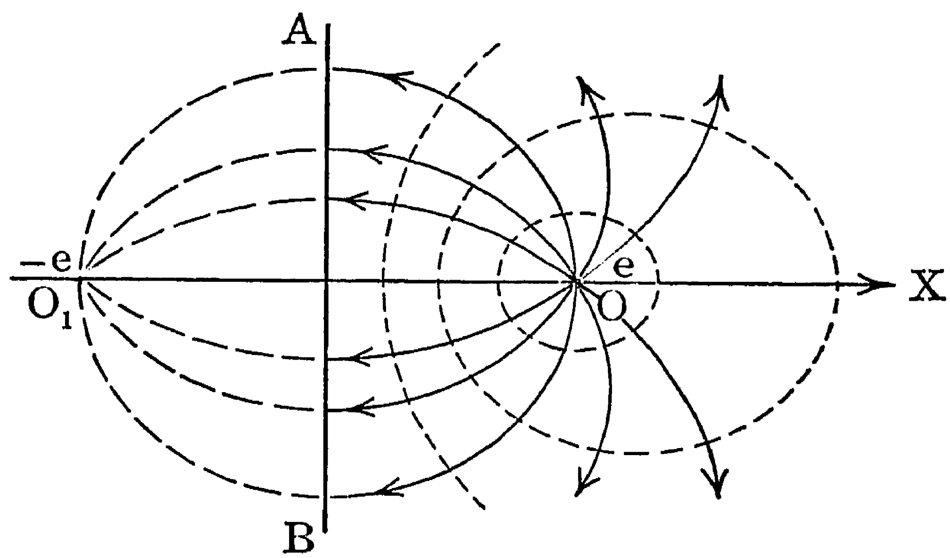

FIG. 50.

surfaces by dotted lines. The level surface AB at the distance $x = -c^2/\phi$ from the charge e terminates the field, for this plane is the locus of those moving-elements which were emitted by the charge at the time $t = -\infty$, and it is obvious that no moving-elements emitted at a later time could have proceeded as far as the plane AB by the time $t = 0$. Although the field terminates at a finite distance from the *present* position O of e, the plane AB is infinitely far from the position occupied by e when the field now in the proximity of AB was emitted. Physically it is unjustifiable to extrapolate the type of motion under consideration indefinitely far back in time, and it is only on account

of this unwarranted procedure that the field terminates in the finite part of the observer's inertial system. If the charge had been during all past time in the finite part of this region, the present field would extend to infinity.

It is interesting to interpret the field described by the analytical expressions we have derived in the region to the left of AB, even though in so doing we are forced to introduce the concept of advanced fields which we have definitely excluded from our physical theory on account of its teleological implications. The continuations of the circular lines of force, represented by broken lines in the figure, meet at O_1, indicating the presence there of a charge $-e$ which moves along the X axis as the mirror image of e in AB. Since the moving-elements are directed *toward* $-e$ instead of away from it, the portion of the field to the left of AB is advanced instead of retarded. Actually we have a combination of two elementary fields, the retarded field of e to the right of AB continuing as the advanced field of the image charge $-e$ to the left. While the former field expands with the time, the latter shrinks.

As $\mathbf{c}$ at the time $t = 0$ is everywhere tangent to the lines of force, a circular line of electric force of radius ρ at time $t = 0$ becomes a circle with the same center and radius $\sqrt{\rho^2 + c^2t^2}$ at time $t \gtrless 0$. Moreover, as $\mathbf{c}$ is not tangent to the lines of electric force at time $t \neq 0$, a magnetic field comes into existence the lines of force of which are circles in planes at right angles to the X axis with centers on this axis. At a time t the electric field of e is identical, except in its extent, with that at the time $-t$, and the magnetic field is the same in magnitude but opposite in sense.

Problem 54a. Obtain (54–3) and (54–4) from (54–1) by means of the transformations (47–5) and (47–6) for $\mathbf{E}$ and $\mathbf{H}$.

Problem 54b. Show that when the charge e (Fig. 50) is at rest in any inertial system S', the image charge $-e$ is also at rest.

Problem 54c. Find the scalar and vector potentials for the field of a point charge moving with constant velocity and obtain (54–3) and (54–4) by differentiation.

$$\textit{Ans.}\quad \Phi = \frac{e}{4\pi r\sqrt{1 - \dfrac{V^2}{c^2}\sin^2\theta}},\quad \mathbf{A} = \frac{e\mathbf{V}}{4\pi rc\sqrt{1 - \dfrac{V^2}{c^2}\sin^2\theta}}.$$

55. Lagrange's Expansion. — We have found simultaneous expressions for $\mathbf{E}$ and $\mathbf{H}$ in closed form for three special types of

motion of an element of charge. In general, however, we must have recourse to an infinite series. The type of series suitable for the purpose of expressing a retarded quantity in terms of the simultaneous values of the quantity and its derivatives was first developed by Lagrange.

In accord with equation (49–1) the time $[t]$ at which the effective position of an element of charge is occupied for calculation of the field at time t is $[t] = t - [r]/c$, where $[r]$ itself is a function of $[t]$ which we can indicate by writing $[r] = f([t])$. Our problem, then, is to express an arbitrary analytic function $u([t])$ as a series in $u(t)$ and its derivatives with respect to t, when $[t]$ and t are connected by a relation of the form $[t] = t - f([t])/c$. We shall formulate the problem in a slightly more general manner, as follows. Let $[t]$ be a function of the independent variables t and α, where

$$[t] = t + \alpha f([t]). \tag{55–1}$$

We wish to expand a function $u([t])$ as a series in $u(t)$ and its derivatives with respect to t.

Since $u([t])$ is a function of the independent variables t and α through $[t]$, we can accomplish our objective at once by expanding $u([t])$ as a Maclaurin series in powers of α, that is,

$$u([t]) = \{u([t])\}_{\alpha=0} + \frac{\alpha}{1!}\left\{\frac{\partial}{\partial\alpha}u([t])\right\}_{\alpha=0} + \frac{\alpha^2}{2!}\left\{\frac{\partial^2}{\partial\alpha^2}u([t])\right\}_{\alpha=0} + \cdots + \frac{\alpha^n}{n!}\left\{\frac{\partial^n}{\partial\alpha^n}u([t])\right\}_{\alpha=0} + \cdots, \tag{55–2}$$

for $[t] = t$ when $\alpha = 0$ according to (55–1). As $\{u([t])\}_{\alpha=0} = u(t)$, all that remains is to express the derivatives of u with respect to α in terms of derivatives with respect to t.

Differentiating (55–1) first partially with respect to t and then partially with respect to α we obtain

$$\frac{\partial[t]}{\partial t}\left\{1 - \alpha\frac{d}{d[t]}f([t])\right\} = 1, \quad \frac{\partial[t]}{\partial\alpha}\left\{1 - \alpha\frac{d}{d[t]}f([t])\right\} = f([t]).$$

Consequently

$$\frac{\partial[t]}{\partial\alpha} = f([t])\frac{\partial[t]}{\partial t},$$

and

$$\frac{\partial}{\partial\alpha}u([t]) = \frac{d}{d[t]}u([t])\frac{\partial[t]}{\partial\alpha} = f([t])\frac{d}{d[t]}u([t])\frac{\partial[t]}{\partial t} = f([t])\frac{\partial}{\partial t}u([t]), \quad (55\text{–}3)$$

which gives for the first derivative

$$\left\{\frac{\partial}{\partial\alpha}u([t])\right\}_{\alpha=0} = f(t)\frac{d}{dt}u(t). \quad (55\text{–}4)$$

If we differentiate (55–3), remembering that this equation holds when $f([t])$ replaces $u([t])$,

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}u([t]) &= \frac{\partial}{\partial\alpha}f([t])\frac{\partial}{\partial t}u([t]) + f([t])\frac{\partial^2}{\partial t\partial\alpha}u([t]) \\ &= f([t])\frac{\partial}{\partial t}f([t])\frac{\partial}{\partial t}u([t]) + f([t])\frac{\partial}{\partial t}\left\{f([t])\frac{\partial}{\partial t}u([t])\right\} \\ &= \frac{\partial}{\partial t}\left\{\overline{f([t])}^2\frac{\partial}{\partial t}u([t])\right\}, \quad (55\text{–}5)\end{aligned}$$

which gives for the second derivative

$$\left\{\frac{\partial^2}{\partial\alpha^2}u([t])\right\}_{\alpha=0} = \frac{d}{dt}\left\{\overline{f(t)}^2\frac{d}{dt}u(t)\right\}. \quad (55\text{–}6)$$

The remaining derivatives can be obtained by mathematical induction. We assume that

$$\frac{\partial^n}{\partial\alpha^n}u([t]) = \frac{\partial^{n-1}}{\partial t^{n-1}}\left\{\overline{f([t])}^n\frac{\partial}{\partial t}u([t])\right\} \quad (55\text{–}7)$$

and differentiate with respect to α, getting

$$\begin{aligned}\frac{\partial^{n+1}}{\partial\alpha^{n+1}}u([t]) &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left\{\frac{\partial}{\partial\alpha}\overline{f([t])}^n\frac{\partial}{\partial t}u([t]) + \overline{f([t])}^n\frac{\partial^2}{\partial t\partial\alpha}u([t])\right\} \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left\{f([t])\frac{\partial}{\partial t}\overline{f([t])}^n\frac{\partial}{\partial t}u([t]) + \overline{f([t])}^n\frac{\partial}{\partial t}\left\{f([t])\frac{\partial}{\partial t}u([t])\right\}\right\} \\ &= \frac{\partial^n}{\partial t^n}\left\{\overline{f([t])}^{n+1}\frac{\partial}{\partial t}u([t])\right\}.\end{aligned}$$

Hence, if the formula (55–7) holds for a positive integer n, it holds for $n + 1$. But we have proved that it holds for $n = 2$. Therefore it is valid for $n = 3, 4, 5, \cdots$.

From (55–7) we have

$$\left\{\frac{\partial^n}{\partial\alpha^n}u([t])\right\}_{\alpha=0} = \frac{d^{n-1}}{dt^{n-1}}\left\{\overline{f(t)}^n\frac{d}{dt}u(t)\right\}. \qquad (55\text{–}8)$$

Finally, then, Lagrange's expansion (55–2) for $u([t])$, where $[t]$ is related to t by (55–1), is

$$u([t]) = u(t) + \frac{\alpha}{1!}f(t)\frac{d}{dt}u(t) + \frac{\alpha^2}{2!}\frac{d}{dt}\left\{\overline{f(t)}^2\frac{d}{dt}u(t)\right\} + \cdots$$
$$+ \frac{\alpha^n}{n!}\frac{d^{n-1}}{dt^{n-1}}\left\{\overline{f(t)}^n\frac{d}{dt}u(t)\right\} + \cdots. \qquad (55\text{–}9)$$

In obtaining simultaneous expansions of retarded quantities we are interested in the case where (55–1) takes the form

$$[t] = t - \frac{[r]}{c}.$$

Hence $\alpha = -1/c, f([t]) = [r]$, and $f(t) = r$. So, if we put $[u] \equiv u([t])$ and $u \equiv u(t)$ in accord with our usual notation, (55–9) becomes

$$[u] = u - \frac{1}{1!c}r\frac{du}{dt} + \frac{1}{2!c^2}\frac{d}{dt}\left(r^2\frac{du}{dt}\right) + \cdots$$
$$+ \frac{(-1)^n}{n!c^n}\frac{d^{n-1}}{dt^{n-1}}\left(r^n\frac{du}{dt}\right) + \cdots, \qquad (55\text{–}10)$$

which, as it is a power series in $1/c$, converges very rapidly when r and its derivatives are small.

56. Simultaneous Expansions of Potentials and Field Intensities. — We are ready now to use Lagrange's series (55–10) to find simultaneous expressions for the scalar potential Φ and the vector potential **A** in the elementary field of an element of charge. Then by differentiation in accord with (50–5) and (50–6) we can obtain simultaneous expressions for **E** and **H**.

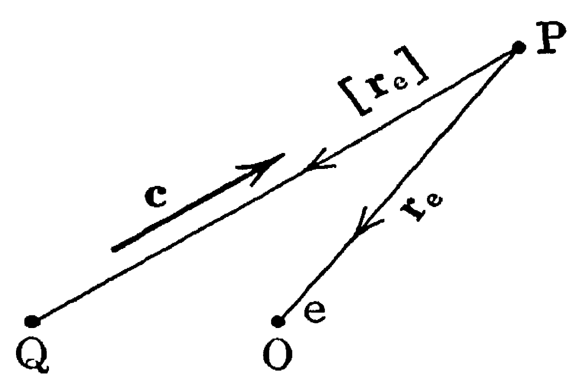

FIG. 51.

Let $[\mathbf{r}_e]$ (Fig. 51) represent the vector from the field-point P at which we wish to evaluate the potentials at time t to the effective position Q ($[x]$, $[y]$, $[z]$) of the charge, and $\mathbf{r}_e$ the

vector from P to the simultaneous position O (x, y, z) of the charge. As here defined $[\mathbf{r}_e]$ and $\mathbf{r}_e$ have the same magnitudes as, but opposite senses to, the vectors previously designated by $[\mathbf{r}]$ and $\mathbf{r}$, respectively, and therefore $[\mathbf{r}_e]$ is in the opposite sense to $\mathbf{c}$. Consequently $[\mathbf{c}\cdot\mathbf{V}/c^2] = -[\dot{r}_e/c]$ and $[\mathbf{V}] = [\dot{\mathbf{r}}_e]$, the dot over a letter representing the derivative with respect to the time. So (50–7) and (50–8) become

$$\Phi = \left[\frac{e}{4\pi r_e\left(1 + \dfrac{\dot{r}_e}{c}\right)}\right], \tag{56–1}$$

$$\mathbf{A} = \left[\frac{e\dot{\mathbf{r}}_e}{4\pi c r_e\left(1 + \dfrac{\dot{r}_e}{c}\right)}\right]. \tag{56–2}$$

Let w stand for any one of the quantities $1/r_e$, $\dot{x}/r_e$, $\dot{y}/r_e$, $\dot{z}/r_e$. Then, using the binomial theorem first, and afterwards applying Lagrange's expansion to the whole series obtained,

$$\left[\frac{w}{1 + \dfrac{\dot{r}_e}{c}}\right] = \left[w - w\frac{\dot{r}_e}{c} + w\left(\frac{\dot{r}_e}{c}\right)^2 + \cdots + (-1)^n w\left(\frac{\dot{r}_e}{c}\right)^n + \cdots\right]$$

$$= w - w\frac{\dot{r}_e}{c} + w\left(\frac{\dot{r}_e}{c}\right)^2 + \cdots + (-1)^n w\left(\frac{\dot{r}_e}{c}\right)^n + \cdots$$

$$-\frac{1}{1!}\frac{r_e}{c}\dot{w} + \frac{1}{1!}\frac{r_e}{c}\frac{d}{dt}\left\{w\frac{\dot{r}_e}{c}\right\} + \cdots$$

$$-\frac{(-1)^{n-1}}{1!}\frac{r_e}{c}\frac{d}{dt}\left\{w\left(\frac{\dot{r}_e}{c}\right)^{n-1}\right\} + \cdots$$

$$+\frac{1}{2!}\frac{d}{dt}\left\{\left(\frac{r_e}{c}\right)^2\dot{w}\right\} + \cdots$$

$$+\frac{(-1)^{n-2}}{2!}\frac{d}{dt}\left\{\left(\frac{r_e}{c}\right)^2\frac{d}{dt}\left\{w\left(\frac{\dot{r}_e}{c}\right)^{n-2}\right\}\right\} + \cdots$$

and so on.

Adding by columns we get

$$\left[\frac{w}{1+\frac{\dot{r}_e}{c}}\right] = w - \frac{1}{1!}\frac{d}{dt}\left\{w\frac{r_e}{c}\right\} + \frac{1}{2!}\frac{d^2}{dt^2}\left\{w\left(\frac{r_e}{c}\right)^2\right\} + \cdots$$
$$+ \frac{(-1)^n}{n!}\frac{d^n}{dt^n}\left\{w\left(\frac{r_e}{c}\right)^n\right\} + \cdots. \tag{56-3}$$

The combination of the terms in the first three columns is obvious; to prove the correctness of the nth derivative we note that

$$w\left(\frac{\dot{r}_e}{c}\right)^n + \frac{1}{1!}\frac{r_e}{c}\frac{d}{dt}\left\{w\left(\frac{\dot{r}_e}{c}\right)^{n-1}\right\} + \frac{1}{2!}\frac{d}{dt}\left\{\left(\frac{r_e}{c}\right)^2\frac{d}{dt}\left\{w\left(\frac{\dot{r}_e}{c}\right)^{n-2}\right\}\right\} + \cdots$$
$$= \frac{d}{dt}\left\{\frac{r_e}{c}w\left(\frac{\dot{r}_e}{c}\right)^{n-1}\right\} + \frac{1}{2!}\frac{d}{dt}\left\{\left(\frac{r_e}{c}\right)^2\frac{d}{dt}\left\{w\left(\frac{\dot{r}_e}{c}\right)^{n-2}\right\}\right\} + \cdots$$
$$= \frac{1}{2!}\frac{d^2}{dt^2}\left\{\left(\frac{r_e}{c}\right)^2 w\left(\frac{\dot{r}_e}{c}\right)^{n-2}\right\} + \cdots$$
$$= \frac{1}{n!}\frac{d^n}{dt^n}\left\{\left(\frac{r_e}{c}\right)^n w\right\}.$$

Putting $1/r_e$ for w in (56–3) we get for the scalar potential the simultaneous series

$$\Phi = \frac{e}{4\pi}\left\{\frac{1}{r_e} - 0 + \frac{1}{2!}\frac{d^2}{dt^2}\left\{\left(\frac{r_e}{c}\right)^2\frac{1}{r_e}\right\} + \cdots\right.$$
$$\left. + \frac{(-1)^n}{n!}\frac{d^n}{dt^n}\left\{\left(\frac{r_e}{c}\right)^n\frac{1}{r_e}\right\} + \cdots\right\}, \tag{56-4}$$

and putting $\dot{x}/r_e$, $\dot{y}/r_e$, $\dot{z}/r_e$ in turn for w to get the three components of the vector potential, we have, on recombination into a single vector equation,

$$\mathbf{A} = \frac{e}{4\pi c}\left\{\frac{\mathbf{V}}{r_e} - \frac{1}{1!}\frac{d}{dt}\left\{\left(\frac{r_e}{c}\right)\frac{\mathbf{V}}{r_e}\right\} + \frac{1}{2!}\frac{d^2}{dt^2}\left\{\left(\frac{r_e}{c}\right)^2\frac{\mathbf{V}}{r_e}\right\} + \cdots\right.$$
$$\left. + \frac{(-1)^n}{n!}\frac{d^n}{dt^n}\left\{\left(\frac{r_e}{c}\right)^n\frac{\mathbf{V}}{r_e}\right\} + \cdots\right\}. \tag{56-5}$$

In differentiating (56–4) and (56–5) to find **E** and **H** we must be careful to distinguish between the coordinates x_P, y_P, z_P of the point P

at which we wish to evaluate the field, and the coordinates x, y, z of the charge. Both sets of coordinates appear in the simultaneous expressions (56–4) and (56–5) for the scalar and vector potentials only through r_e, which represents the function

$$r_e = \sqrt{(x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2},$$

the coordinates x, y, z being functions of the time t. So the differential operator ∇ appearing in (50–5) and (50–6) is

$$\nabla = i\frac{\partial}{\partial x_P} + j\frac{\partial}{\partial y_P} + k\frac{\partial}{\partial z_P} \tag{56–6}$$

in our present notation, and the differential operator with respect to the time, which is identical with that appearing in (56–4) and (56–5), acts on x, y, z, V_x, V_y, V_z.

We shall calculate only the first four terms in the simultaneous series for **E** and the first two in that for **H**. We find

$$\frac{\partial \Phi}{\partial x_P} = \frac{e}{4\pi}\left\{\frac{x - x_P}{r_e^3} - 0 - \frac{1}{2c^2}\frac{d^2}{dt^2}\left(\frac{x - x_P}{r_e}\right)\right.$$
$$\left. + \frac{1}{3c^3}\frac{d^3}{dt^3}(x - x_P) + \cdots\right\},$$

$$\frac{\partial A_x}{\partial t} = \frac{e}{4\pi c}\left\{\frac{d}{dt}\left(\frac{\dot{x}}{r_e}\right) - \frac{1}{c}\frac{d^2}{dt^2}(\dot{x}) + \cdots\right\},$$

$$\frac{\partial A_z}{\partial y_P} = \frac{e}{4\pi c}\left\{\frac{y - y_P}{r_e^3}\dot{z} - 0 + \cdots\right\},$$

$$\frac{\partial A_y}{\partial z_P} = \frac{e}{4\pi c}\left\{\frac{z - z_P}{r_e^3}\dot{y} - 0 + \cdots\right\}.$$

Since $\dot{x} = V_x$, $\dot{r}_e = \dfrac{\mathbf{V}\cdot\mathbf{r}_e}{r_e}$, $\ddot{x} = f_x$, $\ddot{r}_e = \dfrac{\mathbf{f}\cdot\mathbf{r}_e}{r_e} + \dfrac{V^2}{r_e} - \dfrac{\overline{\mathbf{V}\cdot\mathbf{r}_e}^2}{r_e^3}$, $\dddot{x} = \dot{f}_x$, these give

$$E_x = -\frac{\partial \Phi}{\partial x_P} - \frac{1}{c}\frac{\partial A_x}{\partial t}$$

$$= -\frac{e}{4\pi}\frac{x - x_P}{r_e^3} + 0$$
$$- \frac{e}{8\pi c^2}\left\{\frac{f_x}{r_e} + \left(\frac{\mathbf{f}\cdot\mathbf{r}_e}{r_e^2} + \frac{V^2}{r_e^2} - 3\frac{\overline{\mathbf{V}\cdot\mathbf{r}_e}^2}{r_e^4}\right)\frac{x - x_P}{r_e}\right\} + \frac{e}{6\pi c^3}\dot{f}_x + \cdots,$$

$$H_x = \frac{\partial A_z}{\partial y_P} - \frac{\partial A_y}{\partial z_P}$$

$$= \frac{e}{4\pi c}\left\{\frac{y - y_P}{r_e^3} V_z - \frac{z - z_P}{r_e^3} V_y\right\} - 0 + \cdots.$$

Combining these with similar expressions for the other components of **E** and **H** we have the simultaneous vector expressions:

$$\mathbf{E} = -\frac{e}{4\pi r_e^2}\frac{\mathbf{r}_e}{r_e} + 0$$

$$-\frac{e}{8\pi c^2}\left\{\frac{\mathbf{f}}{r_e} + \left(\frac{\mathbf{f}\cdot\mathbf{r}_e}{r_e^3} + \frac{V^2}{r_e^3} - 3\frac{\overline{\mathbf{V}\cdot\mathbf{r}_e}^2}{r_e^5}\right)\mathbf{r}_e\right\} + \frac{e}{6\pi c^3}\dot{\mathbf{f}} + \cdots, \quad (56\text{–}7)$$

$$\mathbf{H} = -\frac{e}{4\pi r_e^2}\frac{\mathbf{V}\times\mathbf{r}_e}{c r_e} + 0 + \cdots, \quad (56\text{–}8)$$

where $\mathbf{r}_e$ is the position vector of the elementary charge e relative to the field-point P at the time t at which the formulas give the electric and magnetic fields at P. If we distinguish terms in the series by designating the term in $1/c^n$ as the nth order term, the series for **E** has been carried through third order terms and that for **H** through second order terms.

Generally it is more convenient to write the expressions (56–7) and (56–8) for **E** and **H** in terms of the position vector **r** of the field-point P relative to the charge e. This necessitates the substitution of $-\mathbf{r}$ for $\mathbf{r}_e$, giving

$$\mathbf{E} = \frac{e}{4\pi r^2}\frac{\mathbf{r}}{r} + 0$$

$$-\frac{e}{8\pi c^2}\left\{\frac{\mathbf{f}}{r} + \left(\frac{\mathbf{f}\cdot\mathbf{r}}{r^3} - \frac{V^2}{r^3} + 3\frac{\overline{\mathbf{V}\cdot\mathbf{r}}^2}{r^5}\right)\mathbf{r}\right\} + \frac{e}{6\pi c^3}\dot{\mathbf{f}} + \cdots, \quad (56\text{–}9)$$

$$\mathbf{H} = \frac{e}{4\pi r^2}\frac{\mathbf{V}\times\mathbf{r}}{cr} + 0 + \cdots. \quad (56\text{–}10)$$

If the charge is momentarily at rest in the observer's inertial system, $\mathbf{V} = 0$, and these equations can be written in the form

$$\mathbf{E} = -\frac{1}{4\pi}\nabla\left(\frac{e}{r}\right) + 0 - \frac{\mathbf{f}}{4\pi c^2}\frac{e}{r} + \frac{1}{8\pi c^2}\nabla\left(\frac{e\mathbf{f}\cdot\mathbf{r}}{r}\right) + \frac{e\dot{\mathbf{f}}}{6\pi c^3} + \cdots, \quad (56\text{–}11)$$

$$\mathbf{H} = 0 + 0 + \cdots, \tag{56-12}$$

since

$$\nabla\left(\frac{\mathbf{f}\cdot\mathbf{r}}{r}\right) = \frac{\mathbf{f}}{r} - \frac{\mathbf{f}\cdot\mathbf{rr}}{r^3},$$

where ∇ is given by (56–6).

57. The Lorentz Electron and the Force Equation. — Although we shall discuss specifically the electron in this and the next two articles, our analysis is applicable to any elementary charged particle, whether positive or negative. A vast amount of experimental evidence has been accumulated to show that all elementary charged particles carry charges of the same magnitude, namely $4.80(10)^{-10}$ electrostatic units, but no precise experimental evidence exists as to the volume occupied by this charge or as to its geometrical distribution. Symmetry suggests, however, that, relative to the inertial system in which the electron is momentarily at rest, the charge is distributed through the volume of a sphere in such a manner that the density of charge is a function only of the distance from the center, and, in order to simplify calculations, it is convenient to suppose that the charge is confined to the surface of the sphere. The latter assumption does not limit the generality of the *qualitative* conclusions to be drawn from our analysis, for it merely fixes the coefficients of the expressions to be deduced within a range extending some twenty-five per cent to the one side or to the other. We define the *Lorentz electron*, then, relative to the inertial system in which its center is momentarily at rest, as a uniformly charged spherical shell of radius a and charge e, every point of which is simultaneously at rest. On account of the Fitzgerald-Lorentz contraction, the shape of the electron relative to any other inertial system is approximately that of an oblate spheroid with the short axis in the direction of its velocity.

First we shall compute the field of a Lorentz electron relative to the inertial system S in which it is momentarily at rest. As $\mathbf{V} = 0$, (56–11) and (56–12) give for the field intensities

$$\mathbf{E} = -\frac{1}{4\pi}\nabla\int_e \frac{de}{r} - \frac{\mathbf{f}}{4\pi c^2}\int_e \frac{de}{r} + \frac{1}{8\pi c^2}\nabla\int_e \frac{\mathbf{f}\cdot\mathbf{r}\,de}{r} + \frac{e\dot{\mathbf{f}}}{6\pi c^3}, \tag{57-1}$$

$$\mathbf{H} = 0, \tag{57-2}$$

to the degree of approximation to which the series have been carried.

Let **R** (Fig. 52) be the vector distance of the field-point P from the center O of the electron, and **a** the vector distance of de from O. Then

$$\mathbf{r} = \mathbf{R} - \mathbf{a}$$

and, if θ is the angle between **a** and **R**,

$$r^2 = R^2 - 2Ra \cos \theta + a^2. \tag{57-3}$$

Consequently

$$de = \frac{e}{4\pi} \sin \theta \, d\theta \, d\phi = \frac{e}{4\pi a R} r \, dr \, d\phi \tag{57-4}$$

where ϕ is the azimuth measured about **R**.

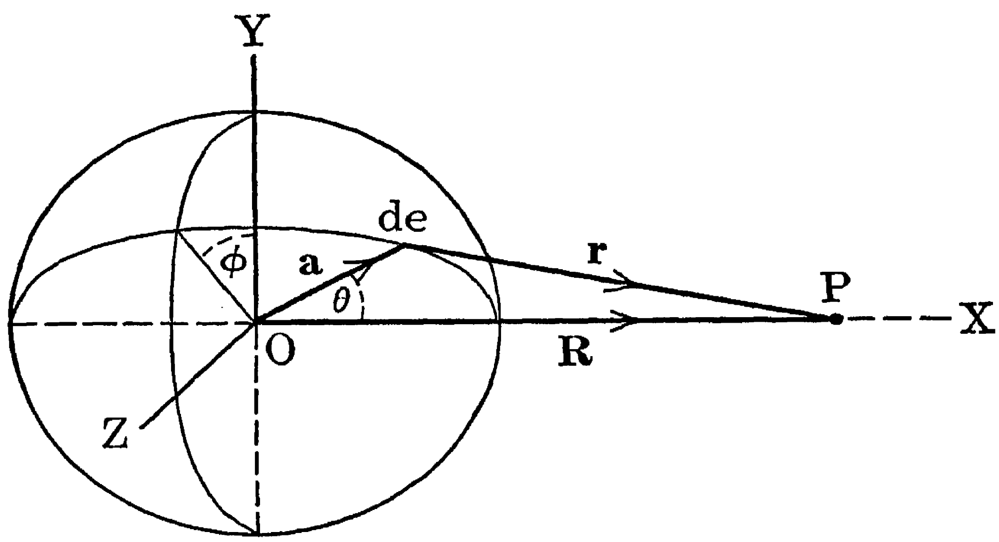

FIG. 52.

Therefore

$$\int_e \frac{de}{r} = \frac{e}{2aR} \left| r \right|_{r_1}^{r_2} = \begin{cases} \dfrac{e}{R} \text{ outside,} \\ \\ \dfrac{e}{a} \text{ inside,} \end{cases}$$

since the limits r_1 and r_2 of r for an outside field-point are $R - a$ and $R + a$, and for an inside field-point $a - R$ and $a + R$.

As $\mathbf{f} \cdot \mathbf{r} = \mathbf{f} \cdot \mathbf{R} - \mathbf{f} \cdot \mathbf{a}$,

$$\int_e \frac{\mathbf{f} \cdot \mathbf{r} \, de}{r} = \mathbf{f} \cdot \mathbf{R} \int_e \frac{de}{r} - \int_e \frac{\mathbf{f} \cdot \mathbf{a} \, de}{r}.$$

Now

$$\mathbf{a} = a(\boldsymbol{i} \cos \theta + \boldsymbol{j} \sin \theta \cos \phi + \boldsymbol{k} \sin \theta \sin \phi), \tag{57-5}$$

$$\mathbf{f} \cdot \mathbf{a} = a(f_x \cos \theta + f_y \sin \theta \cos \phi + f_z \sin \theta \sin \phi). \tag{57-6}$$

Using (57–4) for de, and integrating first with respect to ϕ,

$$\int_e \frac{\mathbf{f}\cdot\mathbf{a}\,de}{r} = \frac{e\mathbf{f}\cdot\mathbf{R}}{2R^2}\int_r \cos\theta\,dr.$$

Next, if we express $\cos\theta$ in terms of r by means of (57–3),

$$\int_r \cos\theta\,dr = \frac{1}{2aR}\int_r (R^2+a^2-r^2)dr = \begin{cases} \dfrac{2a^2}{3R} \text{ outside,} \\[2ex] \dfrac{2R^2}{3a} \text{ inside,} \end{cases} \tag{57–7}$$

which gives

$$\int_e \frac{\mathbf{f}\cdot\mathbf{a}\,de}{r} = \begin{cases} \dfrac{ea^2}{3R^3}\mathbf{f}\cdot\mathbf{R} \text{ outside,} \\[2ex] \dfrac{e}{3a}\mathbf{f}\cdot\mathbf{R} \text{ inside.} \end{cases}$$

Hence, for an outside field-point,

$$\begin{aligned} \mathbf{E} &= -\frac{e}{4\pi}\nabla\left(\frac{1}{R}\right) - \frac{e}{4\pi c^2}\frac{\mathbf{f}}{R} + \frac{e}{8\pi c^2}\nabla\left(\frac{\mathbf{f}\cdot\mathbf{R}}{R} - \frac{a^2\mathbf{f}\cdot\mathbf{R}}{3R^3}\right) + \frac{e\dot{\mathbf{f}}}{6\pi c^3} + \cdots \\ &= \frac{e}{4\pi}\frac{\mathbf{R}}{R^3} - \frac{e}{8\pi c^2}\left\{\left(\frac{1}{R}+\frac{a^2}{3R^3}\right)\mathbf{f} + \left(\frac{1}{R^3}-\frac{a^2}{R^5}\right)\mathbf{f}\cdot\mathbf{RR}\right\} + \frac{e\dot{\mathbf{f}}}{6\pi c^3} + \cdots, \end{aligned} \tag{57–8}$$

whereas for an inside field-point,

$$\begin{aligned} \mathbf{E} &= -\frac{e}{4\pi}\nabla\left(\frac{1}{a}\right) - \frac{e}{4\pi ac^2}\mathbf{f} + \frac{e}{12\pi ac^2}\nabla(\mathbf{f}\cdot\mathbf{R}) + \frac{e\dot{\mathbf{f}}}{6\pi c^3} + \cdots \\ &= -\frac{e\mathbf{f}}{6\pi ac^2} + \frac{e\dot{\mathbf{f}}}{6\pi c^3} + \cdots. \end{aligned} \tag{57–9}$$

Except for the zero order term, the field is continuous at the surface of the electron, as indeed is required by the boundary conditions (52–1) and (52–2), which are satisfied by the zero order term alone. Of course (57–8) represents the field only for small values of R, since the series of which it constitutes the first few terms does not converge at large distances from the electron.

Relative to the inertial system in which it is momentarily at rest, we define the *force* exerted on an element of charge de by a field $\mathbf{E}$ as the product $\mathbf{E}\,de$. If, then, a Lorentz electron is momentarily at rest in system S, the resultant force to which it is subject is $\int_e \mathbf{E}_T de$, where $\mathbf{E}_T$ is the vector sum of the external field $\mathbf{E}_1$ and the field $\mathbf{E}$ due to the electron's own charge.

So far the development of our theory has consisted in formulating definitions consonant with the principle of relativity. A collection of definitions cannot in itself constitute a physical theory. We now make the essential assumption which completes our theoretical structure and makes it into a physical theory capable of correlation with observation. This assumption, which we shall call the *dynamical assumption*, asserts that, relative to the inertial system in which it is momentarily at rest, *the resultant force on an electron always vanishes.* In other words, the sum of the external force and the force exerted on the electron by its own field is equal to zero. The latter force we call the *kinetic reaction* of the electron.

To calculate the force on the electron due to the continuous part of its own field, which is given at the surface of the electron by (57–9), all that is necessary is to multiply by de and integrate over the surface of the particle. As regards the discontinuous part of the field, which is zero inside the surface and equal to the first term in (57–8) outside, we must multiply the arithmetic mean of the field just inside and that just outside the surface by de and integrate over the surface. The reason for this is clear if we think of the charge of the electron as distributed between two concentric shells of radii $a - \frac{1}{2}\Delta a$ and $a + \frac{1}{2}\Delta a$ and recall the first of the two boundary conditions (52–1), which tells us that the normal component of $\mathbf{E}$ increases at the same rate as the charge when we pass through the surface of the electron.

Evidently the discontinuous part of the field leads to no force on the electron as a whole in this case, and we have from (57–9)

$$e\mathbf{E}_1 = \frac{e^2}{6\pi ac^2}\,\mathbf{f} - \frac{e^2}{6\pi c^3}\,\dot{\mathbf{f}} + \cdots, \qquad (57\text{–}10)$$

provided $\mathbf{E}_1$ is sensibly constant over the small volume occupied by the electron.

Equation (57–10) is the *equation of motion* of the electron relative to the inertial system in which it is momentarily at rest, the coefficient

of $\mathbf{f}$ being known as the *rest mass* of the electron. If we put

$$m \equiv \frac{e^2}{6\pi a c^2}, \quad n \equiv \frac{e^2}{6\pi c^3},$$

the equation may be written

$$e\mathbf{E}_1 = m\mathbf{f} - n\dot{\mathbf{f}} + \cdots. \tag{57-11}$$

Generally the second term on the right is so small compared with the first as to be negligible. Then the dynamical assumption leads directly to Newton's second law of motion, which may be considered to be a deduction from electromagnetic theory. As the experimentally measured value of the rest mass of the electron is $9.0(10)^{-28}$ gm, our formula gives $1.9(10)^{-13}$ cm for its radius. Since the radius is inversely proportional to the mass, the radius of a proton, if it is an elementary particle, must be only one eighteen-hundredth as great. As regards order of magnitude, these values of the linear dimensions of elementary particles are in good accord with our knowledge of the structure of atoms, in particular with the information acquired from the study of the deflection of alpha particles by thin foils, which indicates that the radius of the nucleus is of the order of magnitude of $(10)^{-12}$ cm.

If the electron has a velocity $\mathbf{V}$ relative to the observer's inertial system S, it is momentarily at rest in an inertial system S' moving with velocity $\mathbf{v} = \mathbf{V}$ relative to S. Hence

$$e\mathbf{E}_1' = m\mathbf{f}' - n\dot{\mathbf{f}}' + \cdots.$$

Transforming to S by means of (47–2), and (43–7) and (43–8), we get, after putting $\mathbf{V}$ for $\mathbf{v}$, for the three components of the equation of motion,

$$eE_{1x} = \frac{m}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} f_x - \frac{n}{\left(1 - \frac{V^2}{c^2}\right)^2} \dot{f}_x - \frac{3n}{\left(1 - \frac{V^2}{c^2}\right)^3} \frac{\mathbf{f}\cdot\mathbf{V}}{c^2} f_x + \cdots,$$

$$e\left\{E_{1y} - \frac{V}{c} H_{1z}\right\}$$

$$= \frac{m}{\sqrt{1 - \frac{V^2}{c^2}}} f_y - \frac{n}{\left(1 - \frac{V^2}{c^2}\right)} \dot{f}_y - \frac{3n}{\left(1 - \frac{V^2}{c^2}\right)^2} \frac{\mathbf{f}\cdot\mathbf{V}}{c^2} f_y + \cdots,$$

$$e\left\{E_{1z} + \frac{V}{c} H_{1y}\right\}$$

$$= \frac{m}{\sqrt{1 - \frac{V^2}{c^2}}} f_z - \frac{n}{\left(1 - \frac{V^2}{c^2}\right)} \dot{f}_z - \frac{3n}{\left(1 - \frac{V^2}{c^2}\right)^2} \frac{\mathbf{f}\cdot\mathbf{V}}{c^2} f_z + \cdots,$$

where $\mathbf{H}_1$ is the external magnetic intensity. These can be combined into the single vector equation

$$e\left\{\mathbf{E}_1 + \frac{1}{c}\mathbf{V} \times \mathbf{H}_1\right\} = \frac{m}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} \left\{\mathbf{f} + \frac{1}{c^2}(\mathbf{f} \times \mathbf{V}) \times \mathbf{V}\right\}$$

$$- \frac{n}{\left(1 - \frac{V^2}{c^2}\right)^2} \left\{\dot{\mathbf{f}} + \frac{1}{c^2}(\dot{\mathbf{f}} \times \mathbf{V}) \times \mathbf{V}\right\}$$

$$- \frac{3n}{\left(1 - \frac{V^2}{c^2}\right)^3} \frac{\mathbf{f}\cdot\mathbf{V}}{c^2} \left\{\mathbf{f} + \frac{1}{c^2}(\mathbf{f} \times \mathbf{V}) \times \mathbf{V}\right\} + \cdots. \quad (57\text{–}12)$$

The left-hand member of this equation we may name by definition the *external force* on the electron, for it reduces to $e\mathbf{E}_1$ when $\mathbf{V} = 0$. We see then that, when the acceleration is in the direction of the velocity, the coefficient of the acceleration in the first term on the right of (57–12) is

$$m_l \equiv \frac{m}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}},$$

whereas, when the acceleration is at right angles to the velocity, this coefficient is

$$m_t \equiv \frac{m}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

The first is called the *longitudinal mass* and the second the *transverse mass* of the electron. Both are monotonically increasing functions of the velocity, becoming infinite as V approaches c. Hence no finite

electromagnetic field can ever give an electron a velocity as great as that of light.

In cases where terms involving the squares of **V** and **f** are negligible, (57–12) reduces to the simpler form

$$e\left\{\mathbf{E}_1 + \frac{1}{c}\mathbf{V}\times\mathbf{H}_1\right\} = m\mathbf{f} - n\dot{\mathbf{f}} + \cdots, \tag{57–13}$$

whereas in cases where it is possible to neglect terms in the square or time derivative of **f** but necessary to retain those in V^2/c^2,

$$e\left\{\mathbf{E}_1 + \frac{1}{c}\mathbf{V}\times\mathbf{H}_1\right\} = \frac{d}{dt}(m_t\mathbf{V}) \tag{57–14}$$

since

$$\frac{d}{dt}\left(\frac{m\mathbf{V}}{\sqrt{1-\frac{V^2}{c^2}}}\right) = m\left\{\frac{\mathbf{f}}{\sqrt{1-\frac{V^2}{c^2}}} + \frac{\frac{\mathbf{f}\cdot\mathbf{V}\mathbf{V}}{c^2}}{\left(1-\frac{V^2}{c^2}\right)^{3/2}}\right\}$$

$$= \frac{m}{\left(1-\frac{V^2}{c^2}\right)^{3/2}}\left\{\mathbf{f} + \frac{1}{c^2}(\mathbf{f}\times\mathbf{V})\times\mathbf{V}\right\}.$$

By analogy with the Newtonian dynamics we name the vector quantity $m_t\mathbf{V}$ the *linear momentum* of the electron.

So far our attention has been directed to an isolated charged particle. When a number of elementary charged particles are very close together, as in the nucleus of an atom, we can treat them as a group only if we take account of the forces exerted on each by its neighbors as well as the force due to the external fields $\mathbf{E}_1$ and $\mathbf{H}_1$. If we neglect terms in V^2/c^2 it is clear from (47–5) and (56–10) that the electric field of one of the constituent particles is given by (57–8), and that the magnetic field may be ignored since it is proportional to V/c and therefore leads to no force on a neighboring particle in a lower power than the square of this ratio. If we suppose that the particles revolve about one another in such a way that the line joining any two particles assumes all orientations with equal frequency, the time average of the expression $\mathbf{f}\cdot\mathbf{RR}$ appearing in (57–8) is $\frac{1}{3}R^2\mathbf{f}_c$, where $\mathbf{f}_c$ is the acceleration of the group as a whole. Therefore, if r_{ij} is the distance between the ith and the jth particle, the time average

of the force exerted on the ith particle by neighboring particles is

$$-\sum_j \frac{e_i e_j}{6\pi r_{ij} c^2} \mathbf{f}_c + \sum_j \frac{e_i e_j}{6\pi c^3} \dot{\mathbf{f}}_c$$

from (57–8), where $\dot{\mathbf{f}}_c$ like $\mathbf{f}_c$ refers to the group as a whole.

Let the group of particles be in external fields $\mathbf{E}_1$ and $\mathbf{H}_1$ which will be supposed to be sensibly constant over the small region occupied by the group. Then the time average equation of motion of the ith particle is

$$e_i \left\{ \mathbf{E}_1 + \frac{1}{c} \mathbf{V}_c \times \mathbf{H}_1 \right\} - \sum_j \frac{e_i e_j}{6\pi r_{ij} c^2} \mathbf{f}_c + \sum_j \frac{e_i e_j}{6\pi c^3} \dot{\mathbf{f}}_c = m_i \mathbf{f}_c - n_i \dot{\mathbf{f}}_c,$$

since the time averages of $\mathbf{V}_i$, $\mathbf{f}_i$ and $\dot{\mathbf{f}}_i$ are the velocity $\mathbf{V}_c$, acceleration $\mathbf{f}_c$ and rate of change of acceleration $\dot{\mathbf{f}}_c$ of the group as a whole. Summing up over all the particles, and putting $e \equiv \sum_i e_i$ for the net charge of the group,

$$e\left\{ \mathbf{E}_1 + \frac{1}{c} \mathbf{V}_c \times \mathbf{H}_1 \right\} = \left(\sum_i \frac{e_i^2}{6\pi a_i c^2} + \sum_{ij} \frac{e_i e_j}{6\pi r_{ij} c^2} \right) \mathbf{f}_c - \frac{e^2}{6\pi c^3} \dot{\mathbf{f}}_c, \quad (57\text{–}15)$$

where the double sum is to be taken over all pairs for which $i \neq j$.

It appears from (57–15) that the effective mass of the group exceeds the sum of the masses of the individual particles, when isolated, by the *mutual mass*

$$m_m \equiv \sum_{ij} \frac{e_i e_j}{6\pi r_{ij} c^2}. \qquad (57\text{–}16)$$

Charges e_i and e_j of the same sign lead to positive terms in this sum, whereas charges of opposite sign lead to negative terms. Now the zero order internal forces, specified by the first term in (57–8), tend to bring unlike charges as close together as possible and to separate like charges as far as possible. Hence, if equal numbers of charged particles of each sign are present in the group, we would expect the sum of the negative terms in (57–16) to exceed in absolute magnitude the sum of the positive terms, on account of the factor $1/r_{ij}$ in each term. Thus electromagnetic theory is able to account, at least qualitatively, for the mass defects of atomic nuclei of complex structure. The last term on the right of (57–15) depends only on the total

charge, and vanishes if the number of positive particles is equal to the number of negative particles.

From (57–12) we find for the work done by the external field on an isolated electron in the time $t - t_0$

$$\int_{t_0}^{t} e\mathbf{E}_1 \cdot \mathbf{V} dt$$

$$= m\int_{t_0}^{t} \frac{\mathbf{f}\cdot\mathbf{V}\, dt}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} - n\int_{t_0}^{t} \frac{\dot{\mathbf{f}}\cdot\mathbf{V}\, dt}{\left(1 - \frac{V^2}{c^2}\right)^{2}} - 3n\int_{t_0}^{t} \frac{\frac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}\, dt}{\left(1 - \frac{V^2}{c^2}\right)^{3}} + \cdots$$

$$= \frac{mc^2}{\sqrt{1 - \frac{V^2}{c^2}}} - \frac{mc^2}{\sqrt{1 - \frac{V_0^2}{c^2}}} - \frac{n\mathbf{f}\cdot\mathbf{V}}{\left(1 - \frac{V^2}{c^2}\right)^2} + \frac{n\mathbf{f}_0\cdot\mathbf{V}_0}{\left(1 - \frac{V_0^2}{c^2}\right)^2}$$

$$+ \frac{e^2}{6\pi c^3}\int_{t_0}^{t} \frac{f^2\left(1 - \frac{V^2}{c^2}\right) + \frac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}}{\left(1 - \frac{V^2}{c^2}\right)^3} dt + \cdots \qquad (57\text{–}17)$$

if we integrate the term in $\dot{\mathbf{f}}$ by parts, where the subscript 0 refers to the time t_0. The expression

$$T_V \equiv \frac{mc^2}{\sqrt{1 - \frac{V^2}{c^2}}} = m_t c^2, \qquad (57\text{–}18)$$

already obtained for linear motion in (38–13), is known as the *kinetic energy* of the electron. As here defined the kinetic energy does not vanish when $V = 0$, but this is a matter of no consequence as experiment measures only the change in energy between two states, and little if any significance can be attached to absolute energy values. The term

$$T_f \equiv -\frac{n\mathbf{f}\cdot\mathbf{V}}{\left(1 - \frac{V^2}{c^2}\right)^2} \qquad (57\text{–}19)$$

has been called the *acceleration energy*. Both the kinetic and the acceleration energies represent a reversible storage of energy, since

they can be recovered in the form of work by bringing the electron back to its original state. The last term in (57–17), however, represents an irreversible dissipation of energy, which occurs at the rate

$$\frac{e^2}{6\pi c^3}\frac{f^2\left(1-\dfrac{V^2}{c^2}\right)+\dfrac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}}{\left(1-\dfrac{V^2}{c^2}\right)^3} \tag{57–20}$$

per unit time. As we shall see later this energy is radiated away from the electron in the form of electromagnetic waves. In terms of the acceleration $\mathbf{f}'$ of the electron relative to the inertial system S' in which it is momentarily at rest, the rate of dissipation of energy relative to S takes the simpler form

$$\frac{e^2 f'^2}{6\pi c^3}, \tag{57–21}$$

as is evident at once from (43–6). This shows that the quantity under consideration is an invariant of the Lorentz transformation. As the dissipative term in (57–17) comes from the term in $\dot{\mathbf{f}}$ in the equation of motion (57–10), the latter term (with changed sign) is called the *radiation reaction* in contrast to the term in $\mathbf{f}$ which represents (with changed sign) the *mass reaction*. These are only the first two terms in the series for the kinetic reaction.

From (57–12) we are able to formulate the usual *force equation* of electromagnetic theory. We have shown that $e\{\mathbf{E}_1+(1/c)\mathbf{V}\times\mathbf{H}_1\}$ represents the force on a charge e moving with velocity $\mathbf{V}$ due to the fields $\mathbf{E}_1$ and $\mathbf{H}_1$. Hence the force per unit volume on charge of density ρ per unit volume, moving with velocity $\mathbf{V}$ relative to the observer's inertial system, due to fields $\mathbf{E}$ and $\mathbf{H}$ is

$$\mathscr{F}=\rho\left(\mathbf{E}+\frac{1}{c}\mathbf{V}\times\mathbf{H}\right). \tag{57–22}$$

If $\mathbf{E}$ and $\mathbf{H}$ represent the resultant field intensities, every elementary charge in the moving substance moves in such a way that the integrated force acting on it vanishes.

Combining this equation with the field equations deduced in arti-

cle 51 we have the complete set of electromagnetic equations for charges in empty space. They are

$$\left.\begin{aligned} \nabla\cdot\mathbf{E} &= \rho, \quad (a) & \nabla\cdot\mathbf{H} &= 0, \quad (b) \\ \nabla\times\mathbf{E} &= -\frac{1}{c}\dot{\mathbf{H}}, \quad (c) & \nabla\times\mathbf{H} &= \frac{1}{c}(\dot{\mathbf{E}} + \rho\mathbf{V}), \quad (d) \\ \mathscr{F} &= \rho\left(\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{H}\right). & & \quad (e) \end{aligned}\right\} \quad (57\text{–}23)$$

It should be noted, however, that the force equation rests on a much less secure theoretical foundation than the field equations.

58. The Spinning Electron at Rest. — Modern spectroscopic theories demand an electron spinning around a diameter with a half quantum of angular momentum in order to explain the fine structure of spectral lines. Also there is some evidence that the electron possesses an electric moment as well as a magnetic moment. Therefore we shall modify the model of the electron investigated in the last article so as to provide it with both a simple magnetic moment and a simple electric moment as well as a net charge e. Specifically we shall assume the electron to be, relative to the inertial system in which its center is momentarily at rest, a charged spherical shell of radius a with the uniformly distributed surface charge $de_1 = ed\Omega/4\pi$ in the solid angle $d\Omega$, and the non-uniformly distributed surface charge $de_2 = 3\mathbf{p}_E\cdot\mathbf{a}d\Omega/4\pi a^2$, where $\mathbf{a}$ is a radius vector from the center of the electron to the surface element under consideration and $\mathbf{p}_E$ is a vector independent of the coordinates. The uniformly distributed charge de_1 we shall suppose to be spinning about a diameter with angular velocity $\boldsymbol{\omega}$, supplying the electron with angular momentum and a magnetic moment. On the other hand, we shall suppose that the non-uniformly distributed charge de_2, which is responsible for the electric moment, does not partake of the spin. Since the integral of de_2 over the entire surface of the electron vanishes, the non-uniformly distributed charge contributes nothing to the net charge e.

In this article we shall investigate the electric and magnetic fields of an electron which is permanently at rest in the observer's inertial system and for which the vectors $\boldsymbol{\omega}$ and $\mathbf{p}_E$ are constants in the time, and shall calculate the forces exerted on the electron by external electric and magnetic fields. In the next article we shall compute the reaction exerted on the electron by its own field when it is accelerated and when $\boldsymbol{\omega}$ and $\mathbf{p}_E$ vary with the time.

As we are concerned here with a steady state of motion we can use the simultaneous expressions (50–11) and (50–12) for the scalar and vector potentials. In terms of the notation used in Fig. 52, the linear velocity of the uniform charge at any point on the surface of the electron is $\boldsymbol{\omega} \times \mathbf{a}$, and the two potentials become

$$\Phi = \frac{1}{4\pi}\int_{e_1}\frac{de_1}{r} + \frac{1}{4\pi}\int_{e_2}\frac{de_2}{r}, \tag{58–1}$$

$$\mathbf{A} = \frac{1}{4\pi c}\int_{e_1}\frac{\boldsymbol{\omega}\times\mathbf{a}\,de_1}{r}. \tag{58–2}$$

The first integral in (58–1) we have evaluated in the last article. Since

$$d\Omega = \sin\theta\,d\theta\,d\phi = \frac{1}{aR}r\,dr\,d\phi \tag{58–3}$$

from (57–4), we have, with the aid of (57–5),

$$de_2 = \frac{3}{4\pi a^2 R}(p_{Ex}\cos\theta + p_{Ey}\sin\theta\cos\phi + p_{Ez}\sin\theta\sin\phi)r\,dr\,d\phi \tag{58–4}$$

and, integrating first with respect to ϕ,

$$\int_{e_2}\frac{de_2}{r} = \frac{3\mathbf{p}_E\cdot\mathbf{R}}{2a^2R^2}\int_r \cos\theta\,dr = \begin{cases}\dfrac{\mathbf{p}_E\cdot\mathbf{R}}{R^3}\text{ outside,}\\[2ex] \dfrac{\mathbf{p}_E\cdot\mathbf{R}}{a^3}\text{ inside,}\end{cases}$$

from (57–7). Hence

$$\Phi = \begin{cases}\dfrac{e}{4\pi R} + \dfrac{\mathbf{p}_E\cdot\mathbf{R}}{4\pi R^3}\text{ outside,}\\[2ex] \dfrac{e}{4\pi a} + \dfrac{\mathbf{p}_E\cdot\mathbf{R}}{4\pi a^3}\text{ inside.}\end{cases} \tag{58–5}$$

The *electric moment* of the charge $\rho d\tau$ in an element of volume $d\tau$ relative to an origin O is defined as $\mathbf{r}\rho d\tau$, where $\mathbf{r}$ is the position vector of the volume element relative to O. Evidently the electric moment of a dipole consisting of two equal charges of opposite sign q and $-q$, of which the first is at a vector distance $\mathbf{l}$ from the second, is equal to $q\mathbf{l}$, a quantity which is independent of the position of the origin O.

If α is the angle which the radius vector $\mathbf{a}$ from the center of the electron makes with $\mathbf{p}_E$ and γ is the azimuth, the non-uniform charge

becomes $de_2 = 3p_E \cos\alpha \sin\alpha \, d\alpha \, d\gamma/4\pi a$ and the electric moment of the electron is

$$\frac{3\mathbf{p}_E}{4\pi}\int_0^{2\pi}\int^{\pi}\cos^2\alpha \sin\alpha \, d\alpha \, d\gamma = \mathbf{p}_E. \qquad (58\text{–}6)$$

To evaluate the vector potential (58–2) we need

$$\boldsymbol{\omega} \times \mathbf{a} = a\{\boldsymbol{i}(\omega_y \sin\theta \sin\phi - \omega_z \sin\theta\cos\phi)$$
$$+ \boldsymbol{j}(\omega_z \cos\theta - \omega_x \sin\theta \sin\phi) + \boldsymbol{k}(\omega_x \sin\theta\cos\phi - \omega_y\cos\theta)\} \qquad (58\text{–}7)$$

from (57–5), and de_1 as expressed in (57–4). Then, if we integrate first with respect to ϕ and use (57–7),

$$\int_{e_1}\frac{\boldsymbol{\omega}\times\mathbf{a}\, de_1}{r} = \frac{e}{2R}(\boldsymbol{j}\omega_z - \boldsymbol{k}\omega_y)\int_r \cos\theta\, dr = \begin{cases} \dfrac{ea^2}{3R^3}\,\boldsymbol{\omega}\times\mathbf{R} \text{ outside,} \\[2ex] \dfrac{e}{3a}\,\boldsymbol{\omega}\times\mathbf{R} \text{ inside,}\end{cases}$$

giving

$$\mathbf{A} = \begin{cases} \dfrac{ea^2}{12\pi c R^3}\,\boldsymbol{\omega}\times\mathbf{R} \text{ outside,} \\[2ex] \dfrac{e}{12\pi a c}\,\boldsymbol{\omega}\times\mathbf{R} \text{ inside.}\end{cases} \qquad (58\text{–}8)$$

The *magnetic moment* of the current $\rho\mathbf{V}d\tau$ in an element of volume $d\tau$ relative to an origin O is defined as $(1/2c)\mathbf{r}\times\rho\mathbf{V}d\tau$, where $\mathbf{r}$ is the position vector of the volume element relative to O. Thus the magnetic moment of an element $d\boldsymbol{\lambda}$ of a linear circuit carrying a current i is $(i/2c)\mathbf{r}\times d\boldsymbol{\lambda}$, and the magnetic moment of the entire circuit is equal to the product of i/c by the vector area of the circuit, a quantity which is independent of the position of the origin O.

If α is the angle which a radius vector from the center of the spinning electron makes with the axis of rotation, the charge per unit length of the annular ring subtending the angle $d\alpha$ is $(e/4\pi a)d\alpha$. As this charge has the velocity $\omega a \sin\alpha$, the current flowing around the annular ring is $(e\omega/4\pi)\sin\alpha d\alpha$, and the magnetic moment of the electron is

$$\mathbf{p}_H = \frac{ea^2\boldsymbol{\omega}}{4c}\int_0^{\pi}\sin^3\alpha\, d\alpha = \frac{ea^2}{3c}\boldsymbol{\omega}. \qquad (58\text{–}9)$$

In terms of its magnetic moment the vector potential (58–8) of the spinning electron is

$$\mathbf{A} = \begin{cases} \dfrac{\mathbf{p}_H \times \mathbf{R}}{4\pi R^3} \text{ outside,} \\[2ex] \dfrac{\mathbf{p}_H \times \mathbf{R}}{4\pi a^3} \text{ inside.} \end{cases} \tag{58–10}$$

Outside the electron the field intensities are

$$\left.\begin{aligned} \mathbf{E} &= -\nabla\Phi = \frac{e\mathbf{R}}{4\pi R^3} + \frac{1}{4\pi R^5}\{2\mathbf{p}_E\cdot\mathbf{R}\mathbf{R} + (\mathbf{p}_E \times \mathbf{R}) \times \mathbf{R}\}, \\ \mathbf{H} &= \nabla \times \mathbf{A} = \frac{1}{4\pi R^5}\{2\mathbf{p}_H\cdot\mathbf{R}\mathbf{R} + (\mathbf{p}_H \times \mathbf{R}) \times \mathbf{R}\}, \end{aligned}\right\} \tag{58–11}$$

and inside

$$\left.\begin{aligned} \mathbf{E} &= -\nabla\Phi = -\frac{\mathbf{p}_E}{4\pi a^3}, \\ \mathbf{H} &= \nabla \times \mathbf{A} = \frac{\mathbf{p}_H}{2\pi a^3}, \end{aligned}\right\} \tag{58–12}$$

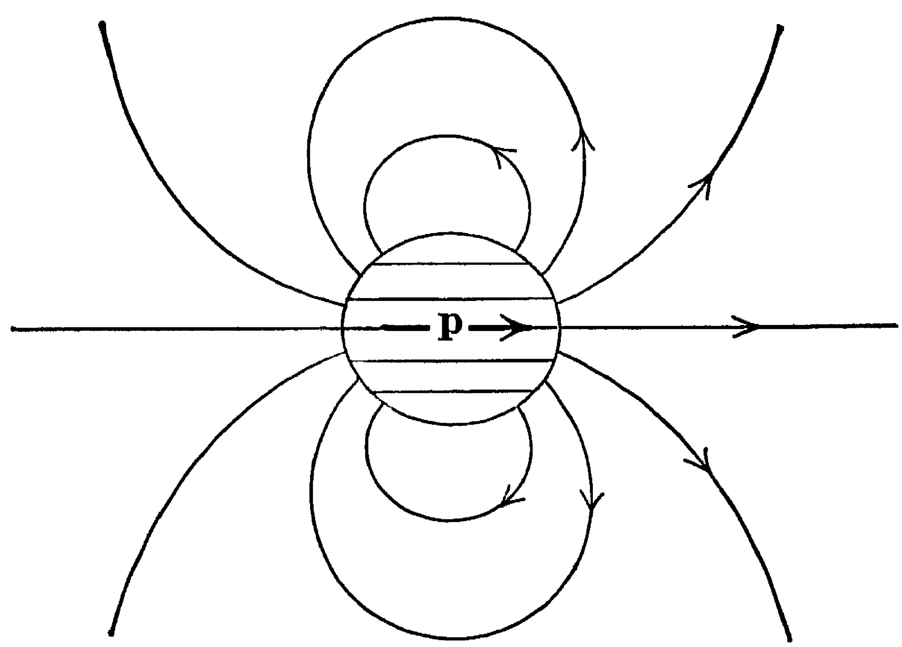

Fig. 53.

from (58–5) and (58–10), satisfying the boundary conditions (52–1) and (52–2). Both fields are uniform in the interior of the electron. The directions of the lines of force of the field of either moment is shown in Fig. 53. In the case of the magnetic field, the lines of force inside the electron have the same sense as outside, whereas in the case of the electric field the sense inside is reversed.

Next we shall calculate the force and torque exerted by external electric and magnetic fields on an electron momentarily at rest in the observer's inertial system. In making this calculation we shall assume that the force equation (57–22) applies to an element of the electron's charge, and we shall take into account both the field and its gradient, but shall omit derivatives of higher order.

If $\mathbf{E}_1$ and $\mathbf{H}_1$ are the external fields at the center of the electron, the fields at a point on the surface whose position vector is $\mathbf{a}$ (Fig. 52) are $\mathbf{E}_1 + \mathbf{a}\cdot\nabla\mathbf{E}_1$ and $\mathbf{H}_1 + \mathbf{a}\cdot\nabla\mathbf{H}_1$ respectively. Hence the force $d\mathcal{K}_1$ on an element of the surface is

$$d\mathcal{K}_1 = (\mathbf{E}_1+\mathbf{a}\cdot\nabla\mathbf{E}_1)(de_1+de_2)+\frac{1}{c}(\boldsymbol{\omega}\times\mathbf{a})\times(\mathbf{H}_1+\mathbf{a}\cdot\nabla\mathbf{H}_1)de_1, \quad (58\text{–}13)$$

since $\mathbf{V} = \boldsymbol{\omega}\times\mathbf{a}$ for the rotating charge, and the torque about the center of the electron is

$$d\mathcal{L}_1 = \mathbf{a}\times d\mathcal{K}_1 = \mathbf{a}\times(\mathbf{E}_1+\mathbf{a}\cdot\nabla\mathbf{E}_1)(de_1+de_2) + \frac{1}{c}(\mathbf{a}\cdot\mathbf{H}_1+\mathbf{a}\cdot\nabla\mathbf{H}_1\cdot\mathbf{a})\boldsymbol{\omega}\times\mathbf{a}\, de_1. \quad (58\text{–}14)$$

It is evident at once that

$$\int_{e_1}\mathbf{E}_1\, de_1 = e\mathbf{E}_1,$$

and from symmetry it is clear that

$$\int_{e_1}\mathbf{a}\cdot\nabla\mathbf{E}_1\, de_1 = 0, \qquad \int_{e_2}\mathbf{E}_1\, de_2 = 0,$$

$$\int_{e_1}(\boldsymbol{\omega}\times\mathbf{a})\times\mathbf{H}_1\, de_1 = \boldsymbol{\omega}\cdot\mathbf{H}_1\int_{e_1}\mathbf{a}\, de_1 - \boldsymbol{\omega}\int_{e_1}\mathbf{a}\cdot\mathbf{H}_1\, de_1 = 0,$$

$$\int_{e_1}\mathbf{a}\times\mathbf{E}_1\, de_1 = 0.$$

From (57–5)

$$\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{E}_1 = a\Big\{&i\left(\frac{\partial E_{1x}}{\partial x}\cos\theta+\frac{\partial E_{1x}}{\partial y}\sin\theta\cos\phi+\frac{\partial E_{1x}}{\partial z}\sin\theta\sin\phi\right)\\ &+j\left(\frac{\partial E_{1y}}{\partial x}\cos\theta+\frac{\partial E_{1y}}{\partial y}\sin\theta\cos\phi+\frac{\partial E_{1y}}{\partial z}\sin\theta\sin\phi\right)\\ &+k\left(\frac{\partial E_{1z}}{\partial x}\cos\theta+\frac{\partial E_{1z}}{\partial y}\sin\theta\cos\phi+\frac{\partial E_{1z}}{\partial z}\sin\theta\sin\phi\right)\Big\}.\end{aligned}$$

(58–15)

If we take the X axis of Fig. 52 in the direction of $\mathbf{p}_E$,

$$de_2 = \frac{3}{4\pi a} p_E \cos\theta \sin\theta \, d\theta \, d\phi, \tag{58–16}$$

and

$$\int_{e_2} \mathbf{a}\cdot\nabla\mathbf{E}_1 \, de_2 = p_E \left\{ \boldsymbol{i}\frac{\partial E_{1x}}{\partial x} + \boldsymbol{j}\frac{\partial E_{1y}}{\partial x} + \boldsymbol{k}\frac{\partial E_{1z}}{\partial x} \right\} = \mathbf{p}_E\cdot\nabla\mathbf{E}_1.$$

The evaluation of the remaining integral in $\mathbf{H}_1$ is simplified if we now take the X axis of Fig. 52 in the direction of $\boldsymbol{\omega}$. Then

$$\boldsymbol{\omega}\times\mathbf{a} = a\omega(-\boldsymbol{j}\sin\theta\sin\phi + \boldsymbol{k}\sin\theta\cos\phi) \tag{58–17}$$

and, as $\mathbf{a}\cdot\nabla\mathbf{H}_1$ is given by (58–15) with $\mathbf{H}_1$ replacing $\mathbf{E}_1$, we get, if we express de_1 as in (57–4),

$$\frac{1}{c}\int_{e_1} ((\boldsymbol{\omega}\times\mathbf{a})\times\mathbf{a}\cdot\nabla\mathbf{H}_1) de_1 = \frac{ea^2\omega}{2c}\left\{ \boldsymbol{i}\left(-\frac{2}{3}\frac{\partial H_{1z}}{\partial z} - \frac{2}{3}\frac{\partial H_{1y}}{\partial y}\right) \right.$$

$$\left. + \boldsymbol{j}\left(\frac{2}{3}\frac{\partial H_{1x}}{\partial y}\right) + \boldsymbol{k}\left(\frac{2}{3}\frac{\partial H_{1x}}{\partial z}\right)\right\} = \frac{ea^2}{3c}\{\nabla\mathbf{H}_1\cdot\boldsymbol{\omega} - \nabla\cdot\mathbf{H}_1\boldsymbol{\omega}\} = \nabla\mathbf{H}_1\cdot\mathbf{p}_H$$

by (58–9), since $\nabla\cdot\mathbf{H}_1 = 0$ by (57–23). Therefore the total external force on the electron is

$$\mathcal{K}_1 = e\mathbf{E}_1 + \mathbf{p}_E\cdot\nabla\mathbf{E}_1 + \nabla\mathbf{H}_1\cdot\mathbf{p}_H. \tag{58–18}$$

If the external field is static $\nabla\times\mathbf{H}_1 = 0$ by (57–23) and $(\nabla\times\mathbf{H}_1)\times\mathbf{p}_H = \mathbf{p}_H\cdot\nabla\mathbf{H}_1 - \nabla\mathbf{H}_1\cdot\mathbf{p}_H = 0$. In this case (58–18) assumes the symmetric form

$$\mathcal{K}_1 = e\mathbf{E}_1 + \mathbf{p}_E\cdot\nabla\mathbf{E}_1 + \mathbf{p}_H\cdot\nabla\mathbf{H}_1. \tag{58–19}$$

In any event $\nabla\mathbf{H}_1\cdot\mathbf{p}_H = \mathbf{p}_H\cdot\nabla\mathbf{H}_1 - \frac{1}{c}\dot{\mathbf{E}}_1\times\mathbf{p}_H$ and $\mathbf{p}_E\cdot\nabla\mathbf{E}_1 = \nabla\mathbf{E}_1\cdot\mathbf{p}_E - \frac{1}{c}\dot{\mathbf{H}}_1\times\mathbf{p}_E$, which enables us to write (58–18) symmetrically in the lengthier form

$$\mathcal{K}_1 = e\mathbf{E}_1 + \tfrac{1}{2}\{\mathbf{p}_E\cdot\nabla\mathbf{E}_1 + \mathbf{p}_H\cdot\nabla\mathbf{H}_1 + \nabla\mathbf{E}_1\cdot\mathbf{p}_E$$

$$+ \nabla\mathbf{H}_1\cdot\mathbf{p}_H - \frac{1}{c}\dot{\mathbf{E}}_1\times\mathbf{p}_H - \frac{1}{c}\dot{\mathbf{H}}_1\times\mathbf{p}_E\}. \tag{58–20}$$

Next we shall compute the torque by integrating (58–14). As

$$\mathbf{a} \times \mathbf{E}_1 = a\{\boldsymbol{i}(E_{1z} \sin\theta \cos\phi - E_{1y} \sin\theta \sin\phi)$$
$$+ \boldsymbol{j}(E_{1x} \sin\theta \sin\phi - E_{1z} \cos\theta) + \boldsymbol{k}(E_{1y} \cos\theta - E_{1x} \sin\theta \cos\phi)\}$$

we get, using (58–16) for de_2,

$$\int_{e_2} \mathbf{a} \times \mathbf{E}_1 \, de_2 = \tfrac{3}{2}p_E\,(-\tfrac{2}{3}\boldsymbol{j}E_{1z} + \tfrac{2}{3}\boldsymbol{k}E_{1y}) = \mathbf{p}_E \times \mathbf{E}_1.$$

From (58–15)

$$\int_{e_1} \mathbf{a} \times (\mathbf{a}\cdot\nabla\mathbf{E}_1)de_1 = \frac{ea^2}{2}\left\{ \boldsymbol{i}\left(\frac{2}{3}\frac{\partial E_{1z}}{\partial y} - \frac{2}{3}\frac{\partial E_{1y}}{\partial z}\right)\right.$$
$$\left. + \boldsymbol{j}\left(\frac{2}{3}\frac{\partial E_{1x}}{\partial z} - \frac{2}{3}\frac{\partial E_{1z}}{\partial x}\right) + \boldsymbol{k}\left(\frac{2}{3}\frac{\partial E_{1y}}{\partial x} - \frac{2}{3}\frac{\partial E_{1x}}{\partial y}\right)\right\}$$
$$= \frac{ea^2}{3}\nabla \times \mathbf{E}_1 = -\frac{ea^2}{3c}\dot{\mathbf{H}}_1$$

by (57–23), and

$$\int_{e_2} \mathbf{a} \times (\mathbf{a}\cdot\nabla\mathbf{E}_1)de_2 = 0.$$

As regards the terms in $\mathbf{H}_1$,

$$\frac{1}{c}\int_{e_1} \mathbf{a}\cdot\mathbf{H}_1\boldsymbol{\omega} \times \mathbf{a}de_1 = \frac{ea^2\omega}{2c}\{-\tfrac{2}{3}\boldsymbol{j}H_{1z} + \tfrac{2}{3}\boldsymbol{k}H_{1y}\}$$
$$= \frac{ea^2}{3c}\boldsymbol{\omega} \times \mathbf{H}_1 = \mathbf{p}_H \times \mathbf{H}_1$$

from (58–9), and

$$\frac{1}{c}\int_{e_1} \mathbf{a}\cdot\nabla\mathbf{H}_1\cdot\mathbf{a}\,\boldsymbol{\omega} \times \mathbf{a}\,de_1 = 0.$$

Therefore the total external torque on the electron is

$$\mathscr{L}_1 = \mathbf{p}_E \times \mathbf{E}_1 + \mathbf{p}_H \times \mathbf{H}_1 - \frac{ea^2}{3c}\dot{\mathbf{H}}_1. \qquad (58\text{–}21)$$

59. The Spinning Electron in Motion. — In this article we shall deduce the equations of motion of a spinning electron possessing simple electric and magnetic moments for both translation and rotation on the dynamical assumption that both the resultant force and the resultant torque, obtained by applying the force equation (57–22) to each element of the electron's charge, vanish relative to the inertial system in which the center of the electron is momentarily at rest. As

we have calculated in the last article the force and the torque due to external electric and magnetic fields, there remains only the computation of the reaction on the electron due to its own field. In this investigation we shall take account of the time rate of change of both the electric and the magnetic moment, but we shall limit ourselves to those terms which are linear in $\dot{\mathbf{p}}_E$ and $\dot{\mathbf{p}}_H$ and which contain only the moments themselves or constants as coefficients. As regards the mass reaction, we shall calculate only those terms which are linear in $\mathbf{f}$, the coefficients of which are constants or functions of the two moments. This limitation is justified by the fact that, although the electric and magnetic moments of the electron may be large, their time rates of change are probably small. Terms in $\dot{\mathbf{f}}$ or higher derivatives will be omitted entirely. Our method, however, will be valid for any peripheral velocities of spin less than the velocity of light.

The first part of the analysis consists in calculating the electric and magnetic fields of the electron at the instant when its center is momentarily at rest in the observer's inertial system S. Then, by integrating over the surface of the electron, we find the force and the torque exerted on it by its own field.

As a preliminary we must find the charge density and current density relative to an inertial system S of the rotating charge of a spinning electron at rest in inertial system S'. Let ρ'' be the volume density of charge at a point on the surface of the electron relative to the inertial system S'' in which this point is momentarily at rest. Then, on account of the Fitzgerald-Lorentz contraction,

$$\rho'' = \rho' \sqrt{1 - \frac{V'^2}{c^2}} = \rho \sqrt{1 - \frac{V^2}{c^2}}.$$

With the aid of (43–1) and (43–3) we find for the transformation of the components of current density and the charge density

$$\left.\begin{aligned} \rho V_x &= \rho' \frac{V_x' + v}{\sqrt{1-\beta^2}}, \\ \rho V_y &= \rho' V_y', \\ \rho V_z &= \rho' V_z', \\ \rho &= \rho' \frac{1 + \dfrac{vV_x'}{c^2}}{\sqrt{1-\beta^2}}. \end{aligned}\right\} \qquad (59\text{–}1)$$

If we neglect terms in β^2, the current and charge densities observed in S are respectively

$$\left.\begin{aligned} \rho\mathbf{V} &= \rho'(\mathbf{V}' + \mathbf{v}), \\ \rho &= \rho'\left(1 + \frac{\mathbf{v}\cdot\mathbf{V}'}{c^2}\right). \end{aligned}\right\} \qquad (59\text{–}2)$$

The last equation states that the rotating uniform charge density ρ' of the spinning electron relative to S' appears non-uniform relative to observers in S in such a way as to give the electron an additional electric moment in a direction at right angles to both its magnetic moment and its velocity. Since, then, the initial uniformly distributed surface charge of a spinning electron whose center is at rest in S becomes non-uniform as the center of the electron acquires speed, the angular velocity of spin must be different for different points on the surface. Let $\mathbf{a}_1$ be the position vector of a point Q on the surface of the electron relative to the center O, and let ω_Q be the angular velocity of spin of Q about O at the time 0 at which the center of the electron is momentarily at rest in S.

Take the X axis in the direction of the vector angular velocity, and let α be the angle which $\mathbf{a}_1$ makes with this axis and γ the azimuth of $\mathbf{a}_1$ measured about the X axis. If ρ_σ is the charge per unit area on the surface of the electron, the equation of continuity (13–4) at time 0 is

$$\frac{1}{a\sin\alpha}\frac{\partial}{\partial\gamma}(\rho_\sigma V_\gamma) + \frac{\partial\rho_\sigma}{\partial t} = 0$$

from (19–10).

Now $V_\gamma = \omega_Q a \sin\alpha$ and, if $\boldsymbol{\omega}$ is the mean angular velocity at time 0, the charge per unit area at the end of a small time t is

$$\rho_\sigma = \frac{e}{4\pi a^2}\left\{1 + \frac{\boldsymbol{\omega}\times\mathbf{a}_1\cdot\mathbf{f}}{c^2}t + \cdots\right\}$$

from (59–2), which gives, when differentiated with respect to the time,

$$\frac{\partial\rho_\sigma}{\partial t} = \frac{e}{4\pi a^2}\frac{\omega a}{c^2}\sin\alpha\,(f_z\cos\gamma - f_y\sin\gamma).$$

Substituting in the equation of continuity we have

$$\frac{\partial\omega_Q}{\partial\gamma} = -\frac{\omega a}{c^2}\sin\alpha\,(f_z\cos\gamma - f_y\sin\gamma),$$

which is satisfied by

$$\boldsymbol{\omega}_Q = \boldsymbol{\omega}\left(1 - \frac{\mathbf{f}\cdot\mathbf{a}_1}{c^2}\right). \tag{59-3}$$

If, then, de_1 is the uniformly distributed charge and $\mathbf{V}_1\, de_1$ the current due to its rotation, in the solid angle $d\Omega$, at the end of the small time t after the center of the electron was at rest in S, we have

$$de_1 = \frac{e}{4\pi}\left\{1 + \frac{\boldsymbol{\omega}\times\mathbf{a}_1\cdot\mathbf{f}}{c^2}\,t + \cdots\right\} d\Omega, \tag{59-4}$$

and

$$\mathbf{V}_1\, de_1 = \frac{e}{4\pi}\left\{\boldsymbol{\omega}\times\mathbf{a}_1 - \frac{\mathbf{f}\cdot\mathbf{a}_1}{c^2}\,\boldsymbol{\omega}\times\mathbf{a}_1 + \dot{\boldsymbol{\omega}}\times\mathbf{a}_1 t + \mathbf{f}t + \cdots\right\} d\Omega \tag{59-5}$$

from (59–2) and (59–3).

Next we consider the non-uniform charge responsible for the electric moment $\mathbf{p}_E$. As its velocity is evidently proportional to $\dot{\mathbf{p}}_E$ we need not carry the approximation so far as in the case of the spinning charge. Our first task is to find the current associated with change of electric moment by setting up and solving the equation of continuity. Now a change in direction of the electric moment $\mathbf{p}_E$ without change in magnitude might be expected to be due to rotation of the entire non-uniform charge responsible for this moment about an axis perpendicular to $\mathbf{p}_E$ and $\dot{\mathbf{p}}_E$ in the sense $\mathbf{p}_E \times \dot{\mathbf{p}}_E$. On the other hand a change in magnitude without change in direction of $\mathbf{p}_E$ must be attributed to a further separation of charge. The simplest way, therefore, to account for both change in magnitude and change in direction of the electric moment of the electron is to combine a reunion of the charges of opposite sign constituting a diminishing electric moment in one direction with a separation of the charges contributing to an increasing electric moment in some other direction. Although we shall select this model, a detailed calculation shows that the final reaction to be computed is the same if we adopt the more complicated hypothesis that change in direction is due to rotation, and change in magnitude to separation, of charge.

If α is the angle which the radius vector $\mathbf{a}_1$ from the center of the electron to a point Q on the surface makes with $\mathbf{p}_E$, the equation of continuity of the charge responsible for the electric moment at time 0 is

$$\frac{1}{a\sin\alpha}\frac{\partial}{\partial\alpha}(\sin\alpha\ \rho_\sigma V_\alpha) + \frac{\partial\rho_\sigma}{\partial t} = 0$$

from (13–4) and (19–10). Now the charge per unit area at the end of a small time t is

$$\rho_\sigma = \frac{3}{4\pi a^3}\{p_E \cos\alpha + (\dot{p}_E \cos\alpha)t\}.$$

Therefore the equation of continuity becomes

$$\frac{\partial}{\partial\alpha}(\sin\alpha\, \rho_\sigma V_\alpha) + \frac{3}{4\pi a^2}\dot{p}_E \sin\alpha\cos\alpha = 0,$$

of which the solution is

$$\rho_\sigma V_\alpha = -\frac{3}{8\pi a^2}\dot{p}_E \sin\alpha.$$

If, then, de_2 is the charge responsible for the electric moment and $\mathbf{V}_2 de_2$ the current due to change of this moment, in the solid angle $d\Omega$, at the end of the small time t after the center of the electron was at rest in S, we have

$$de_2 = \frac{3}{4\pi a^2}\{\mathbf{p}_E\cdot\mathbf{a}_1 + \dot{\mathbf{p}}_E\cdot\mathbf{a}_1\, t + \cdots\}d\Omega, \tag{59–6}$$

$$\mathbf{V}_2 de_2 = \frac{3}{4\pi a^2}\{\tfrac{1}{2}(\mathbf{a}_1\times\dot{\mathbf{p}}_E)\times\mathbf{a}_1 + \mathbf{p}_E\cdot\mathbf{a}_1\mathbf{f}t + \cdots\}d\Omega. \tag{59–7}$$

To find the nearby field of the spinning electron with center momentarily at rest in S, we proceed from the retarded expressions

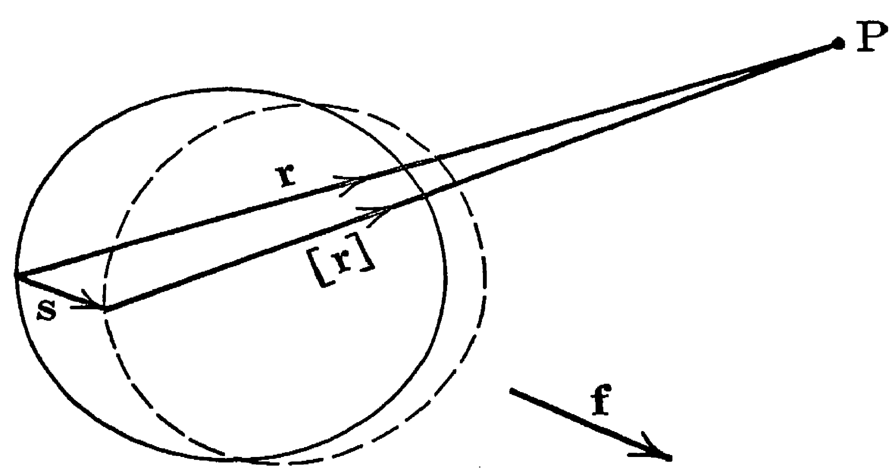

FIG. 54.

(50–9) and (50–10) for the scalar and vector potentials, the integrals being evaluated over the effective electron for the time and point at which the field is to be determined. To find the field at P (Fig. 54) at a small time t later than the instant 0 at which the center of the electron is at rest in S, we must integrate over each volume element

occupied by the charge at the time $[t] \equiv t - [r]/c$. Consequently the simultaneous electron at time 0, represented by the solid circle in the figure, must be replaced by the effective electron at time $[t]$, represented by the broken curve. Moreover, the argument used in deducing (50–9) and (50–10) from (50–7) and (50–8) shows that the charge associated with each surface element must be increased above its simultaneous value in the ratio $1 \Big/ \left[1 - \dfrac{\mathbf{v}\cdot\mathbf{r}}{cr}\right]$, where $[\mathbf{v}] = \mathbf{f}[t] + \cdots$ is the velocity of the center of the electron (not the velocity of the rotating charge) at the time $[t]$. Therefore the integrals to be evaluated are

$$\Phi = \frac{1}{4\pi}\int_{e_1,e_2} \frac{[de_1] + [de_2]}{[r]\left[1 - \dfrac{\mathbf{v}\cdot\mathbf{r}}{cr}\right]}, \tag{59–8}$$

$$\mathbf{A} = \frac{1}{4\pi c}\int_{e_1,e_2} \frac{[\mathbf{V}_1 de_1] + [\mathbf{V}_2 de_2]}{[r]\left[1 - \dfrac{\mathbf{v}\cdot\mathbf{r}}{cr}\right]}, \tag{59–9}$$

where $[de_1]$, $[\mathbf{V}_1 de_1]$, $[de_2]$, $[\mathbf{V}_2 de_2]$ are the expressions (59–4), (59–5), (59–6), (59–7) with $[t]$ replacing t.

If $\mathbf{s}$ (Fig. 54) is the vector displacement of the element of surface under consideration in the time $[t]$, $\mathbf{s} = \frac{1}{2}\mathbf{f}[t]^2 + \cdots$ and

$$[\mathbf{r}] = \mathbf{r} - \mathbf{s} = \mathbf{r} - \tfrac{1}{2}\mathbf{f}[t]^2 + \cdots,$$

$$[r] = r\left\{1 - \frac{1}{2}\frac{\mathbf{f}\cdot\mathbf{r}}{r^2}[t]^2 + \cdots\right\}.$$

Hence

$$\frac{1}{[r]\left[1 - \dfrac{\mathbf{v}\cdot\mathbf{r}}{cr}\right]} = \frac{1}{r}\left\{1 + \mathbf{f}\cdot\mathbf{r}\left(\frac{[t]}{rc} + \frac{1}{2}\frac{[t]^2}{r^2}\right) + \cdots\right\}. \tag{59–10}$$

We need the scalar potential Φ, the vector potential $\mathbf{A}$ and its time derivative at the field-point P at time 0. To get the first, make $[t] = -r/c$ in (59–8), since $[r] = r$ in the terms containing $[t]$ to our degree of approximation, obtaining

$$\Phi = \frac{e}{16\pi^2}\int \frac{d\Omega}{r}\left\{1 - \frac{1}{2}\frac{\mathbf{f}\cdot\mathbf{r}}{c} - \frac{\boldsymbol{\omega}\times\mathbf{a}_1\cdot\mathbf{f}r}{c^3} + \cdots\right\}$$
$$+ \frac{3}{16\pi^2 a^2}\int \frac{d\Omega}{r}\left\{\mathbf{p}_E\cdot\mathbf{a}_1 - \frac{1}{2}\mathbf{p}_E\cdot\mathbf{a}_1\frac{\mathbf{f}\cdot\mathbf{r}}{c} - \dot{\mathbf{p}}_E\cdot\mathbf{a}_1\frac{r}{c} + \cdots\right\}.$$

The negative gradient of this function of r is

$$-\nabla\Phi = -\nabla\left\{\frac{e}{16\pi^2}\int\frac{d\Omega}{r}\right\} - \nabla\left\{\frac{3}{16\pi^2a^2}\int \mathbf{p}_E\cdot\mathbf{a}_1\frac{d\Omega}{r}\right\}$$

$$+\frac{\mathbf{f}}{32\pi^2c^2}\int\left\{e+3\frac{\mathbf{p}_E\cdot\mathbf{a}_1}{a^2}\right\}\frac{d\Omega}{r} - \frac{1}{32\pi^2c^2}\int\left\{e+3\frac{\mathbf{p}_E\cdot\mathbf{a}_1}{a^2}\right\}\mathbf{f}\cdot\mathbf{rr}\frac{d\Omega}{r}+\cdots. \tag{59-11}$$

The vector potential (59–9) is

$$\mathbf{A} = \frac{e}{16\pi^2c}\int\frac{d\Omega}{r}\left\{\boldsymbol{\omega}\times\mathbf{a}_1 - \frac{\mathbf{f}\cdot\mathbf{a}_1}{c^2}\boldsymbol{\omega}\times\mathbf{a}_1 + \dot{\boldsymbol{\omega}}\times\mathbf{a}_1[t] + \mathbf{f}[t]\right.$$

$$\left. + \boldsymbol{\omega}\times\mathbf{a}_1\mathbf{f}\cdot\mathbf{r}\left(\frac{[t]}{rc}+\frac{1}{2}\frac{[t]^2}{r^2}\right)+\cdots\right\}$$

$$+\frac{3}{16\pi^2a^2c}\int\frac{d\Omega}{r}\left\{\tfrac{1}{2}(\mathbf{a}_1\times\dot{\mathbf{p}}_E)\times\mathbf{a}_1 + \mathbf{p}_E\cdot\mathbf{a}_1\mathbf{f}[t]+\cdots\right\}.$$

To get the time derivative of $\mathbf{A}$ at P at time 0 we differentiate with respect to $[t]$, since $dt = d[t]$, and then make $[t] = -r/c$. Thus

$$-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\frac{e}{16\pi^2c^2}\int\dot{\boldsymbol{\omega}}\times\mathbf{a}_1\frac{d\Omega}{r}$$

$$-\frac{\mathbf{f}}{16\pi^2c^2}\int\left\{e+3\frac{\mathbf{p}_E\cdot\mathbf{a}_1}{a^2}\right\}\frac{d\Omega}{r}+\cdots. \tag{59-12}$$

The electric intensity is the sum of (59–11) and (59–12). Expressing $\boldsymbol{\omega}$ in terms of $\mathbf{p}_H$ by means of (58–9),

$$\mathbf{E} = -\nabla\left\{\frac{e}{16\pi^2}\int\frac{d\Omega}{r}\right\} - \nabla\left\{\frac{3}{16\pi^2a^2}\int\mathbf{p}_E\cdot\mathbf{a}_1\frac{d\Omega}{r}\right\}$$

$$-\frac{3}{16\pi^2a^2c}\int\dot{\mathbf{p}}_H\times\mathbf{a}_1\frac{d\Omega}{r} - \frac{\mathbf{f}}{32\pi^2c^2}\int\left\{e+3\frac{\mathbf{p}_E\cdot\mathbf{a}_1}{a^2}\right\}\frac{d\Omega}{r}$$

$$-\frac{1}{32\pi^2c^2}\int\left\{e+3\frac{\mathbf{p}_E\cdot\mathbf{a}_1}{a^2}\right\}\mathbf{f}\cdot\mathbf{rr}\frac{d\Omega}{r}+\cdots. \tag{59-13}$$

Finally, making $[t] = -r/c$ in the expression for $\mathbf{A}$, we get for the vector potential at P at time 0,

$$\mathbf{A} = \frac{e}{16\pi^2 c}\int \frac{d\Omega}{r}\left\{\boldsymbol{\omega}\times\mathbf{a}_1 - \frac{\mathbf{f}\cdot\mathbf{a}_1}{c^2}\boldsymbol{\omega}\times\mathbf{a}_1 - \frac{\dot{\boldsymbol{\omega}}\times\mathbf{a}_1 r}{c} - \frac{\mathbf{f}r}{c} - \frac{\boldsymbol{\omega}\times\mathbf{a}_1\mathbf{f}\cdot\mathbf{r}}{2c^2} + \cdots\right\}$$
$$+ \frac{3}{16\pi^2 a^2 c}\int\frac{d\Omega}{r}\left\{\frac{1}{2}(\mathbf{a}_1\times\dot{\mathbf{p}}_E)\times\mathbf{a}_1 - \frac{\mathbf{p}_E\cdot\mathbf{a}_1\mathbf{f}r}{c} + \cdots\right\}.$$

Therefore, since the magnetic intensity is the curl of the vector potential,

$$\mathbf{H} = \nabla\times\left\{\frac{3}{16\pi^2 a^2}\int \mathbf{p}_H\times\mathbf{a}_1\left(1 - \frac{\mathbf{f}\cdot\mathbf{a}_1}{c^2}\right)\frac{d\Omega}{r}\right\}$$
$$+ \nabla\times\left\{\frac{3}{32\pi^2 a^2 c}\int(\mathbf{a}_1\times\dot{\mathbf{p}}_E)\times\mathbf{a}_1\frac{d\Omega}{r}\right\}$$
$$- \frac{3}{32\pi^2 a^2 c^2}\int \mathbf{f}\times(\mathbf{p}_H\times\mathbf{a}_1)\frac{d\Omega}{r}$$
$$+ \frac{3}{32\pi^2 a^2 c^2}\int \mathbf{r}\times(\mathbf{p}_H\times\mathbf{a}_1)\mathbf{f}\cdot\mathbf{r}\frac{d\Omega}{r^3} + \cdots. \qquad (59\text{–}14)$$

The first two terms in (59–13) and in (59–14), for which the differentiation is indicated but not actually performed, are discontinuous at the surface of the electron. Since these terms alone would survive if the electron were permanently at rest with the assigned distribution of charge and angular velocity, they must satisfy the boundary conditions at the surface. Consequently the remaining terms must be continuous. We shall denote the sum of the discontinuous terms in $\mathbf{E}$ and $\mathbf{H}$ by $\mathbf{E}_D$ and $\mathbf{H}_D$ respectively, and the sum of the continuous terms by $\mathbf{E}_C$ and $\mathbf{H}_C$. Then, from (58–11) and (58–12),

$$\mathbf{E}_D = \begin{cases} \dfrac{e\mathbf{R}}{4\pi R^3} + \dfrac{1}{4\pi R^5}\left\{2\mathbf{p}_E\cdot\mathbf{R}\mathbf{R} + (\mathbf{p}_E\times\mathbf{R})\times\mathbf{R}\right\} \text{ outside,} \\[2ex] -\dfrac{\mathbf{p}_E}{4\pi a^3} \text{ inside.}\end{cases} \qquad (59\text{–}15)$$

We may write $\mathbf{H}_D = \mathbf{H}_{D_1} + \mathbf{H}_{D_2} + \mathbf{H}_{D_3}$ where $\mathbf{H}_{D_1}$, representing the first part of the first term in (59–14), is

$$\mathbf{H}_{D_1} = \begin{cases} \dfrac{1}{4\pi R^5}\left\{2\mathbf{p}_H\cdot\mathbf{RR} + (\mathbf{p}_H \times \mathbf{R}) \times \mathbf{R}\right\} \text{ outside,} \\[2ex] \dfrac{\mathbf{p}_H}{2\pi a^3} \text{ inside,} \end{cases} \tag{59–16}$$

as in (58–11) and (58–12), $\mathbf{H}_{D_2} = \nabla \times \mathbf{A}_{D_2}$ where

$$\mathbf{A}_{D_2} = -\frac{3}{16\pi^2 a^2 c^2}\int \mathbf{f}\cdot\mathbf{a}_1\mathbf{p}_H \times \mathbf{a}_1 \frac{d\Omega}{r}, \tag{59–17}$$

and $\mathbf{H}_{D_3} = \nabla \times \mathbf{A}_{D_3}$, where

$$\mathbf{A}_{D_3} = \frac{3}{32\pi^2 a^2 c}\int (a^2\dot{\mathbf{p}}_E - \dot{\mathbf{p}}_E\cdot\mathbf{a}_1\mathbf{a}_1)\frac{d\Omega}{r}. \tag{59–18}$$

To evaluate the vector potentials (59–17) and (59–18) we need, in terms of the notation of Fig. 52,

$$\begin{aligned}\int_r \cos^2\theta\, dr &= \frac{1}{4\pi R^2}\int_r \left\{(R^2+a^2)^2 - 2(R^2+a^2)r^2 + r^4\right\}dr \\ &= \begin{cases} \dfrac{2a}{15R^2}(5R^2 + 2a^2) \text{ outside,} \\[2ex] \dfrac{2R}{15a^2}(5a^2 + 2R^2) \text{ inside.} \end{cases}\end{aligned} \tag{59–19}$$

Performing the integration, we get

$$\mathbf{A}_{D_2} = \begin{cases} -\dfrac{1}{20\pi c^2}\left\{\left(\dfrac{5}{R} - \dfrac{a^2}{R^3}\right)\mathbf{p}_H \times \mathbf{f} + \dfrac{3a^2}{R^5}\mathbf{f}\cdot\mathbf{R}\mathbf{p}_H \times \mathbf{R}\right\} \text{ outside,} \\[2ex] -\dfrac{1}{20\pi c^2}\left\{\left(\dfrac{5}{a} - \dfrac{R^2}{a^3}\right)\mathbf{p}_H \times \mathbf{f} + \dfrac{3}{a^3}\mathbf{f}\cdot\mathbf{R}\mathbf{p}_H \times \mathbf{R}\right\} \text{ inside,} \end{cases} \tag{59–20}$$

and

$$\mathbf{A}_{D_3} = \begin{cases} \dfrac{1}{4\pi c}\left\{\left(\dfrac{1}{R} + \dfrac{a^2}{10R^3}\right)\dot{\mathbf{p}}_E - \dfrac{3a^2}{10R^5}\dot{\mathbf{p}}_E\cdot\mathbf{RR}\right\} \text{ outside,} \\[2ex] \dfrac{1}{4\pi c}\left\{\left(\dfrac{1}{a} + \dfrac{R^2}{10a^3}\right)\dot{\mathbf{p}}_E - \dfrac{3}{10a^3}\dot{\mathbf{p}}_E\cdot\mathbf{RR}\right\} \text{ inside.} \end{cases} \tag{59–21}$$

Taking the curl of (59–20) and (59–21), and adding to (59–16), we find

$$
\mathbf{H}_D = \begin{cases} \dfrac{1}{4\pi R^5}\left\{2\mathbf{p}_H\cdot\mathbf{RR} + (\mathbf{p}_H\times\mathbf{R})\times\mathbf{R}\right\} + \dfrac{1}{4\pi c R^3}\,\dot{\mathbf{p}}_E\times\mathbf{R} \\ \qquad -\dfrac{1}{20\pi c^2 R^3}\left\{\left(-5+3\dfrac{a^2}{R^2}\right)\mathbf{R}\times(\mathbf{p}_H\times\mathbf{f}) \right. \\ \qquad +\dfrac{3a^2}{R^2}\,\mathbf{f}\times(\mathbf{p}_H\times\mathbf{R}) \\ \qquad \left. -\dfrac{9a^2}{R^2}\,\mathbf{f}\cdot\mathbf{R}\mathbf{p}_H + \dfrac{15a^2}{R^4}\,\mathbf{f}\cdot\mathbf{R}\mathbf{p}_H\cdot\mathbf{RR}\right\} \text{ outside,} \\ \dfrac{\mathbf{p}_H}{2\pi a^3} - \dfrac{\dot{\mathbf{p}}_E\times\mathbf{R}}{8\pi a^3 c} - \dfrac{1}{20\pi a^3 c^2}\left\{-2\mathbf{R}\times(\mathbf{p}_H\times\mathbf{f})\right. \\ \qquad \left. +3\mathbf{f}\times(\mathbf{p}_H\times\mathbf{R}) + 6\mathbf{f}\cdot\mathbf{R}\mathbf{p}_H\right\} \text{ inside.} \end{cases} \tag{59–22}
$$

As the remaining terms in (59–13) and (59–14) are continuous at the surface of the electron, we need calculate them only for a field-point P (Fig. 55) on the surface of the electron at a vector distance $\mathbf{a}_2$ from the origin O at the center. Orient the axes so that

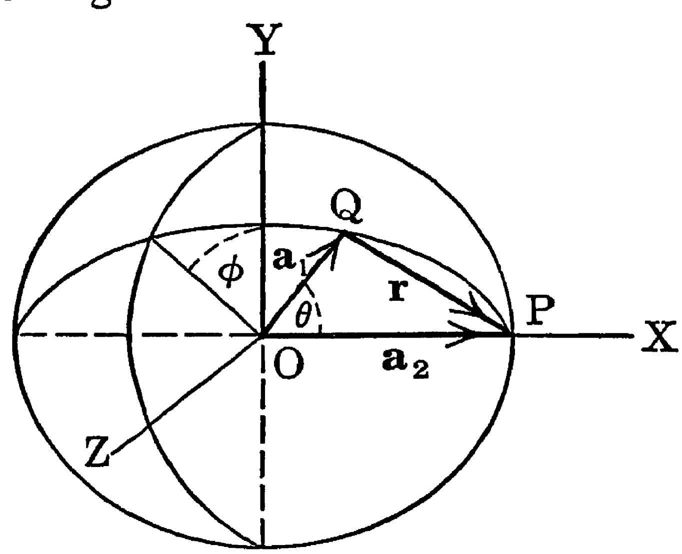

FIG. 55.

$\mathbf{a}_2 = ia$, and let θ be the angle which $\mathbf{a}_1$ makes with $\mathbf{a}_2$ and ϕ the azimuth of $\mathbf{a}_1$ measured about $\mathbf{a}_2$. Then we have

$$
\mathbf{a}_1 = a\left\{i\left(1 - 2\sin^2\frac{\theta}{2}\right) + 2j\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\phi \right. \\ \left. + 2k\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\phi\right\}, \tag{59–23}
$$

$$\mathbf{r} = \mathbf{a}_2 - \mathbf{a}_1 = 2a\left\{i \sin^2\frac{\theta}{2} - j\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\phi\right.$$

$$\left. - k\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\phi\right\}, \tag{59–24}$$

$$r = 2a\sin\frac{\theta}{2}, \tag{59–25}$$

$$d\Omega = 4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right)d\phi, \tag{59–26}$$

$$\mathbf{a}_1\cdot\mathbf{r} = -2a^2\sin^2\frac{\theta}{2}. \tag{59–27}$$

Performing the integration indicated in (59–13) and (59–14) we find for the continuous parts of **E** and **H**,

$$\mathbf{E}_C = -\frac{1}{4\pi a^3 c}\dot{\mathbf{p}}_H \times \mathbf{a}_2 - \frac{e\mathbf{f}}{6\pi a c^2}$$

$$-\frac{1}{20\pi a^3 c^2}\{4\mathbf{p}_E\cdot\mathbf{a}_2\mathbf{f} - \mathbf{f}\cdot\mathbf{p}_E\mathbf{a}_2 - \mathbf{f}\cdot\mathbf{a}_2\mathbf{p}_E\}, \tag{59–28}$$

and

$$\mathbf{H}_C = -\frac{1}{20\pi a^3 c^2}\{4\mathbf{f}\cdot\mathbf{a}_2\mathbf{p}_H - \mathbf{f}\cdot\mathbf{p}_H\mathbf{a}_2 - \mathbf{p}_H\cdot\mathbf{a}_2\mathbf{f}\}. \tag{59–29}$$

Having found the electric and magnetic intensities at the surface of the electron, we are ready to calculate the force and the torque exerted on the electron by its own field from (57–22). The part of the force due to the electric field is

$$\mathcal{K}_E = \frac{1}{4\pi}\int\left\{e + 3\frac{\mathbf{p}_E\cdot\mathbf{a}_2}{a^2}\right\}\mathbf{E}\,d\Omega, \tag{59–30}$$

and the part due to the magnetic field is

$$\mathcal{K}_H = \frac{3}{4\pi a^2}\int\left\{\left(1 - \frac{\mathbf{f}\cdot\mathbf{a}_2}{c^2}\right)\mathbf{p}_H \times \mathbf{a}_2 + \tfrac{1}{2}(a^2\dot{\mathbf{p}}_E - \dot{\mathbf{p}}_E\cdot\mathbf{a}_2\mathbf{a}_2)\right\}\times\mathbf{H}\,d\Omega. \tag{59–31}$$

So far as the force on the rotating uniformly distributed charge due to its own electric and magnetic fields is concerned, we must use the arithmetic mean of the inside and outside value of the discon-

tinuous parts of **E** and **H**, as shown in article 57. The same statement applies to the force on the non-uniformly distributed charge responsible for the electric moment due to its own electric and magnetic fields. When, however, we come to consider the force exerted on the one charge by the field of the other, we must either assume that the one charge lies outside or inside the other, or that the two are uniformly mingled. The latter supposition will be adopted, although a detailed computation shows that the result is the same whichever hypothesis we choose. Specifically we shall assume that both charges are distributed uniformly in the radial direction inside a thin spherical shell of mean radius a. Therefore in the calculation of all the terms in (59–30) and (59–31) we must use the mean of the inside and outside values of **E** and **H**. From (59–15), (59–22), (59–28) and (59–29) these are

$$\overline{\mathbf{E}} = \frac{e}{8\pi a^3}\mathbf{a}_2 + \frac{3}{8\pi a^5}\mathbf{p}_E\cdot\mathbf{a}_2\mathbf{a}_2 - \frac{1}{4\pi a^3}\mathbf{p}_E - \frac{1}{4\pi a^3 c}\dot{\mathbf{p}}_H\times\mathbf{a}_2$$
$$-\frac{e\mathbf{f}}{6\pi a c^2} - \frac{1}{20\pi a^3 c^2}\{4\mathbf{p}_E\cdot\mathbf{a}_2\mathbf{f} - \mathbf{f}\cdot\mathbf{p}_E\mathbf{a}_2 - \mathbf{f}\cdot\mathbf{a}_2\mathbf{p}_E\}, \quad (59\text{–}32)$$

$$\overline{\mathbf{H}} = \frac{3}{8\pi a^5}\mathbf{p}_H\cdot\mathbf{a}_2\mathbf{a}_2 + \frac{1}{8\pi a^3}\mathbf{p}_H + \frac{1}{16\pi a^3 c}\dot{\mathbf{p}}_E\times\mathbf{a}_2$$
$$-\frac{1}{20\pi a^3 c^2}\left\{\mathbf{p}_H\cdot\mathbf{a}_2\mathbf{f} - 4\mathbf{f}\cdot\mathbf{p}_H\mathbf{a}_2 + \frac{7}{2}\mathbf{f}\cdot\mathbf{a}_2\mathbf{p}_H + \frac{15}{2}\frac{\mathbf{f}\cdot\mathbf{a}_2\mathbf{p}_H\cdot\mathbf{a}_2}{a^2}\mathbf{a}_2\right\}\cdot \quad (59\text{–}33)$$

Substituting (59–32) for **E** in (59–30) and integrating, we find for the electric part of the kinetic reaction

$$\mathcal{K}_E = -\frac{1}{4\pi a^3 c}\dot{\mathbf{p}}_H\times\mathbf{p}_E - \frac{e^2\mathbf{f}}{6\pi a c^2} - \frac{1}{10\pi a^3 c^2}\{p_E{}^2\mathbf{f} + (\mathbf{p}_E\times\mathbf{f})\times\mathbf{p}_E\}. \quad (59\text{–}34)$$

The origin of the first term, which is independent of the acceleration of the electron as a whole, is easily revealed. If the angular velocity of a charged sphere rotating about a fixed diameter is increased, the magnetic flux through its equator is augmented with the production of an induced electromotive force in the sense opposite to the angular acceleration. The circuital electric field so produced gives rise to a force on an electric dipole at right angles to the magnetic moment in the sense of the vector $\mathbf{p}_E\times\dot{\mathbf{p}}_H$.

Next, if we put (59–33) in (59–31) and integrate, we find for the magnetic part of the kinetic reaction

$$\mathcal{K}_H = -\frac{1}{4\pi a^3 c}\,\mathbf{p}_H \times \dot{\mathbf{p}}_E - \frac{1}{10\pi a^3 c^2}\{p_H{}^2\mathbf{f} + \tfrac{7}{2}(\mathbf{p}_H \times \mathbf{f}) \times \mathbf{p}_H\}. \quad (59\text{–}35)$$

The first term in this expression is due in part to the force exerted by the magnetic field of the changing electric moment on the rotating charge and in part to the force exerted by the magnetic field of the rotating charge on the current responsible for the change in the electric moment.

Equating the sum of the external force (58–20) and the two parts (59–34) and (59–35) of the kinetic reaction of translation to zero, in accord with the dynamical assumption, we find for the equation of motion of the spinning electron, as regards translation

$$\begin{aligned} e\mathbf{E}_1 + \tfrac{1}{2}\Big\{&\mathbf{p}_E\cdot\nabla\mathbf{E}_1 + \mathbf{p}_H\cdot\nabla\mathbf{H}_1 + \nabla\mathbf{E}_1\cdot\mathbf{p}_E + \nabla\mathbf{H}_1\cdot\mathbf{p}_H \\ &- \frac{1}{c}\dot{\mathbf{E}}_1 \times \mathbf{p}_H - \frac{1}{c}\dot{\mathbf{H}}_1 \times \mathbf{p}_E\Big\} \\ = \frac{1}{4\pi a^3 c}\frac{d}{dt}(\mathbf{p}_H \times \mathbf{p}_E) &+ \frac{e^2\mathbf{f}}{6\pi a c^2} + \frac{1}{10\pi a^3 c^2}\{p_E{}^2\mathbf{f} + (\mathbf{p}_E \times \mathbf{f}) \times \mathbf{p}_E\} \\ + \frac{1}{4\pi a^3 c^2}(\mathbf{p}_H \times \mathbf{f}) \times \mathbf{p}_H &+ \frac{1}{10\pi a^3 c^2}\{p_H{}^2\mathbf{f} + (\mathbf{p}_H \times \mathbf{f}) \times \mathbf{p}_H\} \end{aligned} \quad (59\text{–}36)$$

relative to the inertial system in which the center of the electron is momentarily at rest. In addition to the mass reaction of the non-spinning Lorentz electron, mass reactions depending on the electric and magnetic moments are present, and these reactions are not, in general, opposite to the acceleration. Furthermore a term independent of the acceleration appears in the kinetic reaction. This term exists even in the case of an electron permanently at rest in the observer's inertial system, if the vector product $\mathbf{p}_H \times \mathbf{p}_E$ does not remain constant in time.

Next we shall calculate the torque exerted on the electron by its own field. This torque consists of the electric part

$$\mathcal{L}_E = \frac{1}{4\pi}\int\left\{e + 3\frac{\mathbf{p}_E\cdot\mathbf{a}_2}{a^2}\right\}\mathbf{a}_2 \times \mathbf{E}\,d\Omega \quad (59\text{–}37)$$

and the magnetic part

$$\mathscr{L}_H = \frac{3}{4\pi a^2}\int\Big\{\Big(1 - \frac{\mathbf{f}\cdot\mathbf{a}_2}{c^2}\Big)\mathbf{a}_2 \times \{(\mathbf{p}_H \times \mathbf{a}_2) \times \mathbf{H}\}$$
$$+ \tfrac{1}{2}a^2\mathbf{a}_2 \times (\dot{\mathbf{p}}_E \times \mathbf{H}) - \tfrac{1}{2}\dot{\mathbf{p}}_E\cdot\mathbf{a}_2\mathbf{a}_2 \times (\mathbf{a}_2 \times \mathbf{H})\Big\}d\Omega$$
$$= \frac{3}{4\pi a^2}\int\Big(1 - \frac{\mathbf{f}\cdot\mathbf{a}_2}{c^2}\Big)\mathbf{H}\cdot\mathbf{a}_2\mathbf{p}_H \times \mathbf{a}_2 d\Omega$$
$$+ \frac{3}{8\pi}\int(\mathbf{H}\cdot\mathbf{a}_2\dot{\mathbf{p}}_E - \dot{\mathbf{p}}_E\cdot\mathbf{a}_2\mathbf{H})d\Omega$$
$$- \frac{3}{8\pi a^2}\int\dot{\mathbf{p}}_E\cdot\mathbf{a}_2(\mathbf{H}\cdot\mathbf{a}_2\mathbf{a}_2 - a^2\mathbf{H})d\Omega. \qquad (59\text{–}38)$$

Carrying out the indicated integration,

$$\mathscr{L}_E = -\frac{e}{6\pi ac}\dot{\mathbf{p}}_H - \frac{e}{4\pi ac^2}\mathbf{p}_E \times \mathbf{f}, \qquad (59\text{–}39)$$

$$\mathscr{L}_H = 0, \qquad (59\text{–}40)$$

if we neglect, as hitherto, terms in $\mathbf{f}$ involving $\dot{\mathbf{p}}_E$ in the coefficient.

If we trace the origins of the terms in $\mathscr{L}_E$ we find that $-e\dot{\mathbf{p}}_H/6\pi ac$ is the torque exerted on the uniformly distributed charge by the induced electromotive force due to the change in magnetic flux consequent upon change in magnetic moment, $-e\mathbf{p}_E \times \mathbf{f}/12\pi ac^2$ is the torque on the same charge due to the electric field of the accelerated electric moment as specified by the last term in (59–28), and $-e\mathbf{p}_E \times \mathbf{f}/6\pi ac^2$ represents the torque on the non-uniform charge constituting the electric moment due to the term $-e\mathbf{f}/6\pi ac^2$ in the electric field of the accelerated uniformly distributed charge. The coefficients of these terms are the same no matter whether the one charge distribution lies outside, inside, or is commingled with the other.

Equating the sum of the external torque (58–21) and the two parts (59–39) and (59–40) of the kinetic reaction of rotation to zero in accord with the dynamical assumption, we find for the equation of motion of the spinning electron as regards rotation

$$\mathbf{p}_E \times \mathbf{E}_1 + \mathbf{p}_H \times \mathbf{H}_1 - \frac{ea^2}{3c}\dot{\mathbf{H}}_1 = \frac{e}{6\pi ac}\dot{\mathbf{p}}_H + \frac{e}{4\pi ac^2}\mathbf{p}_E \times \mathbf{f}. \qquad (59\text{–}41)$$

It follows from this equation that the angular momentum of a spinning electron permanently at rest in the observer's inertial system is

$$\mathbf{G}_a = \frac{e}{6\pi a c}\,\mathbf{p}_H. \tag{59-42}$$

60. Equation of Motion of the Spinning Electron. — Since modern spectroscopic theories require an extra-nuclear electron in an atom to have an angular momentum $h/4\pi$, where h is Planck's constant $6.55(10)^{-27}$ erg sec, it follows from (59–42) that the magnetic moment of an atomic electron is

$$p_H = \frac{3ahc}{2e}. \tag{60-1}$$

If, now, the magnetic moment is due to rotation of charge of a single sign, it follows from (58–9) that the ratio of the equatorial linear velocity of spin to the velocity of light is

$$\frac{a\omega}{c} = \frac{9}{2}\left(\frac{hc}{e^2}\right) = \frac{9}{4}\left(\frac{hc}{2\pi e_s{}^2}\right) = 308,$$

where e_s is the elementary charge in electrostatic units. As velocities greater than that of light are inadmissible, we are forced to conclude that the magnetic moment is due to charges of opposite sign rotating in opposite senses about a common axis. In fact we may consider the electron with simple electric and magnetic moments, which we have been discussing, to consist of a non-rotating uniformly distributed charge e, two equal and opposite non-rotating charges responsible for the electric moment, and two equal and opposite uniformly distributed charges giving rise to a magnetic moment by their rotation in opposite senses about a common axis. Obviously the equations of motion for translation and rotation obtained in the last article are in no way altered by this modification in our concept of the nature of the electron.

Since the experimental evidence indicates clearly the presence of a magnetic moment but is less certain regarding the existence of an electric moment, let us consider first an electron which has a resultant charge e and a magnetic moment $\mathbf{p}_H$, but no electric moment. If we denote the impressed force by $\mathscr{K}_1$ and the impressed torque by $\mathscr{L}_1$, the equations of motion (59–36) and (59–41) become

$$\mathscr{K}_1 = \frac{e^2}{6\pi ac^2}\,\mathbf{f} + \frac{1}{20\pi a^3c^2}\left\{9p_H{}^2\mathbf{f} - 7\mathbf{p}_H\cdot\mathbf{f}\mathbf{p}_H\right\}, \tag{60–2}$$

$$\mathscr{L}_1 = \frac{e}{6\pi ac}\,\dot{\mathbf{p}}_H. \tag{60–3}$$

The magnetic mass reaction specified by the second term in (60–2) vanishes for no relative orientation of the vectors $\mathbf{f}$ and $\mathbf{p}_H$, and has a magnitude lying between the limits

$$\frac{p_H{}^2f}{10\pi a^3c^2} \quad\text{and}\quad \frac{9p_H{}^2f}{20\pi a^3c^2}.$$

Taking the smaller of these, in order to be conservative, the ratio of the magnetic mass reaction m_H to the electric mass reaction m_E specified by the first term in (60–2) is

$$\frac{m_H}{m_E} = \frac{3}{5}\frac{p_H{}^2}{e^2a^2} = \frac{27}{20}\left(\frac{hc}{e^2}\right)^2 = 6400,$$

showing that the electric mass reaction is quite negligible compared with the magnetic mass reaction. Ignoring the former, the known mass $9.0(10)^{-28}$ gm of the electron would require a radius

$$a = \frac{9h^2}{40\pi me^2} = \frac{9h^2}{160\pi^2me_s{}^2} = 1.2(10)^{-9}\text{cm},$$

which is one-tenth the kinetic theory radius of the entire atom. Not only does the presence of the magnetic mass reaction demand an electronic radius quite inconsistent with experimental indications, but (60–2) specifies a mass four and one-half times as large when the magnetic moment is perpendicular to the acceleration as when parallel, a phenomenon for which there is no experimental evidence.

Even more convincing evidence against the existence of the magnetic mass reaction is provided by the measurements of the gyromagnetic effect made by Barnett.[1] His experiments yield the ratio μ of the angular momentum to the magnetic moment of the gyroscopic entities responsible for the magnetic properties of matter. In accord with the definition of magnetic moment given in article 58 the magnetic moment due to the orbital motion of an electron around the

[1] S. J. Barnett, Rev. of Mod. Physics, 7, p. 129 (1935).

nucleus of an atom is $(e/2c)\mathbf{r} \times \mathbf{V}$, and the angular momentum is $m\mathbf{r} \times \mathbf{V}$, giving for the gyromagnetic ratio due to orbital motion

$$\mu = \frac{2mc}{e}, \tag{60–4}$$

whereas (59–42) gives for the gyromagnetic ratio due to spin

$$\mu = \frac{e}{6\pi ac} = \frac{mc}{e} \Big/ \left(1 + \frac{m_H}{m_E}\right), \tag{60–5}$$

where $m = m_E + m_H$ is the total mass of the electron. Barnett and others have found for every one of a considerable series of ferromagnetic elements and alloys a value of $\mu e/mc$ a few per cent larger than unity, suggesting very strongly that the magnetic properties of the substances investigated are due almost entirely to spinning electrons with mass $m = m_E$.

The evidence that an electron in the atom has a large angular momentum and at the same time the mass reaction of a non-rotating Lorentz electron leads us to investigate the possibility of eliminating the extra mass reaction by attributing to the electron an electric moment as well as a magnetic moment, a hypothesis made also in Dirac's quantum theory of the electron. Then we must replace the equations of motion (60–2) and (60–3) by (59–36) and (59–41). Furthermore we might justify theoretically our belief that the equation of motion of translation of the spinning electron is the same as that of the non-rotating Lorentz electron by applying the dynamical assumption, that the resultant force and resultant torque vanish relative to the inertial system in which the electron is momentarily at rest, separately (*a*) to the non-rotating uniformly distributed charge e, and (*b*) to the two moments $\mathbf{p}_E$ and $\mathbf{p}_H$. Then each of the equations (59–36) and (59–41) splits into two, giving

$$e\mathbf{E}_1 = \frac{e^2}{6\pi ac^2}\mathbf{f}, \tag{60–6}$$

$$\frac{1}{2}\left\{\mathbf{p}_E\cdot\nabla\mathbf{E}_1 + \mathbf{p}_H\cdot\nabla\mathbf{H}_1 + \nabla\mathbf{E}_1\cdot\mathbf{p}_E + \nabla\mathbf{H}_1\cdot\mathbf{p}_H - \frac{1}{c}\dot{\mathbf{E}}_1\times\mathbf{p}_H - \frac{1}{c}\dot{\mathbf{H}}_1\times\mathbf{p}_E\right\}$$
$$= \frac{1}{4\pi a^3 c}\frac{d}{dt}(\mathbf{p}_H \times \mathbf{p}_E) + \frac{1}{10\pi a^3 c^2}\{p_E{}^2\mathbf{f} + (\mathbf{p}_E \times \mathbf{f}) \times \mathbf{p}_E\}$$
$$+ \frac{1}{4\pi a^3 c^2}(\mathbf{p}_H \times \mathbf{f}) \times \mathbf{p}_H + \frac{1}{10\pi a^3 c^2}\{p_H{}^2\mathbf{f} + (\mathbf{p}_H \times \mathbf{f}) \times \mathbf{p}_H\}, \tag{60–7}$$

for translation, and

$$-\frac{ea^2}{3c}\dot{\mathbf{H}}_1 = \frac{e}{6\pi ac}\dot{\mathbf{p}}_H + \frac{e}{12\pi ac^2}\mathbf{p}_E \times \mathbf{f}, \tag{60–8}$$

$$\mathbf{p}_H \times \mathbf{H}_1 = 0, \tag{60–9}$$

for rotation.

These equations form an interesting basis for speculation, but do not seem to shed any light on the puzzling phenomena of the quantum theory. Therefore we shall dismiss them without further investigation.

A radically different method of eliminating the magnetic mass of the spinning electron which does not require the introduction of an electric moment consists in assuming that the force on an element of charge of the spinning electron, relative to the inertial system in which its center is momentarily at rest, is given by $\mathbf{E}de$ instead of by $\{\mathbf{E} + (1/c)\mathbf{V} \times \mathbf{H}\}de$. Then (59–36) and (59–41) reduce to

$$e\mathbf{E}_1 = \frac{e^2}{6\pi ac^2}\mathbf{f}, \tag{60–10}$$

$$-\frac{ea^2}{3c}\dot{\mathbf{H}}_1 = \frac{e}{6\pi ac}\dot{\mathbf{p}}_H, \tag{60–11}$$

the second of which requires that $\mathbf{p}_H$ be a linear function of $\mathbf{H}$ and therefore not constant in magnitude in a varying magnetic field. Equation (57–12), however, remains valid for an electron in motion relative to the observer's inertial system. This proposal, however, involves such a radical modification of such fundamental concepts of electromagnetic theory as electromagnetic energy, Poynting flux, etc., that it will not be considered further.

In the subsequent development of electromagnetic theory we shall employ the equations of motion (57–10) and (57–12) of the non-rotating Lorentz electron, without committing ourselves to any specific device for eliminating the magnetic mass due to spin. Nevertheless we must recognize the possibility of additional equations, such as (60–7), which may impose a limitation on the types of motion electromagnetically possible.

61. Generalization of the Force Equation. — In article 53 we saw how the field equations could be generalized so as to take account of

the existence of magnetic as well as electric charges. Now we shall effect a corresponding generalization of the force equation (57–22), using the notation of article 53.

We define the force on an element of magnetic charge de_H by $\mathbf{H}de_H$ relative to the inertial system in which the charge is momentarily at rest, where $\mathbf{H}$ is the resultant magnetic intensity. Next we make the dynamical assumption that, relative to the inertial system in which it is momentarily at rest, the resultant force on a magnetic electron always vanishes. This gives an equation of motion analogous to (57–10) for the electric electron, $\mathbf{E}_1$ being replaced by the resultant external magnetic field $\mathbf{H}_1$. To obtain the equation of motion of a magnetic electron relative to an inertial system with respect to which it has a velocity $\mathbf{V}_H$ we proceed exactly as in article 57, using the transformation (53–2). This leads to the force equation

$$\mathscr{F}_H = \rho_H \left\{ \mathbf{H} - \frac{1}{c}\mathbf{V}_H \times \mathbf{E} \right\} \qquad (61\text{–}1)$$

for the force per unit volume on the magnetic charge, which differs only in the sign of the second term from the force equation

$$\mathscr{F}_E = \rho_E \left\{ \mathbf{E} + \frac{1}{c}\mathbf{V}_E \times \mathbf{H} \right\} \qquad (61\text{–}2)$$

specified by (57–22) for the force per unit volume on the electric charge.

Adding (61–1) and (61–2) we have for the total force per unit volume

$$\mathscr{F} = \rho_E \left\{ \mathbf{E} + \frac{1}{c}\mathbf{V}_E \times \mathbf{H} \right\} + \rho_H \left\{ \mathbf{H} - \frac{1}{c}\mathbf{V}_H \times \mathbf{E} \right\}. \qquad (61\text{–}3)$$

It is often stated that no magnetic charges exist in nature, and that therefore the terms in ρ_H in this equation are without physical significance. On the contrary, we shall show now that, if every elementary charged particle contains electric and magnetic charges in the same ratio α, no electromagnetic experiment can reveal the value of α. Under these circumstances $\rho_H = \alpha\rho_E$, where α is a constant, and $\mathbf{V}_E = \mathbf{V}_H \equiv \mathbf{V}$. Therefore the field equations (53–9) and the force equation (61–3) become

$$\left.\begin{aligned}
&\nabla\cdot\mathbf{E} = \rho_E, \quad (a) \qquad \nabla\cdot\mathbf{H} = \alpha\rho_E, \quad (b)\\
&\nabla\times\mathbf{E} = -\frac{1}{c}(\dot{\mathbf{H}} + \alpha\rho_E\mathbf{V}), \ (c) \ \nabla\times\mathbf{H} = \frac{1}{c}(\dot{\mathbf{E}} + \rho_E\mathbf{V}), \ (d)\\
&\mathscr{F} = \rho_E(\mathbf{E} + \alpha\mathbf{H}) + \frac{1}{c}\rho_E\mathbf{V}\times(\mathbf{H} - \alpha\mathbf{E}). \quad (e)
\end{aligned}\right\} (61\text{–}4)$$

If we put

$$\mathbf{A} \equiv \frac{\mathbf{E} + \alpha\mathbf{H}}{\sqrt{1 + \alpha^2}}, \quad \mathbf{B} \equiv \frac{\mathbf{H} - \alpha\mathbf{E}}{\sqrt{1 + \alpha^2}}, \quad \rho \equiv \rho_E\sqrt{1 + \alpha^2} = \sqrt{\rho_E^{\,2} + \rho_H^{\,2}},$$

the fundamental equations (61–4) may be written

$$\left.\begin{aligned}
&\nabla\cdot\mathbf{A} = \rho, \quad (a) \qquad \nabla\cdot\mathbf{B} = 0, \quad (b)\\
&\nabla\times\mathbf{A} = -\frac{1}{c}\dot{\mathbf{B}}, \quad (c) \qquad \nabla\times\mathbf{B} = \frac{1}{c}(\dot{\mathbf{A}} + \rho\mathbf{V}), \quad (d)\\
&\mathscr{F} = \rho\left\{\mathbf{A} + \frac{1}{c}\mathbf{V}\times\mathbf{B}\right\}, \quad (e)
\end{aligned}\right\} (61\text{–}5)$$

which are identical in form with the equations (57–23) obtained on the assumption that only electric charge exists in nature. There is no experimental evidence, therefore, to justify the common assertion that only electric charges and no magnetic charges are present in the world of experience. If the reverse were true, or if electric and magnetic charges occurred combined in any fixed ratio, all electromagnetic phenomena would take place in exactly the same way. No electromagnetic experiment could reveal the proportions in which the two types of charge might exist.

CHAPTER 5

MATERIAL MEDIA

62. Electromagnetic Equations in Material Media. — By the electric and magnetic intensities in a material medium we shall understand the mean values of **E** and **H**, respectively, averaged over a volume $\Delta\tau$ large enough to contain a great many atoms. These definitions ignore the fluctuations in field strength which may occur as we pass from atom to atom, or from electron to proton, and specify the quantities which we are actually able to measure experimentally. Similarly when we speak of the charge density or of the current density in a medium we shall refer to the mean values of ρ or of $\rho\mathbf{V}$ averaged over a volume containing a large number of atoms.

Charges and currents in a medium may conveniently be grouped into three classes. First, we have charged particles which are free to move through the medium under the influence of an impressed field, except in so far as they are hindered by collisions with other charged or uncharged particles. These charges we shall call *free charges* and indicate by letters without subscripts. In a metallic conductor they consist of electrons; in an electrolyte they are positively or negatively charged atoms or groups of atoms known as ions; in an ionized gas they may be in part electrons and in part molecular ions. Free charges accumulated on the surface of a metallic conductor or deposited on the surface of an insulator may be surrounded by a dielectric medium, or the current produced by the motion of free charges in a metallic circuit may be immersed in a permeable medium. In the first case the electric field, and in the second the magnetic field, to which the free charges give rise, polarizes the surrounding medium. The nature of this polarization is an important part of our subsequent investigation.

Second, we have charges which are bound to the atom. These may consist of pairs of equal and opposite charges a fixed distance apart constituting *permanent electric dipoles*, or of pairs of equal and opposite charges whose electrical centers coincide in the absence of an impressed field but which are slightly separated against an elastic

force of restitution by an applied electric field so as to form *induced electric dipoles*. A dipole whose moment changes with the impressed field but does not vanish when the field is removed, may be considered as a combination of a permanent and an induced dipole. The charge in the body or on the surface of a medium due to the orientation of permanent dipoles or to the production of induced dipoles by an impressed field is called a *polarization charge* and will be distinguished by the subscript P. Dielectrics owe their characteristic properties to the presence of such electric dipoles in the medium.

Third, there exist in the atoms or molecules of a medium currents flowing around closed paths. These *Ampèrian currents* may be due in part to the orbital motion of electrons revolving about the nucleus of an atom, but they probably have their chief origin in electron spin. We shall distinguish Ampèrian currents, which are responsible for the magnetic properties of material media, by the subscript I. If the vector sum of the magnetic moments of a group of rigidly connected Ampèrian currents in an ultimate particle of the medium does not vanish, the medium is *paramagnetic* or *ferromagnetic*. Only in this case is a torque exerted on the particle by an impressed magnetic field, but, no matter whether the resultant magnetic moment vanishes or not, a changing magnetic field gives rise to an induced electromotive force which tends to weaken the magnetic moments parallel to $\mathbf{H}$ and to strengthen those opposite to $\mathbf{H}$. This last effect is known as *diamagnetism*.

Finally, since electric charge is conserved, the equation of continuity

$$\nabla \cdot \overline{\rho \mathbf{V}} + \frac{\partial \rho}{\partial t} = 0 \tag{62–1}$$

must be satisfied for each type of charge in a material medium.

First we shall calculate the charge and current density in a medium at rest in the observer's inertial system due to the electric dipoles which it contains. As the normal state of the medium we take that in which the permanent dipoles are oriented at random and the induced dipoles have zero moments. Evidently the mean charge density is everywhere zero in this state. If the permanent dipoles are turned from their random orientations or induced dipoles are formed by an applied electric field, the medium is said to be *polarized*, the *polarization* $\mathbf{P}$ being defined as the vector sum of the electric moments of all the dipoles in a small volume, divided by the volume.

Let q_i and $-q_i$ be two point charges constituting an electric dipole, $\mathbf{r}_{1i}$ being the position vector of q_i and $\mathbf{r}_{2i}$ that of $-q_i$ when the medium is in its normal or unpolarized state. If the dipole is a permanent one, $\mathbf{r}_{2i} \neq \mathbf{r}_{1i}$, whereas if it is an induced dipole, $\mathbf{r}_{2i} = \mathbf{r}_{1i}$. The electric moment of the dipole is then

$$\mathbf{p}_E = q_i(\mathbf{r}_{1i} - \mathbf{r}_{2i})$$

and, when the medium is polarized, the change in the electric moment is

$$\Delta\mathbf{p}_E = q_i(\Delta\mathbf{r}_{1i} - \Delta\mathbf{r}_{2i}).$$

If there are n_i dipoles per unit volume of this type, the total electric moment per unit volume when the medium is polarized is

$$\mathbf{P} = \sum_i n_i q_i(\Delta\mathbf{r}_{1i} - \Delta\mathbf{r}_{2i}),$$

since the mean electric moment per unit volume is zero when the medium is in its normal state.

Now we are ready to calculate the charge passing through a fixed element of area $d\boldsymbol{\sigma}$ from the negative to the positive side when the medium is polarized. Since n_i charges q_i per unit volume are displaced a distance $\Delta\mathbf{r}_{1i}$ and n_i charges $-q_i$ a distance $\Delta\mathbf{r}_{2i}$, this charge is

$$\sum_i n_i q_i \Delta\mathbf{r}_{1i}\cdot d\boldsymbol{\sigma} - \sum_i n_i q_i \Delta\mathbf{r}_{2i}\cdot d\boldsymbol{\sigma} = \mathbf{P}\cdot d\boldsymbol{\sigma}.$$

Therefore, if ρ_P denotes the polarization charge per unit volume, the net charge in a fixed volume τ enclosed in a surface σ, due to the polarization of the medium, is

$$\int_\tau \rho_P d\tau = -\int_\sigma \mathbf{P}\cdot d\boldsymbol{\sigma} = -\int_\tau \nabla\cdot\mathbf{P}d\tau$$

by Gauss' theorem, or

$$\rho_P = -\nabla\cdot\mathbf{P}. \qquad (62\text{–}2)$$

It follows at once from (62–1) and (62–2) that the current density due to polarization of the medium is

$$(\rho\mathbf{V})_P = \dot{\mathbf{P}}. \qquad (62\text{–}3)$$

Often it is convenient to express the polarization charge on the transition layer between two dielectrics (1) and (2) separately from

the charge in the interior. To do this, enclose an area $\Delta\sigma$ of the transition layer (Fig. 56) between the two dielectrics in a pill-box shaped surface with bases parallel to this layer. If $\rho_{\sigma P}$ is the polarization charge per unit area of the transition layer, no free charge being present, and $\mathbf{n}_1$ is a unit vector normal to the layer directed from medium (1) toward medium (2), we have for the net charge in the pill-box,

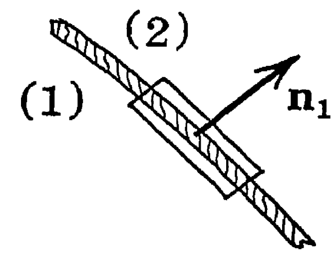

Fig. 56.

$$\rho_{\sigma P}\Delta\sigma = -\int_\sigma \mathbf{P}\cdot d\boldsymbol{\sigma}$$

$$= \mathbf{P}_1\cdot\mathbf{n}_1\Delta\sigma - \mathbf{P}_2\cdot\mathbf{n}_1\Delta\sigma,$$

where the surface integral is taken over the bases of the pill-box, the integral over the short curved portion of the surface being ignored as negligible. Therefore

$$\rho_{\sigma P} = (\mathbf{P}_1 - \mathbf{P}_2)\cdot\mathbf{n}_1. \tag{62–4}$$

If the region (2) is empty space, $\mathbf{P}_2 = 0$ and the polarization charge on the free surface of dielectric (1) is

$$\rho_{\sigma P} = \mathbf{P}_1\cdot\mathbf{n}_1 \tag{62–5}$$

per unit area.

Next we shall calculate the current density and the charge due to the Ampèrian currents in the medium. If i_0 is the current and $\mathbf{s}$ the vector area of an Ampèrian circuit, its magnetic moment $\mathbf{p}_{II}$ is equal to $i_0\mathbf{s}/c$ in accord with the definition given in article 58. Now consider a bounded surface σ lying partly or wholly in the medium. The current i_I through σ, from the negative to the positive side of the surface, due to the Ampèrian circuits which it cuts, is contributed entirely by those circuits which are threaded by the periphery λ of σ, for these are the only Ampèrian circuits which do not intersect σ an even number of times and therefore make equal positive and negative contributions to the current through the surface.

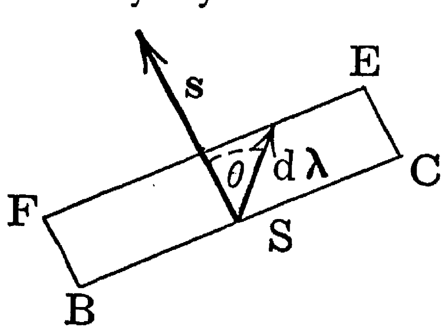

Fig. 57.

Let $d\boldsymbol{\lambda}$ (Fig. 57) be a vector element of the periphery of σ, and let us direct our attention to those Ampèrian currents alone whose parallel vector areas make an angle θ with $d\boldsymbol{\lambda}$. Through the origin and terminus of $d\boldsymbol{\lambda}$ pass planes of area S large compared with s at right angles to $\mathbf{s}$, forming the prism $BCEF$ of volume $Sd\lambda\cos\theta$. If

there are dn Ampèrian circuits of the type considered per unit volume, there are $Sd\lambda \cos\theta\, dn$ in the prism under consideration, and the proportion of these threaded by $d\lambda$ is s/S. Therefore $sd\lambda \cos\theta\, dn$ is the total number threaded, and, as each makes the contribution i_0 to the current through σ,

$$d^2 i_I = i_0 s d\lambda \cos\theta\, dn = c\mathbf{p}_H \cdot d\boldsymbol{\lambda}\, dn.$$

But $\mathbf{p}_H dn$ is the contribution $d\mathbf{I}$ to the magnetic moment $\mathbf{I}$ per unit volume made by the circuits under consideration. Hence

$$d^2 i_I = c d\mathbf{I} \cdot d\boldsymbol{\lambda},$$

and summing up over all orientations

$$di_I = c\mathbf{I} \cdot d\boldsymbol{\lambda}.$$

Finally, integrating around the periphery of σ, we find for the total current through the surface due to the Ampèrian circuits which are cut,

$$i_I = c \oint \mathbf{I} \cdot d\boldsymbol{\lambda}.$$

The vector $\mathbf{I}$ is known as the *intensity of magnetization* and is the magnetic analog of the polarization $\mathbf{P}$.

If, then, $(\rho\mathbf{V})_I$ is the current density in the medium due to the Ampèrian currents,

$$\int_\sigma (\rho\mathbf{V})_I \cdot d\boldsymbol{\sigma} = c \oint \mathbf{I} \cdot d\boldsymbol{\lambda} = c \int_\sigma \nabla \times \mathbf{I} \cdot d\boldsymbol{\sigma}$$

for any surface σ, and consequently

$$(\rho\mathbf{V})_I = c\nabla \times \mathbf{I}. \tag{62–6}$$

As the divergence of the curl is identically zero and the medium is supposed to be initially uncharged, it follows from (62–1) that

$$\rho_I = 0. \tag{62–7}$$

We conclude that magnetization gives rise to no charges in the medium.

Combining these equations with (62–2) and (62–3) we have for the total charge density in a medium containing free charges, bound charges and Ampèrian currents

$$\rho + \rho_P + \rho_I = \rho - \nabla \cdot \mathbf{P}, \tag{62–8}$$

and for the total current density

$$\rho\mathbf{V} + (\rho\mathbf{V})_P + (\rho\mathbf{V})_I = \rho\mathbf{V} + \dot{\mathbf{P}} + c\nabla \times \mathbf{I}. \tag{62–9}$$

The mean magnetic intensity in a material medium is generally called the *magnetic induction* and designated by the letter **B**. If, in accord with this convention, we replace **H** by **B** in the field equations (51–8), and substitute the expressions just found for the charge and current densities, we have, on collecting like terms,

$$\nabla \cdot (\mathbf{E} + \mathbf{P}) = \rho, \quad (a) \qquad \nabla \cdot \mathbf{B} = 0, \quad (b)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \ (c) \qquad \nabla \times (\mathbf{B} - \mathbf{I}) = \frac{1}{c}(\dot{\mathbf{E}} + \dot{\mathbf{P}} + \rho\mathbf{V}). \quad (d)$$

The form of these equations suggests the desirability of defining two new vector functions **D** and **F** by the relations

$$\mathbf{D} \equiv \mathbf{E} + \mathbf{P}, \tag{62–10}$$

$$\mathbf{F} \equiv \mathbf{B} - \mathbf{I}. \tag{62–11}$$

These are known as the *electric displacement* and the *magnetic force* respectively. Writing the field equations in terms of them, and appending the force equation for the free charge alone, we have altogether the electromagnetic equations

$$\left.\begin{aligned} &\nabla \cdot \mathbf{D} = \rho, \quad (a) \qquad \nabla \cdot \mathbf{B} = 0, \quad (b) \\ &\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \quad (c) \qquad \nabla \times \mathbf{F} = \frac{1}{c}(\dot{\mathbf{D}} + \rho\mathbf{V}), \quad (d) \\ &\mathscr{F} = \rho\left\{\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B}\right\}, \quad (e) \end{aligned}\right\} \tag{62–12}$$

for free charges in a material medium which is at rest in the observer's inertial system. These equations reduce to (57–23) for charges in empty space, for there $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{F} = \mathbf{H}$.

As usual the electric field **E** and the magnetic field **B** are expressible as derivatives of potentials in the forms

$$\left.\begin{aligned} \mathbf{E} &= -\nabla\Phi - \frac{1}{c}\dot{\mathbf{A}}, \\ \mathbf{B} &= \nabla \times \mathbf{A}, \end{aligned}\right\} \tag{62–13}$$

where Φ is the sum of the scalar potential of the free charges of density

ρ and the scalar potential of the polarization charges of density $\rho_P = -\nabla \cdot \mathbf{P}$ (there is no contribution from the magnetization since $\rho_I = 0$ by (62–7)), and $\mathbf{A}$ is the sum of the vector potential of the free currents of density $\rho\mathbf{V}$, the vector potential of the polarization currents of density $\dot{\mathbf{P}}$, and the vector potential of the magnetization currents of density $c\nabla \times \mathbf{I}$.

We obtain an elegant physical interpretation of the vector function $\mathbf{F}$ by considering alone the field due to the Ampèrian currents in the medium. Distinguishing this field by the subscript H, the field equations (*a*), (*b*), (*c*), (*d*) of (62–12) become

$$\nabla \cdot \mathbf{E}_H = 0, \quad (a) \qquad \nabla \cdot \mathbf{B}_H = 0, \quad (b)$$

$$\nabla \times \mathbf{E}_H = -\frac{1}{c}\dot{\mathbf{B}}_H, \quad (c) \qquad \nabla \times \mathbf{F}_H = \frac{1}{c}\dot{\mathbf{E}}_H. \quad (d)$$

Making use of the defining relation $\mathbf{F}_H = \mathbf{B}_H - \mathbf{I}$, these may be written

$$\left.\begin{aligned} &\nabla \cdot \mathbf{E}_H = 0, \quad (a) && \nabla \cdot \mathbf{F}_H = -\nabla \cdot \mathbf{I}, \quad (b) \\ &\nabla \times \mathbf{E}_H = -\frac{1}{c}(\dot{\mathbf{F}}_H + \dot{\mathbf{I}}), \quad (c) && \nabla \times \mathbf{F}_H = \frac{1}{c}\dot{\mathbf{E}}_H. \quad (d) \end{aligned}\right\} \quad (62\text{–}14)$$

Comparing with (53–6), we observe that these are exactly the field equations of a distribution of *magnetic charge* of charge density $-\nabla \cdot \mathbf{I}$ per unit volume and current density $\dot{\mathbf{I}}$ per unit cross-section. But comparison with (62–2) and (62–3) reveals the fact that $-\nabla \cdot \mathbf{I}$ and $\dot{\mathbf{I}}$ are just the magnetic charge density and magnetic current density that would exist if the medium were composed of actual magnetic dipoles instead of Ampèrian currents, each dipole having a magnetic moment equal to that of the Ampèrian current which it replaces. Hence $\mathbf{F}_H$ is the magnetic field that would be produced by substituting for each Ampèrian current a magnetic dipole of equal moment. We can therefore write

$$\left.\begin{aligned} \mathbf{F}_H &= -\nabla\Phi_H - \frac{1}{c}\dot{\mathbf{A}}_H, \\ \mathbf{E}_H &= -\nabla \times \mathbf{A}_H, \end{aligned}\right\} \quad (62\text{–}15)$$

in accord with (53–4) and (53–5), where Φ_H is the scalar potential of the magnetic charge density $-\nabla \cdot \mathbf{I}$, and $\mathbf{A}_H$ the vector potential of the magnetic current density $\dot{\mathbf{I}}$. If free and polarization charges and cur-

rents are present as well as Ampèrian currents, the resultant **E** is the vector sum of the electric intensity due to these charges and the field $\mathbf{E}_H$, and the resultant **F** is the vector sum of the magnetic force due to these currents and the field $\mathbf{F}_H$.

The boundary conditions at the surface of discontinuity between two material media, or between a material medium and empty space, are obtained from equations (62–12) in precisely the same manner as those at a charged surface in empty space were obtained from (51–8) or (51–9). We need only write down the results, using the notation of Fig. 47. Corresponding to (52–1) we have

$$\left.\begin{aligned} \mathbf{D}_2\cdot\mathbf{n}_1 &= \mathbf{D}_1\cdot\mathbf{n}_1 + \rho_\sigma, \\ \mathbf{B}_2\cdot\mathbf{n}_1 &= \mathbf{B}_1\cdot\mathbf{n}_1, \end{aligned}\right\} \qquad (62\text{–}16)$$

and corresponding to (52–2)

$$\left.\begin{aligned} \mathbf{E}_2\cdot\mathbf{t}_1 &= \mathbf{E}_1\cdot\mathbf{t}_1, \\ \mathbf{F}_2\cdot\mathbf{t}_1 &= \mathbf{F}_1\cdot\mathbf{t}_1 + \frac{1}{c}\,\rho_\sigma\,\mathbf{V}\times\mathbf{n}_1\cdot\mathbf{t}_1, \end{aligned}\right\} \qquad (62\text{–}17)$$

where ρ_σ is the free charge per unit area on the surface of discontinuity and **V** the velocity with which it is moving.

As before the scalar potential Φ and the vector potential **A** of equations (62–13) are continuous at the interface, and also the potentials Φ_H and $\mathbf{A}_H$ in terms of which we can express the portion of the total field due to the magnetization of the medium. For the latter field alone the boundary conditions may be put in the form

$$\left.\begin{aligned} \mathbf{E}_{H2}\cdot\mathbf{n}_1 &= \mathbf{E}_{H1}\cdot\mathbf{n}_1, \\ \mathbf{F}_{H2}\cdot\mathbf{n}_1 &= \mathbf{F}_{H1}\cdot\mathbf{n}_1 + (\mathbf{I}_1 - \mathbf{I}_2)\cdot\mathbf{n}_1, \end{aligned}\right\} \qquad (62\text{–}18)$$

and

$$\left.\begin{aligned} \mathbf{E}_{H2}\cdot\mathbf{t}_1 &= \mathbf{E}_{H1}\cdot\mathbf{t}_1, \\ \mathbf{F}_{H2}\cdot\mathbf{t}_1 &= \mathbf{F}_{H1}\cdot\mathbf{t}_1, \end{aligned}\right\} \qquad (62\text{–}19)$$

the effective magnetic charge per unit area of the surface separating the two media being $(\mathbf{I}_1 - \mathbf{I}_2)\cdot\mathbf{n}_1$, an expression analogous to (62–4) for a polarized dielectric.

As they stand, equations (62–12) do not suffice to describe the field in a material medium in terms of an assigned distribution of free charge density ρ and free current density $\rho\mathbf{V}$, for they fail to specify the relation between **D** and **E** on the one hand, and that between

B and **F** on the other. The establishment of such relations, which requires more detailed assumptions regarding the physical properties of material media, will occupy our attention in the next few articles. The reader must be warned, however, not to place too much credence in the numerical part of the coefficients calculated on the over-simplified hypotheses we shall make, for we know far too little about the structure of matter to make more than crude approximations to these coefficients. Some improvement in such calculations may be effected by using quantum mechanics, but that is outside the scope of this book.

Problem 62a. Show that **D** in a dielectric is the electric field that would exist if each electric dipole were replaced by a closed magnetic current of electric moment equal to that of the dipole.

Problem 62b. Show that the surface current flowing in the transition layer between two magnetic media is $c(\mathbf{I}_1 - \mathbf{I}_2) \times \mathbf{n}_1$ per unit breadth of the layer, where $\mathbf{n}_1$ is a unit vector normal to the surface of separation, as in Fig. 56. Apply to a uniformly magnetized cylinder.

63. Dielectrics.—In this article we shall deduce constitutive relations between **D** and **E** for a dielectric. The initial step is the determination of the field responsible for the polarization of the medium. By definition the electric intensity **E** in a dielectric is the mean field averaged over a volume $\Delta\tau$ large enough to contain a great many atoms. This field may be analyzed into two parts: the mean field $\mathbf{E}_2$ due to the electric dipoles actually lying in the region $\Delta\tau$ and the field $\mathbf{E}_1$ due to the dipoles and other charges outside this region. We shall calculate $\mathbf{E}_2$, taking $\Delta\tau$ as a sphere of radius b (Fig. 58), since the sphere represents the average shape of a large number of small volumes chosen at random. First consider the mean field $\mathbf{E}_{2q}$ due to the single point charge q situated at a distance **R** from the center O of the sphere. Retaining only the zero order term in (56–9), we have

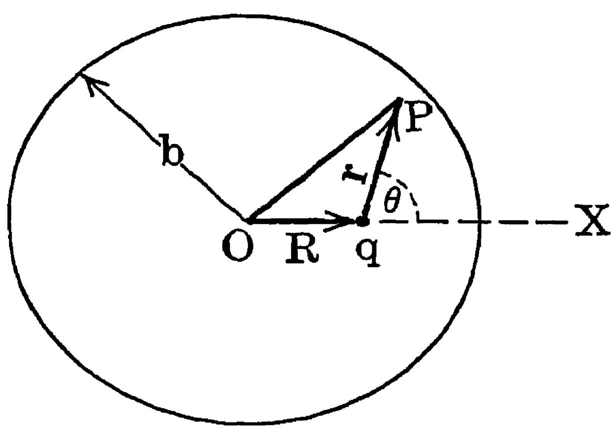

FIG. 58.

$$\mathbf{E}_{2q} = \frac{1}{\Delta\tau}\int_{\Delta\tau} \frac{q}{4\pi r^3}\,\mathbf{r}\,d\tau = \mathbf{i}\,\frac{q}{2\Delta\tau}\int_0^{\pi}\int_0^{r_2} \cos\theta \sin\theta \, dr\, d\theta,$$

where θ is the angle between $\mathbf{r}$ and the common direction of $\mathbf{R}$ and the X axis. The upper limit r_2 of r is

$$r_2 = - R\cos\theta + \sqrt{R^2\cos^2\theta + b^2 - R^2}.$$

Hence

$$\mathbf{E}_{2q} = - \boldsymbol{i}\frac{qR}{3\Delta\tau} = - \frac{q}{3\Delta\tau}\mathbf{R}. \tag{63–1}$$

If, now, there are present inside the sphere two charges q and $-q$ at vector distances $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively from O, the mean field $\mathbf{E}_{2p}$ due to the dipole of moment $\mathbf{p}_E = q(\mathbf{R}_1 - \mathbf{R}_2)$ is

$$\mathbf{E}_{2p} = - \frac{1}{3\Delta\tau}\mathbf{p}_E,$$

independent of the location of the dipole.

In the actual dielectric there are a large number of dipoles inside $\Delta\tau$, the polarization $\mathbf{P}$ being the vector sum of their electric moments divided by $\Delta\tau$. Hence

$$\mathbf{E}_2 = - \frac{1}{3}\frac{\sum_i \mathbf{p}_{Ei}}{\Delta\tau} = - \frac{1}{3}\mathbf{P}. \tag{63–2}$$

Therefore the field responsible for the polarization of the medium in the region $\Delta\tau$ is

$$\mathbf{E}_1 = \mathbf{E} - \mathbf{E}_2 = \mathbf{E} + \tfrac{1}{3}\mathbf{P}. \tag{63–3}$$

On the other hand, if we have a uniform distribution of unbound charges of either one or both signs, as in the case of the free electrons in a metallic conductor, it follows from (63–1) that $\mathbf{E}_2 = 0$ and therefore that the field responsible for the conductivity of a metal is the mean electric intensity $\mathbf{E}$ itself. The difference in the results obtained in the two cases lies in the fact that we are considering the dipole as the ultimate entity in the first case and the charged particle in the second.

We would expect the polarization in $\Delta\tau$ to be a linear vector function of $\mathbf{E}_1$, a supposition which is well confirmed by experiment. Hence we may write for either an anisotropic or an isotropic dielectric

$$\mathbf{P} = \boldsymbol{\alpha}\cdot\mathbf{E}_1, \tag{63–4}$$

where $\boldsymbol{\alpha}$ is the dyadic

$$\begin{aligned}\boldsymbol{\alpha} \equiv \alpha_{11}\boldsymbol{ii} &+ \alpha_{12}\boldsymbol{ij} + \alpha_{13}\boldsymbol{ik}\\ &+ \alpha_{21}\boldsymbol{ji} + \alpha_{22}\boldsymbol{jj} + \alpha_{23}\boldsymbol{jk}\\ &+ \alpha_{31}\boldsymbol{ki} + \alpha_{32}\boldsymbol{kj} + \alpha_{33}\boldsymbol{kk}.\end{aligned}$$

The work done by the field $\mathbf{E}_1$ in increasing the polarization from $\mathbf{P}$ to $\mathbf{P} + d\mathbf{P}$ is

$$d\mathscr{W} = \mathbf{E}_1 \cdot d\mathbf{P} = \mathbf{E}_1 \cdot \boldsymbol{\alpha} \cdot d\mathbf{E}_1.$$

We shall consider only perfect dielectrics in which no dissipation of energy occurs. Then, $d\mathscr{W}$ must be an exact differential and $\alpha_{ij} = \alpha_{ji}$. As $\boldsymbol{\alpha}$ is symmetric in this case, it can be put in the normal form

$$\boldsymbol{\alpha} = \alpha_x \boldsymbol{ii} + \alpha_y \boldsymbol{jj} + \alpha_z \boldsymbol{kk} \tag{63-5}$$

by a proper orientation of the axes. The directions so specified are the principal electrical axes of the dielectric. If the dielectric is isotropic, $\alpha_x = \alpha_y = \alpha_z \equiv \alpha$ and (63–4) assumes the simpler form

$$\mathbf{P} = \alpha \mathbf{E}_1. \tag{63-6}$$

Eliminating $\mathbf{E}_1$ between (63–3) and (63–4) or (63–6) we get

$$\mathbf{P} = \begin{cases} \boldsymbol{\epsilon} \cdot \mathbf{E} & \text{for anisotropic dielectrics,} \\ \epsilon \mathbf{E} & \text{for isotropic dielectrics,} \end{cases} \tag{63-7}$$

where

$$\left.\begin{aligned} \boldsymbol{\epsilon} &\equiv \frac{\alpha_x}{1 - \frac{1}{3}\alpha_x} \boldsymbol{ii} + \frac{\alpha_y}{1 - \frac{1}{3}\alpha_y} \boldsymbol{jj} + \frac{\alpha_z}{1 - \frac{1}{3}\alpha_z} \boldsymbol{kk}, \\ \epsilon &\equiv \frac{\alpha}{1 - \frac{1}{3}\alpha}. \end{aligned}\right\} \tag{63-8}$$

The three elements of $\boldsymbol{\epsilon}$ are known as the *principal electric susceptibilities* of an anisotropic dielectric, and ϵ as the *electric susceptibility* of an isotropic dielectric. Finally, as $\mathbf{D} = \mathbf{E} + \mathbf{P}$,

$$\mathbf{D} = \begin{cases} \mathbf{K} \cdot \mathbf{E} & \text{for anisotropic dielectrics,} \\ \kappa \mathbf{E} & \text{for isotropic dielectrics,} \end{cases} \tag{63-9}$$

where

$$\left.\begin{aligned} \mathbf{K} &\equiv \kappa_x \boldsymbol{ii} + \kappa_y \boldsymbol{jj} + \kappa_z \boldsymbol{kk} \\ &\equiv \frac{1 + \frac{2}{3}\alpha_x}{1 - \frac{1}{3}\alpha_x} \boldsymbol{ii} + \frac{1 + \frac{2}{3}\alpha_y}{1 - \frac{1}{3}\alpha_y} \boldsymbol{jj} + \frac{1 + \frac{2}{3}\alpha_z}{1 - \frac{1}{3}\alpha_z} \boldsymbol{kk}, \\ \kappa &\equiv \frac{1 + \frac{2}{3}\alpha}{1 - \frac{1}{3}\alpha}. \end{aligned}\right\} \tag{63-10}$$

The elements κ_x, κ_y, κ_z of $\mathbf{K}$ are called the *principal permittivities* or *dielectric constants* of an anisotropic dielectric, and κ the *permittivity*

or *dielectric constant* of an isotropic dielectric. Solving (63–10) for $\boldsymbol{\alpha}$ or α we get

$$\left.\begin{aligned} \frac{\kappa_x - 1}{\kappa_x + 2}\,\mathbf{ii} + \frac{\kappa_y - 1}{\kappa_y + 2}\,\mathbf{jj} + \frac{\kappa_z - 1}{\kappa_z + 2}\,\mathbf{kk} &= \frac{1}{3}\boldsymbol{\alpha} \quad \text{for anisotropic dielectrics,} \\ \frac{\kappa - 1}{\kappa + 2} &= \frac{1}{3}\alpha \quad \text{for isotropic dielectrics.} \end{aligned}\right\} \quad (63\text{–}11)$$

Since $\boldsymbol{\alpha}$ and α, rather than $\mathbf{K}$ and κ, describe the physical characteristics of a dielectric, the important quantities to be determined experimentally are not κ_x, κ_y, κ_z or κ but $\dfrac{\kappa_x - 1}{\kappa_x + 2}$, $\dfrac{\kappa_y - 1}{\kappa_y + 2}$, $\dfrac{\kappa_z - 1}{\kappa_z + 2}$ or $\dfrac{\kappa - 1}{\kappa + 2}$.

It is important to note that no polarization charge can exist in the interior of a homogeneous isotropic dielectric in which no free charge is present. Consequently polarization charge in such a dielectric must reside entirely on the surface. To prove this, we have $\nabla\cdot\mathbf{E} = -\nabla\cdot\mathbf{P} = \rho_P$ from (62–10), (62–12*a*) and (62–2). But, as $\mathbf{D} = \kappa\mathbf{E}$, where κ is not a function of the coordinates in the interior of the dielectric on account of the homogeneity of the medium, (62–12*a*) gives $\nabla\cdot\mathbf{E} = 0$. Hence $\rho_P = 0$ everywhere except on the surface. Of course, this conclusion does not imply that the polarization current $\dot{\mathbf{P}}$ must vanish in the interior of the dielectric.

Finally, it follows from (51–10), where ρ represents the total charge per unit volume of whatever classification, that the scalar potential of any static field * must be a solution of Laplace's equation in the portions of a homogeneous isotropic dielectric in which no free charges are present.

We shall not attempt to calculate the values of the coefficients α_x, α_y, α_z for a typical anisotropic dielectric, but shall confine ourselves here to the simpler case of an isotropic dielectric which is polarized by a static or slowly changing electric field. We shall treat, first, a dielectric composed of permanent dipoles of moments $p_E = p_0$ (a constant), second, a dielectric the dipoles of which are induced by the impressed field, and third, a dielectric containing both permanent and induced dipoles.

The torque exerted on a permanent dipole by the field $\mathbf{E}_1$ in which it lies is $\mathbf{p}_0 \times \mathbf{E}_1$ by (58–21). Hence, if θ is the angle which the elec-

* The statement that the potential of the electrostatic field of charges immersed in a homogeneous isotropic dielectric is that of empty space divided by κ, which appears in many texts, is not of general validity, as is shown by the example on p. 293.

tric moment $\mathbf{p}_0$ makes with $\mathbf{E}_1$, the energy of the dipole due to its orientation in the field is

$$U = \int_{\pi/2}^{\theta} p_0 E_1 \sin\theta\, d\theta = -\, p_0 E_1 \cos\theta,$$

if we take the energy when $\mathbf{p}_0$ is at right angles to $\mathbf{E}_1$ as zero. Now the tendency of a field $\mathbf{E}_1$ to orient the permanent dipoles in a gaseous dielectric is opposed only by the disorganizing effect of thermal agitation, and even in many solids thermal agitation seems to be the chief opposing influence. We shall neglect other possible restraints, and make use of the expression

$$dn = Ae^{-U/kT} \sin\theta\, d\theta = Ae^{p_0 E_1 \cos\theta/kT} \sin\theta\, d\theta$$

given by statistical mechanics [1] for the number of dipoles per unit volume which make angles between θ and $\theta + d\theta$ with $\mathbf{E}_1$ when the medium is in thermal equilibrium at the absolute thermodynamic temperature T, where $k = 1.37(10)^{-16}$ erg/1° C is Boltzmann's constant. The total number n of dipoles per unit volume is obtained by integrating this expression with respect to θ between the limits 0 and π.

If we put $x \equiv p_0 E_1/kT$ and $\mu \equiv \cos\theta$, the polarization of the dielectric is

$$\begin{aligned} P &= \int_n p_0 \cos\theta\, dn = Ap_0 \int_0^{\pi} e^{x\cos\theta} \cos\theta \sin\theta\, d\theta \\ &= Ap_0 \int_{-1}^{1} e^{x\mu} \mu\, d\mu. \end{aligned}$$

But

$$n = A \int_0^{\pi} e^{x\cos\theta} \sin\theta\, d\theta = A \int_{-1}^{1} e^{x\mu} d\mu.$$

Therefore

$$\begin{aligned} P &= np_0 \frac{\int_{-1}^{1} e^{x\mu}\mu\, d\mu}{\int_{-1}^{1} e^{x\mu} d\mu} = np_0 \frac{d}{dx} \log \int_{-1}^{1} e^{x\mu} d\mu \\ &= np_0 \left\{ \frac{\cosh x}{\sinh x} - \frac{1}{x} \right\}. \end{aligned} \qquad (63\text{–}12)$$

[1] L. Page, *Theoretical Physics*, Chap. VIII.

Since the ratio cosh x/sinh x approaches unity as x increases without limit, the polarization of a dielectric should approach the saturation value np_0 for very large fields. However the fields available in the laboratory are too weak to saturate ordinary dielectrics and, in fact, the largest values of $x \equiv p_0E_1/kT$ attainable at ordinary temperatures are very small compared with unity. Therefore we can generally use the approximate expression

$$P = \frac{1}{3}\frac{np_0^2}{kT}E_1, \tag{63–13}$$

to which (63–12) reduces for small x. Using the value of α defined by this relation, (63–11) becomes

$$\frac{\kappa - 1}{\kappa + 2} = \frac{1}{9}\frac{np_0^2}{kT}. \tag{63–14}$$

In the case where the polarization of an isotropic dielectric is due to the presence of induced dipoles we can write

$$\mathbf{p}_E = \tau\mathbf{E}_1 \tag{63–15}$$

for the electric moment of an individual dipole, where τ is an atomic or molecular constant known as the *polarizability*. Then, if there are n such dipoles per unit volume,

$$\mathbf{P} = n\,\mathbf{p}_E = n\tau\mathbf{E}_1, \tag{63–16}$$

and putting the value of α so defined in (63–11),

$$\frac{\kappa - 1}{\kappa + 2} = \tfrac{1}{3}\,n\tau. \tag{63–17}$$

We can obtain an approximate value of τ by supposing the N electrons in an atom to be uniformly distributed through the volume of a sphere of radius a. The center of this sphere coincides with the nucleus of the atom when no external field is present, but, under the influence of the field E_1, it becomes displaced a distance $R < a$ sufficient to give rise to a force of restitution equal and opposite to that exerted by the external field. The force of restitution on the negative sphere is most easily obtained by calculating the equal and opposite reaction on the nucleus. The latter, being due to that portion of the charge on the negative sphere lying inside the spherical surface of radius R, considered as located at its center, is

$$\mathscr{K} = \frac{1}{4\pi R^2}\left\{\left(\frac{R}{a}\right)^3 Ne\right\}\left\{Ne\right\}\frac{\mathbf{R}}{R} = \frac{N^2e^2}{4\pi a^3}\mathbf{R}.$$

Hence, as the force exerted on the negative sphere by the impressed field is $Ne\mathbf{E}_1$, we find that $\mathbf{E}_1 = Ne\mathbf{R}/4\pi a^3 = \mathbf{p}_E/4\pi a^3$, or

$$\mathbf{p}_E = 4\pi a^3\mathbf{E}_1. \tag{63–18}$$

This makes $\tau = 4\pi a^3$ and (63–17) becomes

$$\frac{\kappa - 1}{\kappa + 2} = \frac{4}{3}\pi n a^3. \tag{63–19}$$

It is interesting to note that the right-hand side of (63–19) is just the fraction of the volume of the dielectric which is actually filled by the atoms of which it is composed. By measuring κ for a gas, the radius of the molecule can be computed if the number of molecules per unit volume is known.

Finally, let us consider the more general case of an isotropic dielectric containing n atoms or molecules per unit volume, each of which possesses a permanent dipole of moment p_0 and acquires an induced dipole under the action of an external field. Combining (63–14) and (63–17) we have in this case

$$\frac{\kappa - 1}{\kappa + 2} = \tfrac{1}{3}n\left(\tau + \frac{p_0^2}{3kT}\right). \tag{63–20}$$

As both τ and p_0^2 are atomic or molecular constants, this equation states that the ratio of $\kappa - 1$ to $\kappa + 2$ at constant temperature is proportional to the density, a theoretical prediction which has been verified for many gaseous dielectrics. Furthermore, by varying the temperature at constant density it is possible to separate the portion of the dielectric effect due to permanent dipoles from that due to induced dipoles. If $(\kappa - 1)/(\kappa + 2)$ is plotted against $1/kT$ for an actual dielectric, a straight line such as AB (Fig. 59) is generally found. If this line is horizontal, no permanent dipoles are present, whereas if it passes through the origin, the entire effect is due to permanent dipoles. From the intercept on the vertical axis $n\tau$ can be calculated, and from the slope np_0^2 can be obtained.

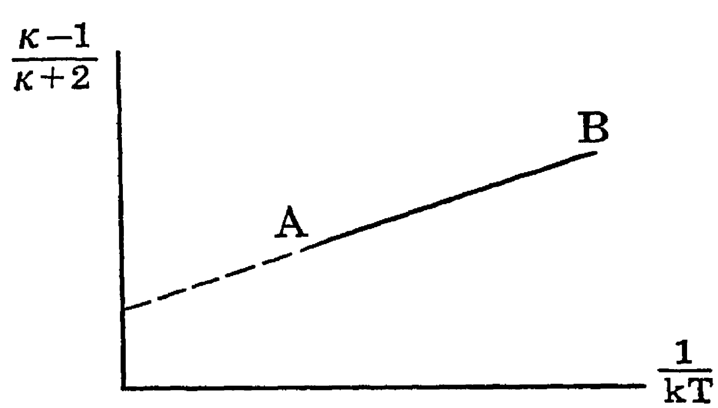

FIG. 59.

64. Steady Current in a Closed Circuit. — Before deducing constitutive relations between **B** and **F** for a magnetic medium composed of Ampèrian currents we must derive expressions for the magnetic field of a closed circuit in which a steady or slowly varying current is flowing, and for the force or torque which such a current experiences when located in an external magnetic field. Whatever the cross-section of the current may be, we can consider it to be made up of current filaments each of which carries a current i which has the same magnitude at any one instant all the way around the closed filament.

As the current is changing slowly if at all we can neglect all the terms except the first in the series (56–5) for the vector potential, a procedure all the more justified by the fact that the second term in the series makes no contribution to the magnetic intensity. If $d\boldsymbol{\lambda}$ is a vector element of the current filament, $e\mathbf{V} = i d\boldsymbol{\lambda}$, and the vector potential at the field-point x_1, y_1, z_1 due to the current element at the point x_2, y_2, z_2 is

$$d\mathbf{A} = \frac{i d\boldsymbol{\lambda}}{4\pi c r},$$

where

$$\mathbf{r} = \boldsymbol{i}(x_1 - x_2) + \boldsymbol{j}(y_1 - y_2) + \boldsymbol{k}(z_1 - z_2).$$

Therefore the vector potential due to the entire filament is

$$\mathbf{A} = \frac{i}{4\pi c} \oint \frac{d\boldsymbol{\lambda}}{r} \tag{64–1}$$

and, if we define ∇_1 and ∇_2 as in article 21,

$$\mathbf{H} = \nabla_1 \times \mathbf{A} = \frac{i}{4\pi c} \oint \nabla_1 \left(\frac{1}{r}\right) \times d\boldsymbol{\lambda} = -\frac{i}{4\pi c} \oint \nabla_2 \left(\frac{1}{r}\right) \times d\boldsymbol{\lambda}$$

since $\nabla_2(1/r) = -\nabla_1(1/r)$. By means of (18–2) we can express **H** as a surface integral over any surface σ bounded by the filament, getting,

$$\mathbf{H} = \frac{i}{4\pi c} \int_\sigma \nabla_2 \left\{ d\boldsymbol{\sigma} \cdot \nabla_2 \left(\frac{1}{r}\right)\right\} - \frac{i}{4\pi c} \int_\sigma \nabla_2 \cdot \nabla_2 \left(\frac{1}{r}\right) d\boldsymbol{\sigma}$$

$$= -\frac{i}{4\pi c} \nabla_1 \int_\sigma d\boldsymbol{\sigma} \cdot \nabla_2 \left(\frac{1}{r}\right) - \frac{i}{4\pi c} \int_\sigma \nabla_2 \cdot \nabla_2 \left(\frac{1}{r}\right) d\boldsymbol{\sigma}. \tag{64–2}$$

We shall consider first the case where the field-point P lies outside the surface σ, and then the more general case where P lies either out-

side or in σ. In the first case the integrand of the second integral vanishes identically, and, if $d\Omega$ is the solid angle at P subtended by $d\boldsymbol{\sigma}$,

$$d\boldsymbol{\sigma}\cdot\nabla_2\left(\frac{1}{r}\right) = \frac{d\boldsymbol{\sigma}\cdot\mathbf{r}}{r^3} = -\, d\Omega.$$

Therefore, omitting the subscript 1 on the operator ∇, the field $\mathbf{H}_o$ outside the surface is

$$\mathbf{H}_o = -\,\nabla\Psi_o, \tag{64–3}$$

where the scalar potential Ψ_o is

$$\Psi_o \equiv -\,\frac{i\Omega}{4\pi c}. \tag{64–4}$$

Incidentally it may be remarked that this is just the potential in Heaviside-Lorentz units of a thin magnetic shell [2] of constant strength i/c coincident with the surface σ.

Provided the linear dimensions of the circuit are small compared with the distance of the field-point P,

$$\frac{i}{c}\int_\sigma d\boldsymbol{\sigma}\cdot\nabla_2\left(\frac{1}{r}\right) = \frac{i\boldsymbol{\sigma}\cdot\mathbf{r}}{cr^3} = \frac{\mathbf{p}_H\cdot\mathbf{r}}{r^3},$$

where $\mathbf{p}_H$ is the magnetic moment of the circuit, and

$$\Psi_o = \frac{\mathbf{p}_H\cdot\mathbf{r}}{4\pi r^3}, \tag{64–5}$$

giving

$$\mathbf{H}_o = -\,\nabla\Psi_o = \frac{1}{4\pi r^5}\left\{2\mathbf{p}_H\cdot\mathbf{r}\mathbf{r} + (\mathbf{p}_H\times\mathbf{r})\times\mathbf{r}\right\} \tag{64–6}$$

in agreement with (58–11).

In order to include the case where the field-point P lies in the surface σ of (64–2) we shall replace the current filament by a current sheet of small width l normal to the surface σ as shown in section in Fig. 60, the current entering the section at F and emerging at G. If $\mathbf{n}_1$ is a unit vector normal to σ, and i the total current in the sheet, (64–2) becomes in this case

$$\mathbf{H} = -\,\frac{i}{4\pi lc}\nabla_1\int_\tau \mathbf{n}_1\cdot\nabla_2\left(\frac{1}{r}\right)d\tau - \frac{i}{4\pi lc}\int_\tau \nabla_2\cdot\nabla_2\left(\frac{1}{r}\right)\mathbf{n}_1 d\tau,$$

[2] L. Page, *Theoretical Physics*, 2nd Edit. p. 412.

the volume integrals being taken through the interior of the shell of surface σ and thickness l bounded along its periphery by the current sheet. If we put $d\tau = dl\, d\sigma$, where dl is measured in the direction of $\mathbf{n}_1$, the first integral becomes

$$\int_\tau \mathbf{n}_1 \cdot \nabla_2 \left(\frac{1}{r}\right) d\tau = \iint \frac{\partial}{\partial l}\left(\frac{1}{r}\right) dl\, d\sigma = \int_{\sigma_1} \frac{1}{r_1}\, d\sigma_1 - \int_{\sigma_2} \frac{1}{r_2}\, d\sigma_2,$$

where r_1 and r_2 are, respectively, the distances from points on the

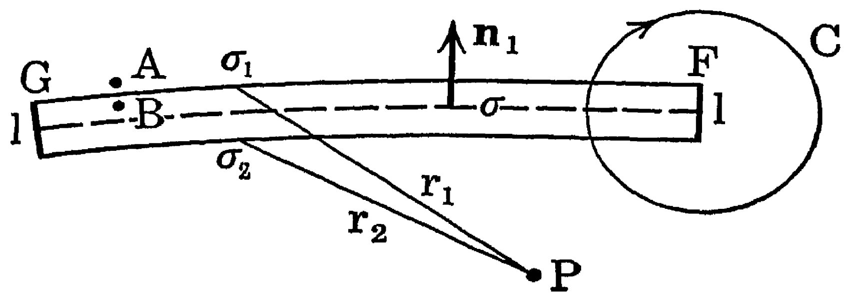

FIG. 60.

bounding surfaces σ_1 and σ_2 of the shell to the field-point P. Therefore, the magnetic field at *any* point P is given by

$$\mathbf{H} = -\nabla\Psi - \frac{i}{4\pi lc}\int_\tau \nabla_2 \cdot \nabla_2 \left(\frac{1}{r}\right) \mathbf{n}_1 d\tau, \tag{64-7}$$

where we have omitted the subscript 1 on the first ∇ as in (64–3), and where

$$\Psi \equiv \frac{1}{4\pi}\left\{\int_{\sigma_1} \frac{i}{lcr_1}\, d\sigma_1 - \int_{\sigma_2} \frac{i}{lcr_2}\, d\sigma_2\right\}. \tag{64-8}$$

But Ψ is just the potential that would be produced by a uniform static magnetic charge i/lc per unit area on the surface σ_1 and $-i/lc$ per unit area on the surface σ_2, the double layer constituting a magnetic shell of strength (magnetic moment per unit area) i/c and thickness l. This shell is known as the *equivalent magnetic shell* for the current circuit under consideration.

The field $\mathbf{H}_o$ outside the shell is given by

$$\mathbf{H}_o = -\nabla\Psi \tag{64-9}$$

since the integrand of the second term in (64–7) vanishes identically, the expression (64–8) for Ψ reducing to (64–4) for any field-point whose distance from the shell is large compared with l. For a field-point inside the shell, however, the integrand of the second term in

(64–7) does not vanish. Instead $-\nabla_2(1/r)$ is the field intensity at x_2, y_2, z_2 due to a static charge 4π at x_1, y_1, z_1 and its divergence is the charge density at x_1, y_1, z_1. Hence

$$\int_\tau \nabla_2 \cdot \nabla_2 \left(\frac{1}{r}\right) \mathbf{n}_1 d\tau = - 4\pi \mathbf{n}_1,$$

where $\mathbf{n}_1$ in the right-hand member is the normal unit vector in the neighborhood of x_1, y_1, z_1. Consequently the magnetic intensity $\mathbf{H}_i$ at a point between the surfaces σ_1 and σ_2 is

$$\mathbf{H}_i = - \nabla\Psi + \frac{i}{lc}\mathbf{n}_1. \tag{64–10}$$

Although the fact that the position of the equivalent magnetic shell is immaterial provided that its periphery coincides with the circuit shows that the magnetic field of the current is everywhere continuous, we shall prove in detail that there is no discontinuity in the field (64–7) as we pass across the surfaces σ_1 and σ_2. The portion of the field specified by $-\nabla\Psi$, since it is identical with that produced by the equivalent magnetic charges on the surfaces σ_1 and σ_2, has a value at a point A (Fig. 60) just outside the surface σ_1 which exceeds its value at a point B just inside by the number i/lc of tubes of force originating on a unit area of the charged surface, that is,

$$(-\nabla\Psi)_A = (-\nabla\Psi)_B + \frac{i}{lc}\mathbf{n}_1.$$

Consequently $\mathbf{H}_A$ and $\mathbf{H}_B$, specified by (64–9) and (64–10) respectively, are equal.

Finally we note that the line integral of $\mathbf{H}$ around any closed curve C encircling the current is

$$\oint \mathbf{H} \cdot d\lambda = - \oint \nabla\Psi \cdot d\lambda + \frac{i}{c} = \frac{i}{c},$$

in accord with Ampère's law (51–9d).

Evidently the magnetic moment per unit volume inside the shell of Fig. 60 is i/lc. Therefore the magnet equivalent to a straight solenoid with n turns per unit length each of which carries a current i is a bar filling the interior of the solenoid which is uniformly magnetized parallel to its axis with intensity of magnetization ni/c. The field due to the solenoid is identical with that of the equivalent mag-

net at all points outside the latter; at inside points the field of the solenoid exceeds that of the magnet by the intensity of magnetization ni/c.

Now we shall turn our attention to the calculation of the force and torque experienced by a rigid closed filament carrying a current i which is located in an external magnetic field $\mathbf{H}$. From the force equation (57–22) we have for the force on a current element $i\,d\boldsymbol{\lambda}$

$$d\mathcal{K} = \frac{i}{c}\,d\boldsymbol{\lambda} \times \mathbf{H}, \tag{64–11}$$

and, using (18–2), the resultant force on the entire circuit is

$$\mathcal{K} = \frac{i}{c}\oint d\boldsymbol{\lambda} \times \mathbf{H} = \frac{i}{c}\int_\sigma \nabla\mathbf{H}\cdot d\boldsymbol{\sigma} \tag{64–12}$$

since $\nabla\cdot\mathbf{H}$ vanishes everywhere by (51–8b). If $N = \int_\sigma \mathbf{H}\cdot d\boldsymbol{\sigma}$ is the magnetic flux through the circuit, this may be written in the form

$$\mathcal{K} = \frac{i}{c}\nabla N \tag{64–13}$$

for a pure translation in which every $d\boldsymbol{\sigma}$ remains constant in direction as well as in magnitude, showing that the circuit tends to translate in such a direction as to increase the flux through it.

The torque about an origin O on a current element at a vector distance $\mathbf{r}$ is

$$d\mathcal{L} = \mathbf{r} \times d\mathcal{K} = \frac{i}{c}\mathbf{r} \times (d\boldsymbol{\lambda} \times \mathbf{H}) \tag{64–14}$$

from (64–11) and, making use of (18–3), the resultant torque on the entire circuit is

$$\begin{aligned}\mathcal{L} &= \frac{i}{c}\oint \mathbf{r} \times (d\boldsymbol{\lambda} \times \mathbf{H})\\ &= -\frac{i}{c}\int_\sigma \nabla \times \overline{\mathbf{rH}}\cdot d\boldsymbol{\sigma} - \frac{i}{c}\int_\sigma \nabla\cdot\overline{\mathbf{Hr}} \times d\boldsymbol{\sigma}.\end{aligned}$$

As $\nabla\cdot\mathbf{H} = 0$ and $\nabla \times \mathbf{r}$ vanishes identically, this becomes

$$\mathcal{L} = \frac{i}{c}\int_\sigma \mathbf{r} \times \nabla\mathbf{H}\cdot d\boldsymbol{\sigma} - \frac{i}{c}\int_\sigma \mathbf{H}\cdot\nabla\mathbf{r} \times d\boldsymbol{\sigma}.$$

Furthermore,

$$\mathbf{H}\cdot\nabla\mathbf{r}\times d\boldsymbol{\sigma} = \left\{H_x\frac{\partial}{\partial x} + H_y\frac{\partial}{\partial y} + H_z\frac{\partial}{\partial z}\right\}\{\boldsymbol{i}(y d\sigma_z - z d\sigma_y) + \text{etc.}\}$$
$$= \boldsymbol{i}\,(H_y d\sigma_z - H_z d\sigma_y) + \text{etc.}$$
$$= \mathbf{H}\times d\boldsymbol{\sigma}.$$

So the torque about O is

$$\mathscr{L} = \frac{i}{c}\int_\sigma \mathbf{r}\times\nabla\mathbf{H}\cdot d\boldsymbol{\sigma} + \frac{i}{c}\int_\sigma d\boldsymbol{\sigma}\times\mathbf{H}. \qquad (64\text{–}15)$$

The first term evidently represents the torque of the force $\mathscr{K}$ given by (64–12) about the origin, and the second, since it is independent of $\mathbf{r}$, is a turning couple.

The rate at which work is done against the force (64–13) during a pure translation with velocity $\mathbf{V}$ in a static field is

$$\frac{dU_T}{dt} = -\mathscr{K}\cdot\mathbf{V} = -\frac{i}{c}\mathbf{V}\cdot\nabla N = -\frac{i}{c}\frac{dN}{dt}, \qquad (64\text{–}16)$$

and for a pure rotation about the origin O with angular velocity $\boldsymbol{\omega}$ the rate at which work is done against the torque (64–15) is

$$\frac{dU_R}{dt} = -\mathscr{L}\cdot\boldsymbol{\omega} = -\frac{i}{c}\int_\sigma(\boldsymbol{\omega}\times\mathbf{r}\cdot\nabla\mathbf{H}\cdot d\boldsymbol{\sigma} + \mathbf{H}\cdot\boldsymbol{\omega}\times d\boldsymbol{\sigma})$$
$$= -\frac{i}{c}\int_\sigma\left(\frac{d\mathbf{H}}{dt}\cdot d\boldsymbol{\sigma} + \mathbf{H}\cdot\frac{d}{dt}d\boldsymbol{\sigma}\right) = -\frac{i}{c}\frac{dN}{dt}, \qquad (64\text{–}17)$$

since $\boldsymbol{\omega}\times\mathbf{r}$ is the linear velocity of a point on the surface σ due to the rotation and $\boldsymbol{\omega}\times d\boldsymbol{\sigma}$ is the time rate of change of the vector $d\boldsymbol{\sigma}$. Hence the total energy of the rigid circuit with respect to an external magnetostatic field is

$$U = -\frac{i}{c}N \qquad (64\text{–}18)$$

provided the current is maintained constant.

If the circuit is so small that $\mathbf{H}$ is effectively constant over the surface bounded by it, (64–12) may be written

$$\mathscr{K} = \nabla\mathbf{H}\cdot\left(\frac{i\boldsymbol{\sigma}}{c}\right) = \nabla\mathbf{H}\cdot\mathbf{p}_H \qquad (64\text{–}19)$$

in agreement with (58–18), and the turning couple specified by the second term in (64–15) becomes

$$\mathscr{L}_c = \left(\frac{i\sigma}{c}\right) \times \mathbf{H} = \mathbf{p}_H \times \mathbf{H} \tag{64–20}$$

in accord with (58–21). Furthermore the energy in a magnetostatic field is

$$U = -\frac{i\sigma}{c} \cdot \mathbf{H} = -\mathbf{p}_H \cdot \mathbf{H} \tag{64–21}$$

from (64–18).

65. Magnetic Media. — The first step in deducing constitutive relations between **B** and **F** for a magnetic medium consists in the determination of the field responsible for the magnetization of the medium, keeping in mind the fact that **B** represents the mean magnetic intensity averaged over a volume $\Delta\tau$ large enough to contain a great many atoms. As in the case of a dielectric, we separate **B** into two parts: the mean field $\mathbf{B}_2$ due to the Ampèrian currents actually lying in the region $\Delta\tau$ and the field $\mathbf{B}_1$ due to currents and moving charges outside this region. As in the dielectric analog, we take for $\Delta\tau$ a sphere of radius b (Fig. 58).

Now the part of the field due to an Ampèrian current represented by the term $-\nabla\Psi$ in (64–7) is identical with that of the equivalent magnetic shell. But this shell is merely a collection of magnetic dipoles, the vector sum of the magnetic moments of all the equivalent shells in $\Delta\tau$ divided by the volume of the region being the intensity of magnetization **I** in the medium. Consequently, following the same analysis as in the case of the dielectric, the contribution to the field $\mathbf{B}_2$ of the term $-\nabla\Psi$ in (64–7) is

$$(\mathbf{B}_2)_1 = -\tfrac{1}{3}\mathbf{I}.$$

All that remains is to calculate the contribution to $\mathbf{B}_2$ due to the second term in (64–7). This term vanishes everywhere outside the equivalent magnetic shell, giving rise to the field $(i/lc)\mathbf{n}_1$ in the interior of the shell, as indicated in (64–10). Its volume integral over the interior of the shell is

$$\frac{i}{lc}\int_\tau \mathbf{n}_1 d\tau = \frac{i}{c}\int_\sigma d\sigma = \frac{i\sigma}{c} = \mathbf{p}_H,$$

where $\mathbf{p}_H$ is the magnetic moment of the Ampèrian current. Therefore the contribution of the second term in (64–7) to $\mathbf{B}_2$ is

$$(\mathbf{B}_2)_2 = \frac{\sum_i \mathbf{p}_{Hi}}{\Delta\tau} = \mathbf{I}.$$

Adding $(\mathbf{B}_2)_1$ and $(\mathbf{B}_2)_2$, we have

$$\mathbf{B}_2 = \tfrac{2}{3}\,\mathbf{I}. \tag{65–1}$$

Consequently the field responsible for the magnetization of the medium in the region $\Delta\tau$ is

$$\begin{aligned} \mathbf{B}_1 = \mathbf{B} - \mathbf{B}_2 &= \mathbf{B} - \tfrac{2}{3}\,\mathbf{I} \\ &= \mathbf{F} + \tfrac{1}{3}\,\mathbf{I} \end{aligned} \tag{65–2}$$

since $\mathbf{F} = \mathbf{B} - \mathbf{I}$. It should be noted that $\mathbf{B}_1$ is given in terms of $\mathbf{F}$ and $\mathbf{I}$ by an expression of the same form as that expressing $\mathbf{E}_1$ in a dielectric in terms of $\mathbf{E}$ and $\mathbf{P}$, as specified by (63–3).

In a paramagnetic or diamagnetic medium the intensity of magnetization is a linear vector function of $\mathbf{B}_1$. Hence we can write

$$\mathbf{I} = \begin{cases} \boldsymbol{\alpha}\cdot\mathbf{B}_1 \text{ for anisotropic media,} \\ \alpha\,\mathbf{B}_1 \text{ for isotropic media,} \end{cases} \tag{65–3}$$

where $\boldsymbol{\alpha}$ is a symmetric dyadic for a perfect magnetic medium and therefore can be put in the form (63–5) by a proper orientation of the axes. Eliminating $\mathbf{B}_1$ between (65–2) and (65–3) we get

$$\mathbf{I} = \begin{cases} \boldsymbol{\epsilon}\cdot\mathbf{F} \text{ for anisotropic media,} \\ \epsilon\,\mathbf{F} \text{ for isotropic media,} \end{cases}$$

where

$$\left.\begin{aligned} \boldsymbol{\epsilon} &\equiv \frac{\alpha_x}{1 - \frac{1}{3}\alpha_x}\,\mathbf{ii} + \frac{\alpha_y}{1 - \frac{1}{3}\alpha_y}\,\mathbf{jj} + \frac{\alpha_z}{1 - \frac{1}{3}\alpha_z}\,\mathbf{kk}, \\ \epsilon &\equiv \frac{\alpha}{1 - \frac{1}{3}\alpha}, \end{aligned}\right\} \tag{65–4}$$

the three elements of $\boldsymbol{\epsilon}$ being the *principal magnetic susceptibilities* of an anisotropic medium, and ϵ being the *magnetic susceptibility* of an isotropic medium. Finally, as $\mathbf{B} = \mathbf{F} + \mathbf{I}$,

$$\mathbf{B} = \begin{cases} \mathbf{M}\cdot\mathbf{F} \text{ for anisotropic media,} \\ \mu\,\mathbf{F} \text{ for isotropic media,} \end{cases} \tag{65–5}$$

where

$$\left.\begin{aligned}\mathbf{M} &\equiv \mu_x \boldsymbol{ii} + \mu_y \boldsymbol{jj} + \mu_z \boldsymbol{kk} \\ &\equiv \frac{1 + \frac{2}{3}\alpha_x}{1 - \frac{1}{3}\alpha_x} \boldsymbol{ii} + \frac{1 + \frac{2}{3}\alpha_y}{1 - \frac{1}{3}\alpha_y} \boldsymbol{jj} + \frac{1 + \frac{2}{3}\alpha_z}{1 - \frac{1}{3}\alpha_z} \boldsymbol{kk}, \\ \mu &\equiv \frac{1 + \frac{2}{3}\alpha}{1 - \frac{1}{3}\alpha}.\end{aligned}\right\} \quad (65\text{–}6)$$

The elements μ_x, μ_y, μ_z of $\mathbf{M}$ are called the *principal permeabilities* of an anisotropic medium, and μ the *permeability* of an isotropic medium. As in the case of a dielectric, if we solve (65–6) for $\boldsymbol{\alpha}$ or α we get

$$\left.\begin{aligned}\frac{\mu_x - 1}{\mu_x + 2} \boldsymbol{ii} + \frac{\mu_y - 1}{\mu_y + 2} \boldsymbol{jj} + \frac{\mu_z - 1}{\mu_z + 2} \boldsymbol{kk} &= \frac{1}{3} \boldsymbol{\alpha} \text{ for anisotropic media,} \\ \frac{\mu - 1}{\mu + 2} &= \frac{1}{3} \alpha \text{ for isotropic media.}\end{aligned}\right\} \quad (65\text{–}7)$$

On account of hysteresis, the magnetic induction $\mathbf{B}$ in a ferromagnetic medium cannot be expressed as a function of the magnetizing force $\mathbf{F}$ alone, since the relation between these two vectors depends upon the past history of the specimen.

It is important to note that we may show from (62–14*b*), by precisely the same argument used for a dielectric in article 63, that no effective magnetization charge of density $-\nabla \cdot \mathbf{I}$ can exist in the interior of a homogeneous isotropic paramagnetic or diamagnetic medium for which the relation $\mathbf{B} = \mu \mathbf{F}$ is valid. However it must not be inferred from this conclusion that the magnetization current $\dot{\mathbf{I}}$ must vanish in such a medium.

Consider a magnetostatic field. Such a field can be produced only by steady currents flowing in closed circuits or by permanent magnets, or by a combination of the two. So let us direct our attention to a group of closed circuits carrying steady currents and of permanent magnets, all immersed in a homogeneous isotropic paramagnetic or diamagnetic medium, and consider the field in the interior of the homogeneous medium. The portion of the field due to the current circuits is given by $\mathbf{H}_o = -\nabla \Psi_o$ by (64–3), and, as $\nabla \cdot \mathbf{H}_o = 0$, the potential Ψ_o satisfies Laplace's equation $\nabla \cdot \nabla \Psi_o = 0$. The magnetic force due to the permanent magnets and the magnetization of the medium is given by $\mathbf{F}_H = -\nabla \Phi_H$ in accord with (62–15), where Φ_H satisfies Laplace's equation $\nabla \cdot \nabla \Phi_H = 0$ by (53–8), since the field is static, and, as we have just shown, there is no magnetization charge

in the interior of the medium. Hence the resultant magnetic force $\mathbf{F} = \mathbf{H}_o + \mathbf{F}_H$ in the medium is given by

$$\mathbf{F} = -\nabla\Phi, \tag{65–8}$$

where the scalar potential $\Phi \equiv \Psi_o + \Phi_H$ is a solution of Laplace's equation

$$\nabla \cdot \nabla\Phi = 0. \tag{65–9}$$

The function Φ is called the *magnetic potential.*

An isotropic paramagnetic medium is the magnetic analog of an isotropic dielectric composed of permanent dipoles, if we suppose the Ampèrian current circuits to have magnetic moments which remain effectively constant in magnitude however much the magnetic field in which they lie may change. As the energy with respect to the field of an Ampèrian current with a permanent magnetic moment $\mathbf{p}_0$ making an angle θ with $\mathbf{B}_1$ is

$$U = -\, p_0 B_1 \cos\theta$$

from (64–21), we can translate (63–13) at once into the magnetic relation

$$I = \frac{1}{3}\frac{np_0{}^2}{kT} B_1, \tag{65–10}$$

which describes the characteristics of many isotropic paramagnetic media with a fair degree of accuracy, since the magnetic fields available in the laboratory are far too weak to produce saturation. This relation is known as *Curie's law.* The value of α which it defines, when substituted in (65–7) gives

$$\frac{\mu - 1}{\mu + 2} = \frac{1}{9}\frac{np_0{}^2}{kT}. \tag{65–11}$$

To explain the diamagnetism of an isotropic medium we shall assume that the dimensions of an Ampèrian circuit, whether consisting of a ring of electrons revolving about the nucleus of an atom, or of the spins of individual electrons about a diameter, remain unaltered when the impressed magnetic field is changed. First we shall consider the orbital motion of an electron in an atom. In the absence of an impressed magnetic field its equation of motion may be written

$$m\mathbf{f} = \mathcal{K} \tag{65–12}$$

relative to the observer's inertial system, where $\mathcal{K}$ is the force, whatever its nature, to which the electron is subject. In the presence of a magnetic field **H**, supposed to be constant over the small region in which the orbit of the electron lies, the equation of motion is

$$m\mathbf{f} = \mathcal{K} + \frac{e}{c}\mathbf{V} \times \mathbf{H}.$$

Taking the origin at the nucleus of the atom, let us refer the motion to a set of axes $X'Y'Z'$ which are rotating about this origin with constant angular velocity $\boldsymbol{\omega}$. If $\mathbf{V}'$ is the velocity and $\mathbf{f}'$ the acceleration of the electron relative to the rotating axes [3]

$$\mathbf{V} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}',$$

$$\mathbf{f} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{V}' + \mathbf{f}',$$

where $\mathbf{r}$ is the position vector of the electron. Hence the equation of motion becomes

$$m\mathbf{f}' = \mathcal{K} + 2m\mathbf{V}' \times \left\{\boldsymbol{\omega} + \frac{e}{2mc}\mathbf{H}\right\} + m(\boldsymbol{\omega} \times \mathbf{r}) \times \left\{\boldsymbol{\omega} + \frac{e}{mc}\mathbf{H}\right\}$$

relative to $X'Y'Z'$. Now, if we make

$$\boldsymbol{\omega} = -\frac{e}{2mc}\mathbf{H}, \tag{65–13}$$

we have

$$\begin{aligned} m\mathbf{f}' &= \mathcal{K} - m\left\{\left(\frac{e}{2mc}\mathbf{H}\right) \times \mathbf{r}\right\} \times \left(\frac{e}{2mc}\mathbf{H}\right) \\ &= \mathcal{K} \end{aligned} \tag{65–14}$$

to a sufficient degree of approximation, since the angular velocity $(e/2mc)H$ is so small compared with the other angular velocities involved that its square may be neglected. Comparing (65–14) with (65–12), we observe that the equation of motion relative to rotating axes in the presence of the magnetic field is the same as that relative to fixed axes in its absence. Hence the effect of the magnetic field is to superpose on the normal motion of an electron a precession about the lines of magnetic force with angular velocity $-(e/2mc)\mathbf{H}$. This is known as *Larmor precession*. If the magnetic moment due to the orbital motion of an electron is in the direction of the impressed magnetic field, the effect of the field is to make the electron revolve less rapidly and therefore diminish its magnetic moment, whereas if the

[3] L. Page, *Theoretical Physics*, 2nd Edit. pp. 101, 103.

original magnetic moment is opposite to the field, the application of the field causes the electron to revolve more rapidly and thereby increase its magnetic moment.

In the absence of an impressed field we must suppose that each elementary particle of a diamagnetic medium contains a number of rigidly connected Ampèrian circuits of such moments and so oriented that the resultant magnetic moment of the particle is zero. Let $\mathbf{p}_i$ be the normal magnetic moment, i_i the normal current and ω_i the normal angular velocity about the nucleus, of the electrons constituting the current in one of these Ampèrian circuits. Then, if $\mathbf{p}_i$ makes an angle θ with an impressed magnetic field $\mathbf{B}_1$, the added magnetic moment Δp_i and the added current Δi_i are given by

$$\frac{\Delta p_i}{p_i} = \frac{\Delta i_i}{i_i} = \frac{-\dfrac{e}{2mc}B_1}{\omega_i}\cos\theta.$$

As the intensity of magnetization is zero in the normal state, the distribution in angle of the n_i Ampèrian circuits per unit volume of the type under consideration must be random. Therefore the intensity of magnetization due to the field $\mathbf{B}_1$ is

$$I = \sum_i \int_0^\pi \Delta p_i \cos\theta \frac{n_i}{2} \sin\theta \, d\theta = -\sum_i \frac{e n_i p_i}{6mc\omega_i} B_1. \quad (65\text{–}15)$$

As the coefficient α of B_1 is very small even for the most strongly diamagnetic substances, such as bismuth, (65–6) gives $\mu = 1 + \alpha$.

To discuss the diamagnetic effect due to electron spin we use (60–8), neglecting the second order term on the right. Then we have

$$\dot{\mathbf{p}}_H = -2\pi a^3 \dot{\mathbf{H}},$$

and integrating, we find for the diamagnetic intensity of magnetization due to the field B_1,

$$I = -2\pi n a^3 B_1, \quad (65\text{–}16)$$

where n is the number of spinning electrons per unit volume.

66. Motion of Ions in Uniform Electric and Magnetic Fields.— Before investigating the theory of metallic conduction, we must examine the motion of free ions or electrons in uniform electric and magnetic fields. We shall suppose the charged particles to be so widely separated that they suffer no collisions with one another and do not modify appreciably the uniform external fields in which they

are situated. Then we can treat each particle as if all the others were absent. We shall neglect the radiation reaction of the field of an ion, but shall take account of the variation of mass with velocity, writing the equation of motion in the form (57–14). We have four cases to consider: (I) a uniform electric field alone, (II) a uniform magnetic field alone, (III) crossed uniform electric and magnetic fields with $E^2 < H^2$, (IV) crossed uniform electric and magnetic fields with $E^2 > H^2$.

(I) *Uniform Electric Field E.* The equation of motion is

$$\frac{d}{dt}\frac{\mathbf{V}}{\sqrt{1-\dfrac{V^2}{c^2}}} = \frac{e}{m}\mathbf{E}. \tag{66–1}$$

If we put $\mathbf{B} \equiv \mathbf{V}/c$ this equation may be written

$$\frac{d}{dt}\frac{\mathbf{B}}{\sqrt{1-\mathrm{B}^2}} = \frac{e}{mc}\mathbf{E}, \tag{66–2}$$

and, if $\mathbf{r}$ is the position vector of the particle, the energy equation is

$$\frac{1}{\sqrt{1-\mathrm{B}^2}} = \frac{e}{mc^2}\mathbf{E}\cdot\mathbf{r}, \tag{66–3}$$

the constant of integration being made zero by a suitable location of the origin.

Integrating (66–2) and using (66–3) to eliminate $\sqrt{1-\mathrm{B}^2}$, we have

$$\mathbf{E}\cdot\mathbf{r}\,\mathbf{B} - \mathbf{E}\cdot\mathbf{r}_0\mathbf{B}_0 = \mathbf{E}ct, \tag{66–4}$$

where $\mathbf{r}_0$ is the position vector of the particle and $\mathbf{B}_0$ is the ratio of $\mathbf{V}$ to c when $t = 0$.

Let us orient the axes so that the Y axis is parallel to $\mathbf{E}$. Splitting (66–4) into its component equations,

$$\left.\begin{aligned} y\frac{dx}{dt} - y_0V_{0x} &= 0, \\ y\frac{dy}{dt} - y_0V_{0y} &= c^2t, \\ y\frac{dz}{dt} - y_0V_{0z} &= 0, \end{aligned}\right\} \tag{66–5}$$

and integrating,

$$\left.\begin{aligned}
x &= y_0 B_{0x} \log \frac{\sqrt{1 + 2\dfrac{V_{0y}}{y_0}t + \dfrac{c^2}{y_0^2}t^2} + B_{0y} + \dfrac{c}{y_0}t}{1 + B_{0y}}, \\
y &= y_0 \sqrt{1 + 2\frac{V_{0y}}{y_0}t + \frac{c^2}{y_0^2}t^2}, \\
z &= y_0 B_{0z} \log \frac{\sqrt{1 + 2\dfrac{V_{0y}}{y_0}t + \dfrac{c^2}{y_0^2}t^2} + B_{0y} + \dfrac{c}{y_0}t}{1 + B_{0y}},
\end{aligned}\right\} \quad (66\text{–}6)$$

where y_0 is related to B_0 by the equation

$$\frac{c}{y_0} = \sqrt{1 - B_0^2}\,\frac{e}{mc}E \qquad (66\text{–}7)$$

obtained from (66–3).

To investigate the nature of the path we can make $B_{0y} = B_{0z} = 0$ without loss of generality. Then the trajectory lies in the XY plane, and its equation is

$$x = \pm\, y_0 B_0 \log \frac{y + \sqrt{y^2 - y_0^2}}{y_0} \qquad (66\text{–}8)$$

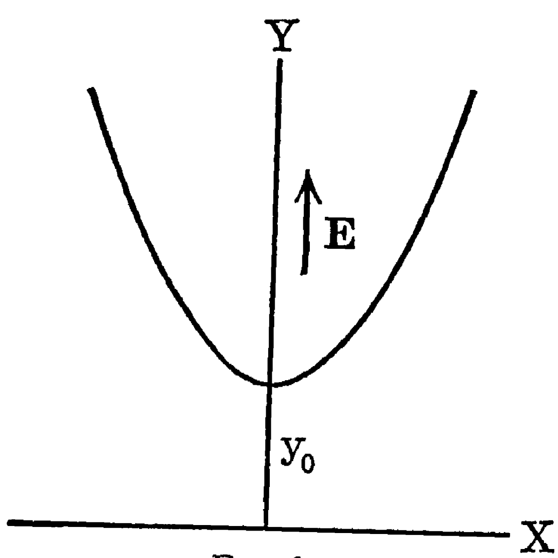

FIG. 61.

with the slope

$$\frac{dy}{dx} = \pm \frac{\sqrt{y^2 - y_0^2}}{y_0 B_0}.$$

The trajectory, illustrated in Fig. 61, is much like the parabola of the Newtonian dynamics.

(II) *Uniform Magnetic Field H.* In this case the equation of motion is

$$\frac{d}{dt}\frac{\mathbf{V}}{\sqrt{1 - \dfrac{V^2}{c^2}}} = \frac{e}{mc}\mathbf{V} \times \mathbf{H}. \qquad (66\text{–}9)$$

Taking the scalar product with $\mathbf{V}$ and integrating, we find

$$\frac{1}{\sqrt{1 - B^2}} = \text{Constant}, \qquad (66\text{–}10)$$

where $\mathrm{B} \equiv \mathrm{V}/c$ as before. Hence the speed remains constant throughout the motion. Orienting the axes so that the Z axis is parallel to **H**, the component equations of (66–9) are

$$\left.\begin{aligned} \frac{d^2}{dt^2}(x + \mathrm{i}y) &= -\mathrm{i}\frac{eH}{mc}\sqrt{1 - \mathrm{B}^2}\,\frac{d}{dt}(x + \mathrm{i}y), \\ \frac{d^2z}{dt^2} &= 0, \end{aligned}\right\} \qquad (66\text{–}11)$$

where $\mathrm{i} \equiv \sqrt{-1}$. The second tells us that the velocity in the Z direction is constant. Putting

$$\Omega \equiv -\sqrt{1 - \mathrm{B}^2}\,\frac{e}{mc}\,\mathbf{H},$$

the first gives on integration

$$x - x_0 + \mathrm{i}(y - y_0) = a e^{\mathrm{i}(\Omega t - \epsilon)}, \qquad (66\text{–}13)$$

where x_0, y_0, a and ϵ are constants of integration. Therefore the projection of the motion on a plane perpendicular to **H** is motion with constant angular velocity $\mathbf{\Omega}$ in a circle of radius a, positive ions describing the path in the negative sense and negative ions in the positive sense. If $\mathrm{B} \ll 1$, the angular velocity Ω is effectively independent of the speed of the ion.

Evidently

$$a^2 = \frac{V_x^{\,2} + V_y^{\,2}}{\Omega^2}, \qquad (66\text{–}14)$$

showing that the radius of the circular projection of the path on the plane perpendicular to **H** is proportional to the projection of the velocity on this plane.

Combining the motion perpendicular to **H** with that parallel to the field, we see that the paths of the ions in space are helices about the lines of magnetic force.

(III) *Crossed Electric and Magnetic Fields*, $E < H$. Orient the axes so that the Y axis is parallel to **E** and the Z axis to **H** in the observer's inertial system S. Then, as shown in article 47, the electric field is zero in an inertial system S' moving in the X direction relative to S with velocity $v = (E/H)c$, and the magnetic field in this inertial system is

$$H' = H\sqrt{1 - \beta^2} = \sqrt{H^2 - E^2} \qquad (66\text{–}15)$$

in the Z' direction. Relative to S', therefore, an ion describes a helical path whose projection on the X' Y' plane is a circle of radius given by

$$a'^2 = \frac{V_x'^2 + V_y'^2}{\Omega'^2}$$

traversed with angular velocity

$$\Omega' = -\sqrt{1 - B'^2}\,\frac{e}{mc}\,\mathbf{H}'.$$

If we take the origin O' of S' on the circumference of this circle, the projection of the motion on the $X'Y'$ plane is specified by

$$x' + iy' = a'\left\{e^{i(\Omega' t' - \epsilon')} - e^{-i\epsilon'}\right\} \qquad (66\text{–}16)$$

in accord with (66–13), and the X and Y components of the initial velocity V_0' of the ion are given by

$$V'_{0x} + iV'_{0y} = ia'\Omega' e^{-i\epsilon'} = a'\Omega' \sin\epsilon' + ia'\Omega' \cos\epsilon'. \qquad (66\text{–}17)$$

Transforming to S by means of (42–2),

$$kx + iy = c\frac{E}{H}kt + a'\left\{e^{i\left\{k\Omega'\left(t - \frac{\beta}{c}x\right) - \epsilon'\right\}} - e^{-i\epsilon'}\right\}, \qquad (66\text{–}18)$$

where

$$\left.\begin{aligned} k &\equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1 - \frac{E^2}{H^2}}}, \\ k\Omega' &= -\sqrt{1 - B'^2}\,\frac{eH}{mc}. \end{aligned}\right\} \qquad (66\text{–}19)$$

We see from (66–18) that the motion at right angles to $\mathbf{H}$ consists of a steady progression in the X direction with a constant velocity $(E/H)c$ determined by the two fields alone and independent of the charge or mass of the ion, accompanied by oscillations in the X and Y directions. As the motion along Z' in S' takes place with constant velocity V'_{0z}, the motion along Z in S is given by the equation

$$z = kV'_{0z}\left(t - \frac{\beta}{c}x\right). \qquad (66\text{–}20)$$

In article 101 we shall obtain the exact trajectory for this type of

motion. Here we shall consider only two limiting cases of interest.

(*a*) $E \ll H$ and therefore $\beta \ll 1$. Then

$$\left.\begin{aligned} x &= c\frac{E}{H}t + a'\{\cos(k\Omega' t - \epsilon') - \cos\epsilon'\}, \\ y &= a'\{\sin(k\Omega' t - \epsilon') + \sin\epsilon'\}, \\ z &= V'_{0z}t. \end{aligned}\right\} \quad (66\text{–}21)$$

The projection of the ion path on the XY plane is a cycloid generated by a circle of radius $\rho = cE/k\Omega' H$ rolling along a line parallel to the X axis. The cycloid is prolate, common, or curtate according as a' is less than, equal to, or greater than ρ. The generating circle lies above (in the sense of the Y axis) the line on which it rolls for a positive ion, and below for a negative ion, but in each case it rolls in the direction of increasing x at a rate independent of the charge or mass of the ion. When $a' = 0$ the prolate cycloid existing for small a' degenerates into a straight line parallel to the X axis, along which the ion moves with constant speed $(E/H)c$ under the action of the equal and opposite forces due to the electric and the magnetic fields respectively. In this case the ion is at rest in S'.

(*b*) $k\Omega'\left(t - \frac{\beta}{c}x\right) \ll 1$. In this case we investigate the initial phases of the motion, making no restrictions on the relative magnitudes of E and H, other than $E < H$, or on the initial velocity V_0 of the ion, other than $V_0 < c$. Expanding the exponential in (66–18) through terms in the cube of the exponent, we have, with the aid of (66–17),

$$\left.\begin{aligned} kx + iy = kvt + (-iV'_{0x} + V'_{0y})\Big\{ & ik\left(t - \frac{\beta}{c}x\right) \\ & - \frac{1}{2}k^2\Omega'\left(t - \frac{\beta}{c}x\right)^2 - \frac{1}{6}ik^3\Omega'^2\left(t - \frac{\beta}{c}x\right)^3 + \cdots\Big\}, \\ z = V'_{0z}k\left(t - \frac{\beta}{c}x\right). & \end{aligned}\right\} \quad (66\text{–}22)$$

Using (43–1) to express the components of $\mathbf{V}_0'$ in terms of those of $\mathbf{V}_0$, and remembering that

$$k^2\Omega' = -\frac{\sqrt{1 - \mathrm{B}_0^2}}{1 - \beta \mathrm{B}_{0x}}\frac{eH}{mc}$$

from (66–19), we find, if we solve for $t - (\beta/c)x$ by successive approximations,

$$t - \frac{\beta}{c}x = (1 - \beta B_{0x})\left\{t - \frac{eE}{2mc}\sqrt{1 - B_0{}^2}B_{0y}t^2 + \frac{e^2HE}{6m^2c^2}(1 - B_0{}^2)B_{0x}t^3 - \frac{e^2E^2}{6m^2c^2}(1 - B_0{}^2)(1 - 3B_{0y}{}^2)t^3 + \cdots\right\} \quad (66\text{–}23)$$

Substituting this series in the expressions (66–22) for x, y, z we get, through terms in t^3,

$$\frac{x}{c} = B_{0x}t + \frac{eH}{2mc}\sqrt{1 - B_0{}^2}B_{0y}t^2 - \frac{eE}{2mc}\sqrt{1 - B_0{}^2}B_{0x}B_{0y}t^2 - \frac{e^2H^2}{6m^2c^2}(1 - B_0{}^2)B_{0x}t^3 + \frac{e^2HE}{6m^2c^2}(1 - B_0{}^2)(1 + B_{0x}{}^2 - 3B_{0y}{}^2)t^3 - \frac{e^2E^2}{6m^2c^2}(1 - B_0{}^2)(1 - 3B_{0y}{}^2)B_{0x}t^3 + \cdots, \quad (66\text{–}24)$$

$$\frac{y}{c} = B_{0y}t - \frac{eH}{2mc}\sqrt{1 - B_0{}^2}B_{0x}t^2 + \frac{eE}{2mc}\sqrt{1 - B_0{}^2}(1 - B_{0y}{}^2)t^2 - \frac{e^2H^2}{6m^2c^2}(1 - B_0{}^2)B_{0y}t^3 + \frac{2e^2HE}{3m^2c^2}(1 - B_0{}^2)B_{0x}B_{0y}t^3 - \frac{e^2E^2}{2m^2c^2}(1 - B_0{}^2)(1 - B_{0y}{}^2)B_{0y}t^3 + \cdots, \quad (66\text{–}25)$$

$$\frac{z}{c} = B_{0z}t - \frac{eE}{2mc}\sqrt{1 - B_0{}^2}B_{0y}B_{0z}t^2 + \frac{e^2HE}{6m^2c^2}(1 - B_0{}^2)B_{0x}B_{0z}t^3 - \frac{e^2E^2}{6m^2c^2}(1 - B_0{}^2)(1 - 3B_{0y}{}^2)B_{0z}t^3 + \cdots. \quad (66\text{–}26)$$

If B_{0x}, B_{0y}, B_{0z} are small enough so that we can neglect their squares and products, these expressions reduce to

$$
\left.\begin{aligned}
x &= V_{0x}t + \frac{eH}{2mc} V_{0y}t^2 - \frac{e^2H^2}{6m^2c^2} V_{0x}t^3 \\
&\quad + \frac{e^2HE}{6m^2c^2} ct^3 - \frac{e^2E^2}{6m^2c^2} V_{0x}t^3 + \cdots, \\
y &= V_{0y}t - \frac{eH}{2mc} V_{0x}t^2 + \frac{eE}{2mc} ct^2 \\
&\quad - \frac{e^2H^2}{6m^2c^2} V_{0y}t^3 - \frac{e^2E^2}{2m^2c^2} V_{0y}t^3 + \cdots, \\
z &= V_{0z}t - \frac{e^2E^2}{6m^2c^2} V_{0z}t^3 + \cdots.
\end{aligned}\right\} \quad (66\text{–}27)
$$

(IV) *Crossed Electric and Magnetic Fields*, $E > H$. As before, we orient the axes so that the Y axis is parallel to $\mathbf{E}$ and the Z axis to $\mathbf{H}$ in the observer's intertial system S. Then the magnetic field vanishes in an inertial system S' moving in the X direction relative to S with velocity $v = (H/E)c$ in accord with (47–4), and the electric field is

$$
E' = E\sqrt{1 - \beta^2} = \sqrt{E^2 - H^2} \quad (66\text{–}28)
$$

in the Y direction. Relative to S' the equations of motion of the ion are given by (66–6) with primes on the coordinates and the components of the initial velocity. Transforming to S by means of (42–2),

$$
\left.\begin{aligned}
k(x - vt) &= y_0'\mathrm{B}_{0x}' \log \frac{P(x, t) + Q(x, t)}{1 + \mathrm{B}_{0y}'}, \\
y &= y_0'P(x, t), \\
z &= y_0'\mathrm{B}_{0z}' \log \frac{P(x, t) + Q(x, t)}{1 + \mathrm{B}_{0y}'},
\end{aligned}\right\} \quad (66\text{–}29)
$$

where

$$
P(x, t) \equiv \sqrt{1 + 2\frac{V_{0y}'}{y_0'} k\left(t - \frac{\beta}{c}x\right) + \frac{c^2}{y_0'^2} k^2 \left(t - \frac{\beta}{c}x\right)^2},
$$

$$
Q(x, t) \equiv \mathrm{B}_{0y}' + \frac{c}{y_0'} k\left(t - \frac{\beta}{c}x\right).
$$

The components of the initial velocity relative to S' are given in terms of those relative to S by (43–1). In terms of E,

$$
\frac{c}{y_0'} k = \sqrt{1 - \mathrm{B}_0'^2} \frac{eE}{mc}. \quad (66\text{–}30)
$$

Let us discuss the same two special cases as in (III).

(*a*) $H \ll E$ and therefore $\beta \ll 1$. We neglect β in the terms on the right of (66–29), and put $k = 1$. Then

$$\left.\begin{aligned} x &= c\frac{H}{E}t + y_0'B'_{0x}\log\frac{\sqrt{1 + 2\dfrac{V'_{0y}}{y_0'}t + \dfrac{c^2}{y_0'^2}t^2} + B'_{0y} + \dfrac{c}{y_0'}t}{1 + B'_{0y}}, \\ y &= y_0'\sqrt{1 + 2\frac{V'_{0y}}{y_0'}t + \frac{c^2}{y_0'^2}t^2}, \\ z &= y_0'B'_{0z}\log\frac{\sqrt{1 + 2\dfrac{V'_{0y}}{y_0'}t + \dfrac{c^2}{y_0'^2}t^2} + B'_{0y} + \dfrac{c}{y_0'}t}{1 + B'_{0y}}, \end{aligned}\right\} \quad (66\text{–}31)$$

which show that the trajectory differs from that of (I) for a uniform electric field alone only in the addition of the constant velocity $(H/E)c$ in the X direction. As in the case where $E \ll H$, this drift along the X axis is independent of the charge or mass of the ion.

(*b*) $k\dfrac{y_0}{c}\left(t - \dfrac{\beta}{c}x\right) \ll 1$. This condition restricts us to the initial phases of the motion, but allows us to consider any initial velocity less than that of light, as well as any value of the ratio H/E less than unity. Expanding the right-hand members of (66–29) through terms in the cube of $t - (\beta/c)x$ and expressing the components of $\mathbf{V}_0'$ in terms of those of $\mathbf{V}_0$ by (43–1), we find, if we solve by successive approximations,

$$\begin{aligned} t - \frac{\beta}{c}x = (1 - \beta B_{0x})t &+ \frac{eH}{2mc}\sqrt{1 - B_0^2}B_{0y}(B_{0x} - \beta)t^2 \\ &+ \frac{e^2HE}{6m^2c^2}(1 - B_0^2)\{(B_{0x} - \beta)(1 - \beta B_{0x}) \\ &- 3B_{0y}^2(B_{0x} - \beta)\}t^3 + \cdots. \end{aligned} \quad (66\text{–}32)$$

Substituting back, we get identically the equations (66–24) to (66–26). Therefore the initial phases of the motion, through terms in the cube of the time, are given by the same equations for all values of the ratio E/H.

67. Conducting Media. — In a metallic conductor the valence electrons are supposed to have become free from the atoms of the

metal, leaving the latter positively charged. Since the mean kinetic energy of translation of a particle at temperature T is $\frac{3}{2}kT$, the electrons, on account of their much smaller masses, have much larger velocities than the residual atom cores. Therefore we can consider the atom cores as approximately stationary, the electrons moving in the interstices between them and suffering collisions with them. On account of the small dimensions of the electrons as compared with the atom cores, we can neglect entirely the infrequent collisions of one electron with another.

When an electric field $\mathbf{E}$ is present in a conductor, the free electrons are accelerated by the field and in consequence they acquire a velocity in the direction of the field in the interval of time elapsing between two successive collisions. This added velocity in the direction of the field is lost, however, at the next collision with an atom core, with the result that a general drift in the direction of the field ensues with a mean velocity which is a linear vector function of $\mathbf{E}$. While in the case of a macroscopically isotropic conductor we have the simple relation

$$\mathbf{j} = \sigma \mathbf{E} \tag{67–1}$$

between the current density $\mathbf{j}$, the conductivity σ and the electric intensity $\mathbf{E}$, in the case of a single crystal we need the more general tensor relation

$$\mathbf{j} = \mathbf{\Sigma} \cdot \mathbf{E}, \tag{67–2}$$

where $\mathbf{\Sigma}$ is the *conductivity dyadic*. The crystal structure of all common metallic conductors requires $\mathbf{\Sigma}$ to be symmetric, although it is possible that it may contain a skew-symmetric part in the case of certain less common conducting crystals. In any event $\mathbf{\Sigma}$ can be written in the form

$$\mathbf{\Sigma} = \sigma_x i_1 i_2 + \sigma_y j_1 j_2 + \sigma_z k_1 k_2 \tag{67–3}$$

by a proper orientation of axes, as shown in article 27.

When we measure the conductivity, or its reciprocal the resistivity, of a conducting substance, we place the latter between two electrodes which are maintained at a constant, known, difference of potential. The electric field due to this potential difference we shall call the applied field, and designate it by $\mathbf{E}_0$. Of necessity the current, as soon as a steady state has been reached, has the direction of $\mathbf{E}_0$. The ratio of E_0 to the measured current density j is the experimentally determined resistivity. In the case of an isotropic conductor this

represents the true resistivity $\rho = 1/\sigma$ since the electric intensity $\mathbf{E}$ inside the conductor is identical with $\mathbf{E}_0$. In the case of an anisotropic conductor, however, the initial current must flow in such a direction as to build up charges on the surface of the conductor sufficient to produce a resultant field in the interior in the direction specified by (67–2). Hence, in the steady state, $\mathbf{E}$ is not parallel to $\mathbf{E}_0$, although the component of $\mathbf{E}$ in the direction of the current is still equal to E_0. Consequently the measured resistivity ρ_m is

$$\rho_m = \frac{\mathbf{E}\cdot\mathbf{j}}{j^2}.$$

Solving (67–2) for $\mathbf{E}$ we get

$$\mathbf{E} = \mathbf{\Sigma}^{-1}\cdot\mathbf{j} = (\rho_x \boldsymbol{i}_2\boldsymbol{i}_1 + \rho_y \boldsymbol{j}_2\boldsymbol{j}_1 + \rho_z \boldsymbol{k}_2\boldsymbol{k}_1)\cdot\mathbf{j}$$

where $\rho_x \equiv 1/\sigma_x$, $\rho_y \equiv 1/\sigma_y$, $\rho_z \equiv 1/\sigma_z$. Therefore

$$\mathbf{E}\cdot\mathbf{j} = \mathbf{j}\cdot(\rho_x \boldsymbol{i}_2\boldsymbol{i}_1 + \rho_y \boldsymbol{j}_2\boldsymbol{j}_1 + \rho_z \boldsymbol{k}_2\boldsymbol{k}_1)\cdot\mathbf{j}.$$

Let the crystal be oriented relative to the electrodes so that the direction cosines of $\mathbf{j}$ with $\boldsymbol{i}_1$, $\boldsymbol{j}_1$, $\boldsymbol{k}_1$ and $\boldsymbol{i}_2$, $\boldsymbol{j}_2$, $\boldsymbol{k}_2$ are l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Then

$$\mathbf{j} = j(l_1\boldsymbol{i}_1 + m_1\boldsymbol{j}_1 + n_1\boldsymbol{k}_1) = j(l_2\boldsymbol{i}_2 + m_2\boldsymbol{j}_2 + n_2\boldsymbol{k}_2)$$

and

$$\rho_m = \rho_x l_1 l_2 + \rho_y m_1 m_2 + \rho_z n_1 n_2. \tag{67–4}$$

We are most interested in the case where $\mathbf{\Sigma}$ is symmetric. Then $l_1 = l_2 \equiv l$, $m_1 = m_2 \equiv m$, $n_1 = n_2 \equiv n$ and

$$\rho_m = \rho_x l^2 + \rho_y m^2 + \rho_z n^2. \tag{67–5}$$

Making the current parallel to each of the principal axes of the crystal in turn, we measure the principal resistivities ρ_x, ρ_y, ρ_z.

If two of the principal resistivities are equal, as is often the case, (67–5) assumes the simpler form

$$\begin{aligned}\rho_m &= \rho_x l^2 + \rho_y(m^2 + n^2)\\ &= \rho_x \cos^2\alpha + \rho_y \sin^2\alpha,\end{aligned} \tag{67–6}$$

where α is the angle which the current makes with the unique axis of the crystal.

We shall calculate both the electrical conductivity σ and the thermal conductivity σ_h of an isotropic conductor which, to make the results more general, we shall assume to be in a uniform magnetic

field at right angles to both the electric field and the temperature gradient. As we are interested more in the interrelations of the various phenomena involved than in the numerical factors of the physical coefficients, we shall apply the Maxwellian law of distribution of velocities to the conduction electrons, even though we thereby miss the improvement in the numerical parts of the coefficients which can be obtained by the use of the newer quantum statistics developed by Fermi.

Consider an electron which has just suffered a collision with an atom core and starts off on a free path with initial velocity **V**. Orienting our axes with the Y axis parallel to **E** and the Z axis to **H**, as in article 66, we may write for the components of the initial velocity

$$\begin{aligned} V_x &= V \sin\theta \cos\phi, \\ V_y &= V \sin\theta \sin\phi, \\ V_z &= V \cos\theta, \end{aligned}$$

where θ is the angle which **V** makes with the Z axis, and ϕ is the azimuth measured from the XZ plane. If, then, we put

$$\gamma \equiv \frac{eE}{mc}, \qquad \omega \equiv \frac{eH}{mc},$$

equations (66–27) for the initial phases of the motion become

$$\begin{aligned} x &= Vt \sin\theta \cos\phi + \tfrac{1}{2}\omega V t^2 \sin\theta \sin\phi - \tfrac{1}{6}\omega^2 V t^3 \sin\theta \cos\phi \\ &\qquad + \tfrac{1}{6}\omega\gamma c t^3 - \tfrac{1}{6}\gamma^2 V t^3 \sin\theta \cos\phi + \cdots, \\ y &= Vt \sin\theta \sin\phi - \tfrac{1}{2}\omega V t^2 \sin\theta \cos\phi + \tfrac{1}{2}\gamma c t^2 \\ &\qquad - \tfrac{1}{6}\omega^2 V t^3 \sin\theta \sin\phi - \tfrac{1}{2}\gamma^2 V t^3 \sin\theta \sin\phi + \cdots, \\ z &= Vt \cos\theta - \tfrac{1}{6}\gamma^2 V t^3 \cos\theta + \cdots, \end{aligned}$$

through terms in t^3. These equations provide a sufficiently good approximation to the motion, since the velocities of the conduction electrons in a metal are small compared with the velocity of light, and the free paths are short.

Differentiating with respect to t we find the element $ds = \sqrt{dx^2 + dy^2 + dz^2}$ of the path as a function of t, and integrating

$$s = Vt + \tfrac{1}{2}\gamma c t^2 \sin\theta \sin\phi - \tfrac{1}{6}\omega\gamma c t^3 \sin\theta \cos\phi + \frac{1}{6}\frac{\gamma^2 c^2}{V} t^3 (1 - \sin^2\theta \sin^2\phi) + \cdots,$$

provided we discard terms in V^2/c^2 as compared with unity.

Solving the last equation for t by successive approximations,

$$t = \frac{s}{V} - \frac{1}{2}\frac{\gamma c}{V^3}s^2 \sin\theta \sin\phi + \frac{1}{6}\frac{\omega\gamma c}{V^4}s^3 \sin\theta\cos\phi - \frac{1}{6}\frac{\gamma^2 c^2}{V^5}s^3(1 - 4\sin^2\theta\sin^2\phi) + \cdots.$$

Putting this in the expressions above for x, y, z the coordinates of the displacement of the electron are expressed as functions of s by the following series:

$$\left.\begin{aligned}
x &= s\sin\theta\cos\phi + \frac{1}{2}\frac{\omega}{V}s^2\sin\theta\sin\phi - \frac{1}{2}\frac{\gamma c}{V^2}s^2\sin^2\theta\sin\phi\cos\phi \\
&\quad - \frac{1}{6}\frac{\omega^2}{V^2}s^3\sin\theta\cos\phi \\
&\quad + \frac{1}{6}\frac{\omega\gamma c}{V^3}s^3(1 + \sin^2\theta\cos^2\phi - 3\sin^2\theta\sin^2\phi) \\
&\quad - \frac{1}{6}\frac{\gamma^2}{V^2}s^3\sin\theta\cos\phi \\
&\quad - \frac{1}{6}\frac{\gamma^2 c^2}{V^4}s^3(1 - 4\sin^2\theta\sin^2\phi)\sin\theta\cos\phi + \cdots, \\
y &= s\sin\theta\sin\phi - \frac{1}{2}\frac{\omega}{V}s^2\sin\theta\cos\phi \\
&\quad + \frac{1}{2}\frac{\gamma c}{V^2}s^2(1 - \sin^2\theta\sin^2\phi) - \frac{1}{6}\frac{\omega^2}{V^2}s^3\sin\theta\sin\phi \\
&\quad + \frac{2}{3}\frac{\omega\gamma c}{V^3}s^3\sin^2\theta\sin\phi\cos\phi - \frac{1}{2}\frac{\gamma^2}{V^2}s^3\sin\theta\sin\phi \\
&\quad - \frac{2}{3}\frac{\gamma^2 c^2}{V^4}s^3(1 - \sin^2\theta\sin^2\phi)\sin\theta\sin\phi + \cdots, \\
z &= s\cos\theta - \frac{1}{2}\frac{\gamma c}{V^2}s^2\sin\theta\cos\theta\sin\phi + \frac{1}{6}\frac{\omega\gamma c}{V^3}s^3\sin\theta\cos\theta\cos\phi \\
&\quad - \frac{1}{6}\frac{\gamma^2}{V^2}s^3\cos\theta - \frac{1}{6}\frac{\gamma^2 c^2}{V^4}s^3(1 - 4\sin^2\theta\sin^2\phi)\cos\theta + \cdots.
\end{aligned}\right\} \quad (67\text{–}7)$$

Next we must find an expression for the number of electrons originating in the volume element $dx\ dy\ dz$ at O (Fig. 62) which pass

through the area $(dy\, dz)_P$ at P per unit time. Since equations (67–7) give x, y, z as functions of s, θ, ϕ we have

$$(dy\, dz)_P \frac{\partial x}{\partial s} ds = J\, ds\, d\theta\, d\phi$$

where J is the Jacobian

$$J \equiv \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}.$$

Hence

$$(dy\, dz)_P = \frac{J}{\frac{\partial x}{\partial s}} d\theta\, d\phi,$$

and, if the number of electrons per unit volume per unit time which start out on a new free path with initial speed in the range V to $V + dV$ after collision with an atom core is denoted by dN, then the number leaving the volume element $dx\, dy\, dz$ directed toward the area $(dy\, dz)_P$ is, per unit area at P,

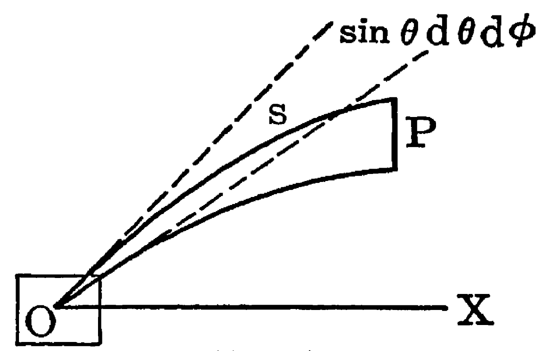

FIG. 62.

$$\frac{dN}{4\pi} \frac{\sin\theta\, d\theta\, d\phi}{(dy\, dz)_P} dx\, dy\, dz = \frac{dN}{4\pi} \frac{\partial x}{\partial s} \frac{\sin\theta}{J} dx\, dy\, dz.$$

But $dx\, dy\, dz = J\, ds\, d\theta\, d\phi$. Therefore this number is

$$\frac{dN}{4\pi} \frac{\partial x}{\partial s} \sin\theta\, ds\, d\theta\, d\phi.$$

Of these, however, only the fraction * $e^{-s/l}$ will reach P before suffering a collision, where l is the mean free path of the electrons. Furthermore, if dn_0 is the number of electrons per unit volume at O

* The derivation of the Kinetic Theory expressions used in this article may be found in L. Page, *Theoretical Physics*, 2nd Edit. Art. 102 and 103.

with speeds between V and $V + dV$, dN is related to dn_0 by the equation

$$dN = \frac{V}{l}\, dn_0.$$

Consequently the number of electrons of initial speed between V and $V + dV$, whose paths originate in the volume element at O, which pass through a unit area at right angles to the X axis at P per unit time is

$$\frac{dn_0}{4\pi}\frac{V}{l}\,\mathrm{e}^{-s/l}\frac{\partial x}{\partial s}\sin\theta\, ds\, d\theta\, d\phi. \tag{67–8}$$

If there is a temperature gradient at right angles to the Z axis, as we have assumed, dn_0 varies with the position of the point O. Hence, if dn is the number of electrons per unit volume at P with speeds between V and $V + dV$, we must put

$$dn_0 = dn - x\frac{\partial}{\partial x}(dn) - y\frac{\partial}{\partial y}(dn)$$

before integrating over s, θ and ϕ to find the total number of electrons passing through the unit area under consideration at P. Next, integrating ϕ from 0 to 2π, θ from 0 to π, and s from 0 to ∞ in (67–8) and similar expressions for unit areas at P perpendicular to the Y and Z axes, we get for the numbers $d\nu_x$, $d\nu_y$, $d\nu_z$ of electrons passing through unit areas perpendicular to the X, Y, Z axes respectively at P per unit time:

$$\left.\begin{aligned} d\nu_x &= \frac{1}{3}\frac{dn\, e^2 l^2}{m^2 V^2 c}EH - \frac{1}{3}Vl\frac{\partial}{\partial x}(dn) - \frac{1}{3}\frac{el^2}{mc}\frac{\partial}{\partial y}(dn)H,\\ d\nu_y &= \frac{2}{3}\frac{dn\, el}{mV}E - \frac{1}{3}Vl\frac{\partial}{\partial y}(dn) + \frac{1}{3}\frac{el^2}{mc}\frac{\partial}{\partial x}(dn)H,\\ d\nu_z &= 0. \end{aligned}\right\} \tag{67–9}$$

To obtain the three components of the electric current density $\mathbf{j}$ we must multiply these expressions by the charge e of an electron and integrate with respect to n, whereas to obtain the three components of the heat current density $\mathbf{j}_h$ we must multiply by the kinetic energy $\frac{1}{2}mV^2$ of an electron and integrate. Now dn is given

in terms of the total number n of electrons per unit volume by Maxwell's distribution law

$$dn = \frac{4n}{\sqrt{\pi}} \tau^{3/2} e^{-\tau V^2} V^2 dV,$$

where $\tau \equiv m/2kT$, k being the Boltzmann constant and T the absolute thermodynamic temperature. Hence

$$\frac{\partial}{\partial x}(dn) = (-\tfrac{3}{2} + \tau V^2)\frac{1}{T}\frac{\partial T}{\partial x} dn,$$

and similarly for the derivative with respect to y. Substituting in (67–9) and integrating with respect to V from 0 to ∞, we get for the electric current density

$$j_x = \int e dv_x = \frac{1}{3}\frac{ne^3 l^2}{mckT} EH - \frac{2}{3}\frac{nelk}{\sqrt{2\pi mkT}}\frac{\partial T}{\partial x},$$

$$j_y = \int e dv_y = \frac{4}{3}\frac{ne^2 l}{\sqrt{2\pi mkT}} E - \frac{2}{3}\frac{nelk}{\sqrt{2\pi mkT}}\frac{\partial T}{\partial y},$$

$$j_z = \int e dv_z = 0,$$

and for the heat current density

$$j_{hx} = \int \frac{1}{2} mV^2 dv_x = \frac{1}{6}\frac{ne^2 l^2}{mc} EH - 2nlk\sqrt{\frac{2kT}{\pi m}}\frac{\partial T}{\partial x} - \frac{1}{2}\frac{nel^2 k}{mc} H \frac{\partial T}{\partial y},$$

$$j_{hy} = \int \frac{1}{2} mV^2 dv_y = \frac{2}{3} nel\sqrt{\frac{2kT}{\pi m}} E - 2nlk\sqrt{\frac{2kT}{\pi m}}\frac{\partial T}{\partial y} + \frac{1}{2}\frac{nel^2 k}{mc} H \frac{\partial T}{\partial x},$$

$$j_{hz} = \int \frac{1}{2} mV^2 dv_z = 0,$$

which we can express more simply in the vector forms

$$\mathbf{j} = \frac{4}{3}\frac{ne^2 l}{\sqrt{2\pi mkT}}\left\{\mathbf{E} - \frac{k}{2e}\nabla T\right\} + \frac{1}{3}\frac{ne^3 l^2}{mckT}\mathbf{E}\times\mathbf{H}, \quad (67\text{–}10)$$

$$\mathbf{j}_h = \frac{2}{3}nel\sqrt{\frac{2kT}{\pi m}}\left\{\mathbf{E} - \frac{3k}{e}\nabla T\right\} + \frac{1}{6}\frac{ne^2l^2}{mc}\mathbf{E}\times\mathbf{H}$$
$$+ \frac{1}{2}\frac{nel^2k}{mc}\mathbf{H}\times\nabla T. \quad (67\text{–}11)$$

First we consider the special case where no transverse magnetic field is present. As the electrical conductivity σ is the ratio of j to E when all parts of the conductor are at the same temperature, and the thermal conductivity σ_h is the ratio of j_h to $|-\nabla T|$ when no electric current is flowing,

$$\sigma = \frac{4}{3}\frac{ne^2l}{\sqrt{2\pi mkT}},$$

$$\sigma_h = \frac{5}{3}nlk\sqrt{\frac{2kT}{\pi m}}.$$

Although there are too many unknown quantities in these expressions to make comparisons with experiment possible, if we take the ratio of the thermal to the electrical conductivity we are led to the simple expression

$$\frac{\sigma_h}{\sigma} = \frac{5}{2}\left(\frac{k}{e}\right)^2 T, \quad (67\text{–}12)$$

known as the *law of Wiedemann and Franz.* A relation of this form holds well for pure metallic conductors, but with a numerical coefficient slightly greater than 3 instead of $\frac{5}{2}$.

The term in $\mathbf{j}_h$ involving $\mathbf{E}$ represents the heat current due to the excess of the energy gained in the course of a free path by electrons moving in the direction of $e\mathbf{E}$ over that lost by electrons moving in the opposite direction. The coefficient $k/(-2e)$ of ∇T in the expression for the electric current $\mathbf{j}$ is the *Thomson coefficient* or *specific heat of electricity*. To investigate its significance, consider a metallic conductor in which both an electric field and a temperature gradient are present. Multiplying (67–10) by the resistivity $\rho = 1/\sigma$ of the metal,

$$\rho\mathbf{j} = \mathbf{E} + \frac{k}{(-2e)}\nabla T. \quad (67\text{–}13)$$

When no current is flowing, there is an electric field in the opposite sense to ∇T, since e is negative. This is due to the transpiration [4]

[4] L. Page, *Theoretical Physics*, 2nd Edit. p. 347.

of electrons from the hotter to the cooler end of the conductor. If we apply an equal and opposite field by external means, so that the resultant $\mathbf{E}$ in the metal is zero, a current will flow from the cooler to the warmer end, the energy necessary to maintain the current being supplied by the heat lost by the electrons in traveling from the hotter to the cooler end of the conductor. So when a resultant $\mathbf{E}$ in the direction of ∇T is present in the metal, a greater current flows than would exist if all parts of the conductor were at the same temperature. The energy relations involved are made more apparent by taking the scalar product of (67–13) with $\mathbf{j}$. Then

$$\rho j^2 = \mathbf{E}\cdot\mathbf{j} + \frac{k}{(-2e)}\,\mathbf{j}\cdot\nabla T,$$

showing that the rate ρj^2 of production of Joule heat per unit volume exceeds the power $\mathbf{E}\cdot\mathbf{j}$ supplied per unit volume by an amount exactly accounted for by attributing a specific head $k/(-2e)$ to each unit charge. The experimentally measured values of this coefficient are, however, much smaller than that given by our theory.

Next we consider the special case where a transverse magnetic field is present but there is no temperature gradient in the metal. The better to visualize the phenomena involved we may think of the conductor as a thin strip of metal with its surface at right angles to the magnetic field. In terms of the electrical conductivity σ we can write (67–10) in the form

$$\mathbf{j} = \sigma\left\{\mathbf{E} + \frac{3\pi}{8nec}\,\sigma\mathbf{E}\times\mathbf{H}\right\}. \tag{67–14}$$

Solving for $\mathbf{E}$,

$$\mathbf{E} = \rho\mathbf{j} - \frac{3\pi}{8nec}\,\sigma\mathbf{E}\times\mathbf{H},$$

where ρ is the resistivity $1/\sigma$. As the second term is very small compared with the first, we can substitute $\mathbf{j}$ for $\sigma\mathbf{E}$ in it, obtaining

$$\mathbf{E} = \rho\mathbf{j} + \frac{3\pi}{8nec}\,\mathbf{H}\times\mathbf{j}. \tag{67–15}$$

This equation shows that the electric field is not in the direction of the current, the factor $3\pi/8nec$ in the transverse component being known as the *Hall coefficient*. This coefficient is measured experimentally by finding the angle which $\mathbf{E}$ makes with $\mathbf{j}$, which is equal to the product of the Hall coefficient by the magnetic intensity divided by the resistivity of the conductor.

As the charge of the electron is negative, the Hall coefficient should be negative for all metals. Unfortunately for the simpie theory which we have developed, this coefficient is positive in nearly as many cases as it is negative.

The term involving $\mathbf{E} \times \mathbf{H}$ in (67–11) shows that an electric current flowing along a strip of metal placed in a magnetic field is accompanied by a transverse heat current the direction of which is independent of the sign of the carrier. This heat current produces a difference in temperature between the two edges of the strip, known as the *Ettinghausen effect.* While the temperature difference has the sense indicated by the theory in the case of most conductors, it has the reverse sense in the case of iron, a result which may be due to the unusual magnetic properties of this metal.

Finally, consider an electrically insulated metallic strip, along which a temperature gradient exists, which is placed in a magnetic field perpendicular to its surface. Putting the value of $\mathbf{E}$ obtained by equating (67–10) to zero, in (67–11), we find for the heat current

$$\mathbf{j}_h = -\frac{5}{3}nlk\sqrt{\frac{2kT}{\pi m}}\nabla T + \frac{7}{12}\frac{nel^2k}{mc}\mathbf{H} \times \nabla T. \qquad (67\text{–}16)$$

The temperature difference between the edges of the strip due to the transverse heat current proportional to $\mathbf{H} \times \nabla T$ is known as the *Righi-Leduc effect.* Although exceptions exist, the sense of the effect is generally that predicted by the theory.

The electric field in the strip, obtained by equating (67–10) to zero, is

$$\mathbf{E} = \frac{k}{2e}\nabla T + \frac{l}{8c}\sqrt{\frac{2\pi k}{mT}}\mathbf{H} \times \nabla T. \qquad (67\text{–}17)$$

The transverse component of $\mathbf{E}$ may be measured by observing the current through a galvanometer connected to opposite points on the edges of the strip. This is known as the *Nernst effect.* While the Nernst effect is in the sense predicted by the theory in bismuth, it is in the opposite sense in iron.

68. Moving Media. — If a material medium has a constant velocity $\mathbf{v}$ relative to the observer's inertial system S, it is generally simpler to discuss any specific problem involving the medium relative to the inertial system S' in which it is at rest and to which the equations derived in this chapter apply, than to discuss it relative to the observer's reference system. Then, if desired, the solution of the

problem can be expressed in terms of measurements made in S by means of the transformations already deduced.

In some cases, however, it may be more convenient to make use of the electromagnetic equations relative to S, which we shall now derive, limiting our investigation to the case of an isotropic, but not necessarily homogeneous, medium at rest in S', for which the relations $\mathbf{D}' = \kappa\mathbf{E}'$ and $\mathbf{B}' = \mu\mathbf{F}'$ hold, where κ and μ may be functions of the coordinates and even of the time. Since $\mathbf{E}'$ and $\mathbf{B}'$ represent, respectively, the true electric and true magnetic fields in S', we shall first express the electromagnetic equations relative to S' in terms of these vectors alone. We have from (62–12),

$$\left.\begin{aligned}
\nabla'\cdot(\kappa\mathbf{E}') &= \rho', & (a)\\
\nabla'\cdot\mathbf{B}' &= 0, & (b)\\
\nabla'\times\mathbf{E}' &= -\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t'}, & (c)\\
\nabla'\times\left(\frac{\mathbf{B}'}{\mu}\right) &= \frac{1}{c}\left\{\frac{\partial}{\partial t'}(\kappa\mathbf{E}') + \rho'\mathbf{V}'\right\}, & (d)\\
\mathscr{F}' &= \rho'\left\{\mathbf{E}' + \frac{1}{c}\mathbf{V}'\times\mathbf{B}'\right\}\cdot & (e)
\end{aligned}\right\} \qquad (68\text{–}1)$$

As usual, we shall take the X and X' axes in the direction of the velocity $\mathbf{v}$ relative to S of the reference system S' in which the medium is at rest. The transformations for the components of $\mathbf{E}'$ and $\mathbf{B}'$ are given by (47–2) and (47–4) respectively with the components of the magnetic induction replacing those of the magnetic intensity. From the Lorentz transformation (42–2) we find for the differential operators

$$\left.\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{1}{\sqrt{1-\beta^2}}\left\{\frac{\partial}{\partial x} + \frac{\beta}{c}\frac{\partial}{\partial t}\cdot\right\},\\
\frac{\partial}{\partial y'} &= \frac{\partial}{\partial y},\\
\frac{\partial}{\partial z'} &= \frac{\partial}{\partial z},\\
\frac{\partial}{\partial t'} &= \frac{1}{\sqrt{1-\beta^2}}\left\{\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right\},
\end{aligned}\right\} \qquad (68\text{–}2)$$

and from (59–1) we have for the current and charge density

$$\left.\begin{aligned} \rho' V_x' &= \rho\frac{V_x - v}{\sqrt{1-\beta^2}}, \\ \rho' V_y' &= \rho V_y, \\ \rho' V_z' &= \rho V_z, \\ \rho' &= \rho\frac{1 - \dfrac{vV_x}{c^2}}{\sqrt{1-\beta^2}}. \end{aligned}\right\} \tag{68–3}$$

The transformation, although laborious, is straight-forward. The equation for the force per unit volume on the free charge retains, of course, the same form relative to S as relative to S'. The field equations (a), (b), (c), (d) become respectively:

$$\nabla\cdot\left\{\kappa\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\right\} = \rho - \frac{1}{c^2}\mathbf{v}\cdot\left\{\frac{\partial}{\partial t}(\kappa\mathbf{E}) + \rho\mathbf{V}\right\}, \tag{a}$$

$$\nabla\cdot\left\{\mathbf{B} - \frac{1}{c}\mathbf{v}\times\mathbf{E}\right\} = -\frac{1}{c^2}\mathbf{v}\cdot\frac{\partial\mathbf{B}}{\partial t}, \tag{b}$$

$$\nabla\times\mathbf{E} + \frac{1}{c}\mathbf{v}\nabla\cdot\mathbf{B} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{c}$$

$$\begin{aligned} \nabla\times&\left\{\frac{\mathbf{B}}{\mu} + \frac{k^2}{c}\left(\kappa - \frac{1}{\mu}\right)\mathbf{v}\times\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\right\} \\ &- \frac{k^2}{c^2}\mathbf{v}\nabla\cdot\left\{\left(\kappa - \frac{1}{\mu}\right)\mathbf{v}\times\left(\mathbf{B} - \frac{1}{c}\mathbf{v}\times\mathbf{E}\right)\right\} - \frac{1}{c}\mathbf{v}\nabla\cdot(\kappa\mathbf{E}) \\ = \frac{1}{c}\frac{\partial}{\partial t}&\left\{\kappa\mathbf{E} + \frac{k^2}{c}\left(\kappa - \frac{1}{\mu}\right)\mathbf{v}\times\left(\mathbf{B} - \frac{1}{c}\mathbf{v}\times\mathbf{E}\right)\right\} + \frac{1}{c}\rho(\mathbf{V} - \mathbf{v}), \end{aligned} \tag{d}$$

where, as usual, $k \equiv 1/\sqrt{1-\beta^2}$.

Eliminating first $\rho\mathbf{V}$ and then ρ from (a) and (d), and first $\nabla\times\mathbf{E}$ and then $\nabla\cdot\mathbf{B}$ from (b) and (c), we find that the five electromagnetic equations assume the standard form (62–12) relative to S provided we now define **D** and **F** by the relations

$$\mathbf{D} \equiv \kappa\mathbf{E} + \frac{k^2}{c}\left(\kappa - \frac{1}{\mu}\right)\mathbf{v}\times\left(\mathbf{B} - \frac{1}{c}\mathbf{v}\times\mathbf{E}\right), \tag{68–4}$$

$$\mathbf{F} \equiv \frac{\mathbf{B}}{\mu} + \frac{k^2}{c}\left(\kappa - \frac{1}{\mu}\right)\mathbf{v}\times\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right). \tag{68–5}$$

If we solve this pair of equations for **D** and **B** as functions of **E** and **F** we get the more symmetrical relations

$$\mathbf{D} = \kappa\mathbf{E} + \frac{1}{c}\frac{\kappa\mu - 1}{1 - \beta^2\kappa\mu}\mathbf{v} \times \left\{\mathbf{F} - \frac{1}{c}\mathbf{v} \times \kappa\mathbf{E}\right\}, \qquad (68\text{–}6)$$

$$\mathbf{B} = \mu\mathbf{F} - \frac{1}{c}\frac{\kappa\mu - 1}{1 - \beta^2\kappa\mu}\mathbf{v} \times \left\{\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mu\mathbf{F}\right\}. \qquad (68\text{–}7)$$

This pair of equations can also be expressed in the simpler but less useful form

$$\mathbf{D} + \frac{1}{c}\mathbf{v} \times \mathbf{F} = \kappa\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right), \qquad (68\text{–}8)$$

$$\mathbf{B} - \frac{1}{c}\mathbf{v} \times \mathbf{E} = \mu\left(\mathbf{F} - \frac{1}{c}\mathbf{v} \times \mathbf{D}\right), \qquad (68\text{–}9)$$

first obtained by Minkowski.

Implicit in (68–4) and (68–5) are the transformations for the components of **D** and **F**. For instance, taking the X axis in the direction of **v** and using (47–5) and (47–6) to transform the components of **E** and **B** respectively, we have from (68–4)

$$D_x = \kappa E_x = \kappa E_x' = D_x',$$

$$D_y = \kappa E_y + k^2\left(\kappa - \frac{1}{\mu}\right)(\beta^2 E_y - \beta B_z)$$

$$= k\left(\kappa E_y' + \beta\frac{B_z'}{\mu}\right) = k(D_y' + \beta F_z'),$$

and similarly for D_z and the components of **F**. Rewriting (47–5) and (47–6) with the components of **B** in place of those of **H**, we have then for the four field vectors the transformations:

$$\left.\begin{array}{ll}
E_x = E_x', & B_x = B_x', \\
E_y = k\{E_y' + \beta B_z'\}, & B_y = k\{B_y' - \beta E_z'\}, \\
E_z = k\{E_z' - \beta B_y'\}; & B_z = k\{B_z' + \beta E_y'\}; \\
 & \\
D_x = D_x', & F_x = F_x', \\
D_y = k\{D_y' + \beta F_z'\}, & F_y = k\{F_y' - \beta D_z'\}, \\
D_z = k\{D_z' - \beta F_y'\}; & F_z = k\{F_z' + \beta D_y'\}.
\end{array}\right\} \qquad (68\text{–}10)$$

It is worthy of note that the pair of transformations for **D** and **F** are identical in form with the pair for **E** and **B**. The inverse transformations are obtained from these by interchanging the primed and unprimed components and changing the sign of the terms in β.

An experiment performed by H. A. Wilson in 1904 affords a test of (68–4). In this experiment a slab of dielectric κ (Fig. 63) is translated with velocity **v** between the plates AB and CD of a condenser through a magnetic field B_z parallel to the Z axis. Neglecting terms in β^2 in (68–4) we have for this case

$$D_y = \kappa E_y - (\kappa - 1)\,\beta B_z$$

since $\mu = 1$. Now the electric intensity outside the dielectric, which is due entirely to the polarization charges of equal magnitude and

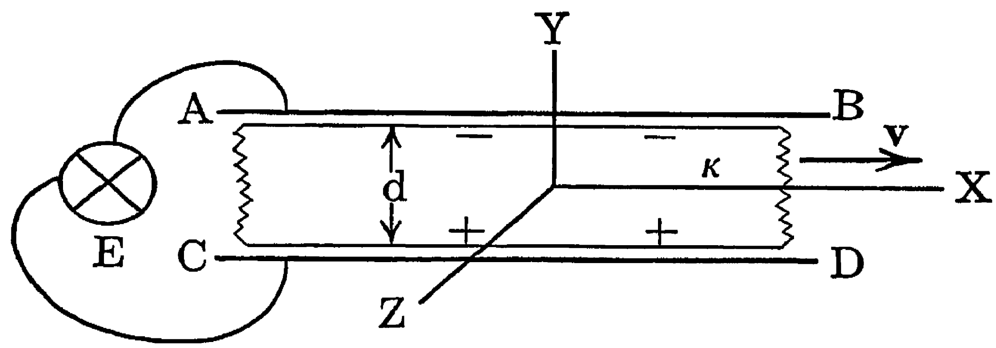

FIG. 63.

opposite sign on the two surfaces of the medium, is negligibly small. Hence the electric displacement D_y, since it is continuous at each surface, must vanish inside the dielectric as well as outside. Consequently

$$E_y = \frac{\kappa - 1}{\kappa}\,\beta B_z, \tag{68–11}$$

which is $(\kappa - 1)/\kappa$ times the electric field due to the charges produced on the surfaces of a conducting slab moving with the same velocity through the same magnetic field. Each plate of the condenser assumes the potential of the adjacent face of the dielectric, and the potential difference $E_y d$ is measured by the electrometer E. The formula (68–11) was well verified by the experimental observations.

The reader's attention must be drawn to the fact that, when a dielectric of permittivity κ is moving with velocity **v** relative to the observer's inertial system, **B** and **F** are not in general the same inside the medium even if its permeability μ is unity. This is easily made evident in the situation described in Fig. 63. Relative to the reference

system S', in which the dielectric is at rest, $D_y' = \kappa E_y'$ and $B_z' = F_z'$. But relative to S we find from (68–10) that $B_z = k\{F_z' + \beta E_y'\}$ and $F_z = k\{F_z' + \beta\kappa E_y'\}$. Hence F_z exceeds B_z by

$$F_z - B_z = \beta k(\kappa - 1)E_y'.$$

This difference, of course, is due to the currents produced by translating with velocity $\mathbf{v}$ the dipoles in the medium which have been lined up by the electric field E_y'. The true magnetic intensity B_z includes the fields due to these dipole currents, whereas F_z does not. Similarly $\mathbf{D}$ and $\mathbf{E}$ are not in general the same in a permeable medium with $\kappa = 1$ which is in motion relative to the observer.

If the moving medium is conducting, the current density relative to the reference system S' in which the medium is at rest is $\rho'\mathbf{V}' = \sigma\mathbf{E}'$, where σ is the conductivity. Hence

$$\begin{aligned} \sigma E_x &= k(\rho V_x - \rho v), \\ \sigma k\{E_y - \beta B_z\} &= \rho V_y, \\ \sigma k\{E_z + \beta B_y\} &= \rho V_z, \end{aligned}$$

by (68–3) and (68–10). These may be combined in the vector equation

$$\rho(\mathbf{V} - \mathbf{v}) = \frac{\sigma}{k}\mathbf{E} + \frac{\sigma k}{c}\mathbf{v} \times \left\{\mathbf{B} - \frac{1}{c}\mathbf{v} \times \mathbf{E}\right\}, \qquad (68\text{–}12)$$

where $\rho(\mathbf{V} - \mathbf{v})$ is the current density relative to the moving conductor, as measured in S. Even when $\mathbf{B} = 0$, the current is not exactly in the direction of $\mathbf{E}$ unless $\mathbf{E}$ is either parallel or perpendicular to $\mathbf{v}$. We can, when $\mathbf{B} = 0$, write (68–12) in the form

$$\rho(\mathbf{V} - \mathbf{v}) = \mathbf{\Sigma}\cdot\mathbf{E} \qquad (68\text{–}13)$$

where $\mathbf{\Sigma}$ is the symmetric dyadic

$$\mathbf{\Sigma} \equiv \frac{\sigma}{k}\boldsymbol{ii} + \sigma k\boldsymbol{jj} + \sigma k\boldsymbol{kk}. \qquad (68\text{–}14)$$

Problem 68a. From (68–10) obtain the transformations for $\mathbf{P} \equiv \mathbf{D} - \mathbf{E}$ and $\mathbf{I} \equiv \mathbf{B} - \mathbf{F}$.

CHAPTER 6

ENERGY, STRESS, MOMENTUM, WAVE MOTION

69. Energy Equation. — In this article we shall derive expressions for the electromagnetic energy and the flux of energy in an electromagnetic field. Inasmuch as we have established the electromagnetic equations for material media, we shall be able to treat concurrently the case of charges and currents in empty space and the case of charges and currents in a material medium at rest in the observer's inertial system. Specifically we shall develop the theory for the latter case, that for the former requiring only the substitution of **E** for **D** and **H** for **B** and **F**.

We proceed from the field equations (62–12*a*) to (62–12*d*). Taking the scalar product of **F** by (62–12*c*) and of **E** by (62–12*d*) and combining, we have

$$\mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{F}\cdot\dot{\mathbf{B}} + c\nabla\cdot(\mathbf{E}\times\mathbf{F}) + \rho\mathbf{V}\cdot\mathbf{E} = 0. \qquad (69\text{–}1)$$

At first we shall exclude ferromagnetic media, and limit ourselves to anisotropic media in which the relations $\mathbf{D} = \mathbf{K}\cdot\mathbf{E}$ and $\mathbf{B} = \mathbf{M}\cdot\mathbf{F}$ hold, where **K** and **M** are symmetric dyadics, or to the simpler case of isotropic media where $\mathbf{D} = \kappa\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{F}$. In an anisotropic medium

$$\mathbf{E}\cdot\dot{\mathbf{D}} = \mathbf{E}\cdot\mathbf{K}\cdot\dot{\mathbf{E}} = \frac{\partial}{\partial t}\left\{\tfrac{1}{2}\mathbf{E}\cdot\mathbf{K}\cdot\mathbf{E}\right\} = \frac{\partial}{\partial t}\left\{\tfrac{1}{2}\mathbf{E}\cdot\mathbf{D}\right\}$$

since the medium is at rest and therefore the elements of **K** are not functions of the time, or in an isotropic medium

$$\mathbf{E}\cdot\dot{\mathbf{D}} = \kappa\mathbf{E}\cdot\dot{\mathbf{E}} = \frac{\partial}{\partial t}\left\{\tfrac{1}{2}\kappa\mathbf{E}\cdot\mathbf{E}\right\} = \frac{\partial}{\partial t}\left\{\tfrac{1}{2}\mathbf{E}\cdot\mathbf{D}\right\},$$

and similarly for $\mathbf{F}\cdot\dot{\mathbf{B}}$. So, in either case, if we integrate (69–1) over a fixed volume τ,

$$\frac{d}{dt}\int_\tau (\tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} + \tfrac{1}{2}\mathbf{F}\cdot\mathbf{B})d\tau + c\int_\sigma \mathbf{E}\times\mathbf{F}\cdot d\sigma + \int_\tau \rho\mathbf{V}\cdot\mathbf{E}d\tau = 0, \quad (69\text{–}2)$$

the second term having been expressed as a surface integral over the surface σ bounding τ by Gauss' theorem.

Now the last term in (69–2) represents the rate at which work is done on the free charges in the region τ by the electromagnetic field, since it is the volume integral of the scalar product of the force $\mathcal{F}$ per unit volume expressed by (62–12*e*) and the velocity $\mathbf{V}$ of the free charge ρ. Even if the volume τ is large enough to contain a completely isolated group of charges, this integral does not, in general, vanish. Therefore, if we wish to retain the law of conservation of energy, we must conclude that work may be performed by an electromagnetic field on a group of charges at the expense of other forms of energy analogous to the kinetic and potential energies of dynamics, which we may consider to be electromagnetic in character. Furthermore, we must recognize the fact that, if $\mathbf{E}$ and $\mathbf{F}$ do not vanish over the surface σ bounding the volume τ, we cannot consider the region τ to constitute an isolated system in so far as the electromagnetic field is concerned, but must take into consideration the possibility of a flow of energy out of or into the region τ through its boundary. The law of conservation of energy may therefore be expressed in the following words: the time rate of increase of the electromagnetic energy in the region τ, plus the time rate of flow of energy out of τ through the bounding surface σ, plus the rate at which work is done by the electromagnetic field on the charges within τ, is equal to zero. This statement leads to the following interpretation of the quantities appearing in (69–2): in each unit volume there is present an electric energy

$$u_E = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{D} \tag{69–3}$$

and a magnetic energy

$$u_H = \tfrac{1}{2}\mathbf{F}\cdot\mathbf{B}, \tag{69–4}$$

and through the boundary there is a flow of energy given in magnitude and direction by the *Poynting flux*

$$\mathbf{s} = c\mathbf{E}\times\mathbf{F} \tag{69–5}$$

per unit cross-section per unit time. We need not consider alternative interpretations, for all interpretations consistent with (69–2) must lead to the same results in so far as measurable energy changes are concerned, and it is obvious that we have chosen the interpretation most directly suggested by the form of the equation.

In an isotropic medium these expressions become

$$u_E = \tfrac{1}{2}\kappa E^2, \quad u_H = \tfrac{1}{2}\mu F^2, \quad \mathbf{s} = c\mathbf{E} \times \mathbf{F}, \tag{69–6}$$

and for charges and currents in empty space

$$u_E = \tfrac{1}{2}E^2, \quad u_H = \tfrac{1}{2}H^2, \quad \mathbf{s} = c\mathbf{E} \times \mathbf{H}. \tag{69–7}$$

The restriction on the type of medium which we made in deducing (69–2) from (69–1) was necessary to enable us to express the first two terms in (69–1) as derivatives with respect to the time. If we do not introduce this restriction we still have for the time rate of increase of electromagnetic energy in the volume τ the expression

$$\int_\tau (\mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{F}\cdot\dot{\mathbf{B}})d\tau,$$

which gives for the electric and magnetic energies per unit volume

$$u_E = \int \mathbf{E}\cdot d\mathbf{D}, \quad u_H = \int \mathbf{F}\cdot d\mathbf{B}. \tag{69–8}$$

These expressions are valid in any medium, even ferromagnetic. The expression for the Poynting flux, of course, remains as before, whatever the nature of the medium may be.

We shall now consider some examples, the medium being either one for which the constitutive relations assumed in deriving (69–2) are valid, or empty space.

In the case of an electrostatic system of charges all at rest in the observer's inertial system, $\mathbf{F}$ is everywhere zero and the Poynting flux vanishes. In this case the total electromagnetic energy is

$$U_E = \int \tfrac{1}{2}\mathbf{E}\cdot\mathbf{D}d\tau \tag{69–9}$$

integrated through all space. This represents the *potential energy* of the group of charges. For example, consider an isolated sphere of radius a with a uniformly distributed surface charge Q, the sphere being surrounded by an infinitely extended isotropic dielectric of permittivity κ. Then E vanishes inside the sphere and is given by $Q/4\pi\kappa r^2$ outside. Hence

$$U_E = \frac{\kappa}{2}\int_a^\infty E^2 4\pi r^2 dr = \frac{Q^2}{8\pi\kappa a}. \tag{69–10}$$

We recognize this as the expression calculated by elementary methods

for the potential energy of a charged sphere. From it we conclude that the capacity of the sphere is $4\pi\kappa a$.

Again, in the case of a purely magnetostatic system of current circuits all at rest in the observer's inertial system, **E** is everywhere zero, the Poynting flux vanishes, and the total electromagnetic energy is

$$U_H = \int \tfrac{1}{2}\mathbf{F}\cdot\mathbf{B}d\tau \tag{69–11}$$

integrated through all space. This may be interpreted as the *kinetic energy* of the currents since it reduces to zero if the moving charges are brought to rest. For instance, consider a circuit consisting of two coaxial cylindrical shells of radii a and b ($b > a$), connected at their ends by perpendicular planes, between which there is present a paramagnetic or diamagnetic medium of permeability μ. If a constant current i flows around this circuit, (62–12d) shows that $F = i/2\pi cr$ between the shells and is zero elsewhere, r being the distance from the axis. Therefore if l is the length of either cylinder,

$$U_H = \frac{\mu l}{2}\int_a^b F^2 2\pi r\, dr = \frac{\mu l i^2}{4\pi c^2}\log\frac{b}{a}, \tag{69–12}$$

giving $(\mu l/2\pi c^2)\log b/a$ for the self-inductance of the circuit. This expression for the magnetic energy of the circuit agrees with that calculated by elementary methods.

In general, however, we do not secure agreement with our ordinary definitions if we interpret electric energy as potential and magnetic energy as kinetic. To illustrate this fact we shall calculate the energy of a Lorentz electron moving through empty space with constant velocity **v** relative to the observer's inertial system S. The electric and magnetic energies are $\int \tfrac{1}{2}E^2 d\tau$ and $\int \tfrac{1}{2}H^2 d\tau$ respectively, which we need integrate only from the surface of the electron to infinity since the field inside the electron vanishes. The integration can be carried out most conveniently in the inertial system S' in which the electron is at rest, for in that inertial system the electron is a sphere of radius a. Taking the X axis in the direction of **v**, we have

$$E_x = E_x', \quad H_x = 0,$$
$$E_y = kE_y', \quad H_y = -k\beta E_z',$$
$$E_z = kE_z', \quad H_z = k\beta E_y',$$

from (47–5) and (47–6). Also, on account of the Fitzgerald-Lorentz contraction, $d\tau = d\tau'/k$. Consequently

$$U_E = \int \tfrac{1}{2}E^2 d\tau = \frac{1}{2k}\int \{E_x'^2 + k^2(E_y'^2 + E_z'^2)\}d\tau',$$

$$U_H = \int \tfrac{1}{2}H^2 d\tau = \frac{k\beta^2}{2}\int (E_y'^2 + E_z'^2)d\tau'.$$

From symmetry

$$\int E_x'^2 d\tau' = \int E_y'^2 d\tau' = \int E_z'^2 d\tau' = \tfrac{1}{3}\int E'^2 d\tau' = \frac{e^2}{12\pi a}.$$

Therefore

$$U_E = \frac{e^2}{6\pi a}\left\{\tfrac{1}{4}\sqrt{1-\beta^2} + \frac{1}{2\sqrt{1-\beta^2}}\right\},$$

$$U_H = \frac{e^2}{6\pi a}\left\{\frac{\beta^2}{2\sqrt{1-\beta^2}}\right\}.$$

Both U_E and U_H are functions of β, and U_H alone does not accord with (57–18). Even if we add U_E and U_H to get the total energy U, and subtract from U the energy U_0 for $v = 0$, we get

$$U - U_0 = \frac{e^2}{6\pi a}\left\{\frac{1}{\sqrt{1-\beta^2}} - \tfrac{1}{4}\sqrt{1-\beta^2} - \tfrac{3}{4}\right\} \qquad (69\text{–}13)$$

which does not agree with the change in kinetic energy

$$T_v - T_{v_0} = \frac{e^2}{6\pi a}\left\{\frac{1}{\sqrt{1-\beta^2}} - 1\right\} \qquad (69\text{–}14)$$

obtained from (57–18).

The discrepancy is due to the fact that the electron is not a rigid structure, but contracts as its velocity relative to the observer increases. Since $U - U_0$ represents the increase in total energy it includes, in addition to the increase in kinetic energy, the work done against the electromagnetic stress on the surface of the electron as the volume of the electron diminishes.

We shall calculate this stress now. Let the broken circle of radius a (Fig. 64) represent a section of the electron as viewed from the inertial system S' in which it is at rest, and the solid ellipse the same section of the electron as it appears to an observer in the inertial

system S. If α is the angle which the normal to the surface element $d\sigma$ makes with the X axis, evidently $\tan\alpha = \sqrt{1-\beta^2}\tan\theta'$

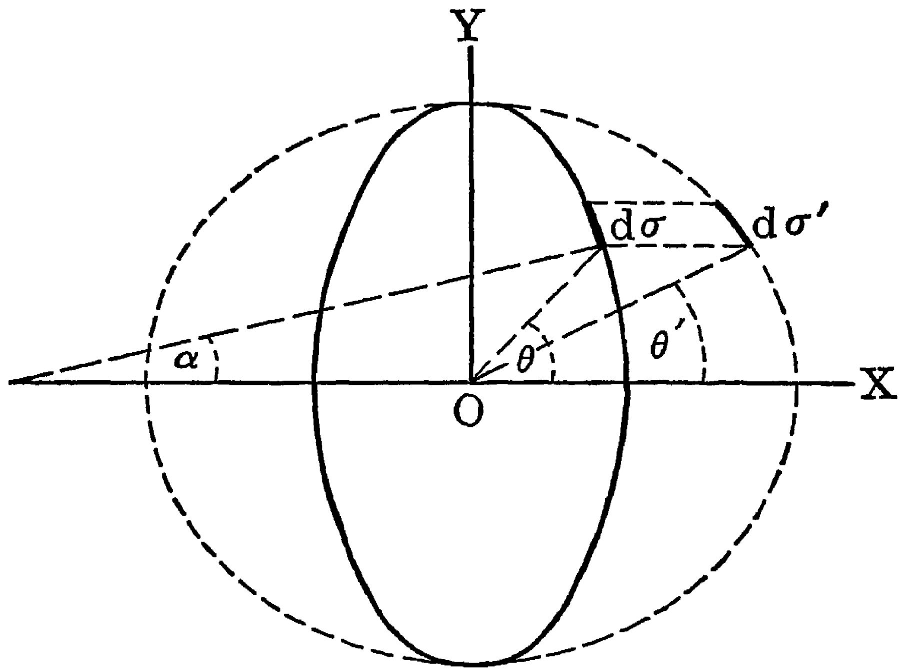

FIG. 64.

$= (1-\beta^2)\tan\theta$. The components of the force $\mathfrak{k}$ per unit charge just outside $d\sigma$ in S are

$$\mathfrak{k}_x = E_x = E_x',$$

$$\mathfrak{k}_y = E_y - \beta H_z = \frac{E_y'}{k}.$$

Hence

$$\frac{\mathfrak{k}_y}{\mathfrak{k}_x} = \frac{1}{k}\frac{E_y'}{E_x'} = \frac{1}{k}\tan\theta' = \tan\alpha,$$

showing that $\mathfrak{k}$ is normal to the surface. As the electromagnetic field vanishes inside the electron, the force to which a unit charge on the surface is subject is

$$\tfrac{1}{2}\mathfrak{k} = \tfrac{1}{2}\sqrt{E_x'^2 + \frac{E_y'^2}{k^2}} = \frac{e}{8\pi a^2}\sqrt{\cos^2\theta' + \frac{\sin^2\theta'}{k^2}}. \qquad (69\text{–}15)$$

To find the charge ρ_σ per unit area on the surface of the electron, we have $\rho_\sigma d\sigma = \rho_\sigma' d\sigma'$ and $\rho_\sigma' = e/4\pi a^2$. Therefore, as

$$d\sigma = d\sigma'\sqrt{\cos^2\theta' + \frac{\sin^2\theta'}{k^2}},$$

we find

$$\rho_\sigma = \frac{e}{4\pi a^2} \frac{1}{\sqrt{\cos^2 \theta' + \dfrac{\sin^2 \theta'}{k^2}}}. \tag{69–16}$$

Multiplying (69–15) by (69–16) we get for the force per unit area or stress

$$\mathscr{S} = \frac{e^2}{32\pi^2 a^4} \tag{69–17}$$

along the normal to the surface. It should be noted that this is a hydrostatic tension independent of the velocity of the electron. The work W done against this stress when the velocity increases is measured by the product of the stress by the decrease in volume, that is,

$$W = \frac{e^2}{6\pi a} \left\{ \tfrac{1}{4} - \tfrac{1}{4}\sqrt{1 - \beta^2} \right\}. \tag{69–18}$$

Adding this to (69–14) gives (69–13).

The Poynting flux finds its chief field of usefulness in radiation problems. Here we will give only one simple illustration. Consider a long, straight conductor of circular cross-section carrying a steady current i. If R is the resistance per unit length, the electric intensity just outside the conductor is $E = Ri$ parallel to the current. The magnetic force just outside is $F = i/2\pi ac$ perpendicular to the current, where a is the radius of the conductor. Therefore there is a Poynting flux through the surface of the conductor directed inwards amounting to $s = cEF = Ri^2/2\pi a$. This represents a flow of energy into the conductor through its surface in the amount Ri^2 per unit length per unit time, accounting quantitatively for the Joule heat produced. Evidently this calculation is valid whether the conductor is surrounded by a material medium or not.

70. Stress and Momentum in a Homogeneous Medium. — In this article we shall limit ourselves to a *homogeneous* medium at rest in the observer's inertial system in which the constitutive relations $\mathbf{D} = \mathbf{K}\cdot\mathbf{E}$ and $\mathbf{B} = \mathbf{M}\cdot\mathbf{F}$ or $\mathbf{D} = \kappa\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{F}$ hold, including empty space as a special case. If, then, a material medium is under consideration, the only free charges which may be contemplated are elementary charges, such as electrons, which may be present in the medium without disturbing its homogeneity. If, on the other hand, the medium is empty space, our formulas are valid for extended free charges of any dimensions.

Eliminating ρ and $\rho\mathbf{V}$ from the equation (62–12e) for the force per unit volume on free charges immersed in a material medium by means of (62–12a) and (62–12d) we get

$$\mathscr{F} = \nabla\cdot\mathbf{DE} + (\nabla\times\mathbf{F})\times\mathbf{B} - \frac{1}{c}\dot{\mathbf{D}}\times\mathbf{B}.$$

Also, from (62–12b) and (62–12c),

$$0 = \nabla\cdot\mathbf{BF} + (\nabla\times\mathbf{E})\times\mathbf{D} + \frac{1}{c}\dot{\mathbf{B}}\times\mathbf{D}.$$

Adding, expanding the triple vector products, and combining terms,

$$\mathscr{F} = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \nabla\cdot(\mathbf{DE}+\mathbf{BF}) - \nabla\mathbf{E}\cdot\mathbf{D} - \nabla\mathbf{F}\cdot\mathbf{B}. \quad (70\text{–}1)$$

If $\mathbf{D} = \mathbf{K}\cdot\mathbf{E}$,

$$\nabla\mathbf{E}\cdot\mathbf{D} = \nabla\mathbf{E}\cdot\mathbf{K}\cdot\mathbf{E} = \tfrac{1}{2}\nabla\{\mathbf{E}\cdot\mathbf{K}\cdot\mathbf{E}\} = \tfrac{1}{2}\nabla\{\mathbf{E}\cdot\mathbf{D}\},$$

as the medium is homogeneous and therefore the elements of $\mathbf{K}$ are not functions of the coordinates; or, if $\mathbf{D} = \kappa\mathbf{E}$,

$$\nabla\mathbf{E}\cdot\mathbf{D} = \kappa\nabla\mathbf{E}\cdot\mathbf{E} = \tfrac{1}{2}\nabla\{\kappa\mathbf{E}\cdot\mathbf{E}\} = \tfrac{1}{2}\nabla\{\mathbf{E}\cdot\mathbf{D}\},$$

and similarly with $\nabla\mathbf{F}\cdot\mathbf{B}$. In either case

$$\mathscr{F} = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \nabla\cdot(\mathbf{DE}+\mathbf{BF}) - \tfrac{1}{2}\nabla(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{F}). \quad (70\text{–}2)$$

The electromagnetic force on all the free charges in the portion of a homogeneous medium lying inside a fixed volume τ is, therefore, given by the integral

$$\mathscr{K} = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{D}\times\mathbf{B}\,d\tau + \int_\sigma (\mathbf{ED}+\mathbf{FB})\cdot d\boldsymbol{\sigma}$$
$$-\tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{B})\,d\boldsymbol{\sigma}, \quad (70\text{–}3)$$

where the last two terms have been expressed as surface integrals with the aid of (17–4) and (17–3). In this expression the force is specified entirely in terms of the field vectors without explicit reference to the free charges and currents on which it acts. The part

$$\mathscr{K}_\tau = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{D}\times\mathbf{B}\,d\tau \quad (70\text{–}4)$$

remaining as a volume integral represents a body force distributed through the volume τ, while the part

$$\mathcal{K}_\sigma = \int_\sigma (\mathbf{ED} + \mathbf{FB})\cdot d\sigma - \tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{B})d\sigma \qquad (70\text{–}5)$$

expressed as surface integrals represents stresses acting on the surface σ bounding τ.

We can write (70–5) in the form

$$\mathcal{K}_\sigma = \int_\sigma d\sigma\cdot\mathbf{\Psi} \qquad (70\text{–}6)$$

where $\mathbf{\Psi}$ is the *stress dyadic*

$$\mathbf{\Psi} \equiv \mathbf{DE} + \mathbf{BF} - \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{B})\mathbf{I}, \qquad (70\text{–}7)$$

$\mathbf{I}$ being the unit dyadic $\boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk}$. In terms of its elements,

$$\left.\begin{aligned}\mathbf{\Psi} = \boldsymbol{ii}\{\tfrac{1}{2}(E_xD_x - E_yD_y - E_zD_z) + \tfrac{1}{2}(F_xB_x - F_yB_y - F_zB_z)\}\\ +\boldsymbol{ij}\{E_yD_x + F_yB_x\} + \boldsymbol{ik}\{E_zD_x + F_zB_x\}\\ + \boldsymbol{ji}\{E_xD_y + F_xB_y\} + \boldsymbol{jj}\{\tfrac{1}{2}(E_yD_y - E_zD_z - E_xD_x)\\ +\tfrac{1}{2}(F_yB_y - F_zB_z - F_xB_x)\} + \boldsymbol{jk}\{E_zD_y + F_zB_y\}\\ + \boldsymbol{ki}\{E_xD_z + F_xB_z\} + \boldsymbol{kj}\{E_yD_z + F_yB_z\}\\ +\boldsymbol{kk}\{\tfrac{1}{2}(E_zD_z - E_xD_x - E_yD_y)\\ + \tfrac{1}{2}(F_zB_z - F_xB_x - F_yB_y)\}.\end{aligned}\right\} \qquad (70\text{–}8)$$

As the stress on the surface element $d\sigma$ is $d\sigma\cdot\mathbf{\Psi}$, the force per unit volume due to the stress system is $\nabla\cdot\mathbf{\Psi}$ from (25–4).

In any electrostatic or magnetostatic field the body force $\mathcal{K}_\tau$ vanishes, since $\mathbf{D}\times\mathbf{B}$ is not a function of the time. If we are interested only in the *mean* force the same is true for any steady radiation field, for, although $\mathbf{D}\times\mathbf{B}$ varies during a single period, its mean value remains unchanged. In either of these cases the mean resultant force is given completely by integrating the stresses over the boundary of the region τ.

On the other hand, if the electric and magnetic fields vanish everywhere on the surface σ the entire force on the free charges within this surface is given by (70–4). This condition can always be met by making the boundary σ sufficiently remote provided $\mathbf{E}$ and $\mathbf{F}$ fall off at least as rapidly as the inverse square of the radius vector r for

all $r > R$, for then products of two components of $\mathbf{E}$ or of $\mathbf{F}$ fall off at least as rapidly as r^{-4} for $r > R$ and the surface integrals in (70–3) vanish with r^{-2}. This is the case with any static or uniformly convected system of charges, or with any radiation field which does not extend to infinity. Under these circumstances we may write (70–3) in the form

$$\mathcal{K} + \frac{1}{c}\frac{d}{dt}\int \mathbf{D} \times \mathbf{B} d\tau = 0, \tag{70–9}$$

where the volume integral is taken over all space. We see from this equation that the electromagnetic forces which the free charges exert on one another do not, in general, form a system in equilibrium, as do the mechanical forces between particles on the Newtonian dynamics. Hence the law of action and reaction no longer holds, if we limit our consideration to the forces on charges alone. We can reestablish this important law, however, if we introduce in addition to forces on charges a force

$$\frac{1}{c}\frac{d}{dt}\int \mathbf{D} \times \mathbf{B} d\tau$$

on the electromagnetic field. The fact that this force appears as a time derivative enables us to interpret

$$\mathbf{G}_l \equiv \frac{1}{c}\int \mathbf{D} \times \mathbf{B} d\tau \tag{70–10}$$

as the linear momentum of the electromagnetic field, and

$$\mathbf{g}_l \equiv \frac{1}{c}\mathbf{D} \times \mathbf{B} \tag{70–11}$$

as the linear momentum per unit volume. Provided we always take into account this electromagnetic momentum, the law of action and reaction and its corollary, the law of conservation of linear momentum, remain valid for charges in an electromagnetic field.

The concept of a force acting on an electromagnetic field in which the volume integral of $\mathbf{D} \times \mathbf{B}$ is not constant in time is not limited, however, to the case where the surface integrals in (70–3) vanish. For, if we write this equation in the form

$$\mathcal{K} + \frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{D} \times \mathbf{B} d\tau$$
$$= \int_\sigma (\mathbf{ED} + \mathbf{FB}) \cdot d\sigma - \tfrac{1}{2}\int_\sigma (\mathbf{E} \cdot \mathbf{D} + \mathbf{F} \cdot \mathbf{B}) d\sigma, \tag{70–12}$$

we can still interpret the second term on the left as the force on the electromagnetic field inside the region τ. Then the equation states that the sum of the force on the free charges in τ and the time rate of increase of the linear momentum of the field in τ is equal to the force due to the stresses on the surface σ bounding τ.

In an isotropic medium (70–11) becomes

$$\mathbf{g}_l = \frac{\kappa\mu}{c}\mathbf{E} \times \mathbf{F} = \frac{\kappa\mu}{c^2}\mathbf{s} \tag{70–13}$$

and in empty space

$$\mathbf{g}_l = \frac{1}{c}\mathbf{E} \times \mathbf{H} = \frac{1}{c^2}\mathbf{s}, \tag{70–14}$$

where $\mathbf{s}$ is the Poynting flux.

As an example we shall calculate the electromagnetic linear momentum of the field of a Lorentz electron moving through empty space with constant velocity $\mathbf{v}$ relative to the observer's inertial system S. It is clear from symmetry that $G_{ly} = G_{lz} = 0$. Using the relations given in the last article,

$$G_{lx} = \frac{1}{c}\int (E_yH_z - E_zH_y)d\tau = \frac{\beta k}{c}\int_a^\infty (E_y'^2 + E_z'^2)d\tau$$

$$= \frac{e^2}{6\pi ac^2}\frac{v}{\sqrt{1 - \beta^2}}.$$

We have, then, for the vector momentum of the field

$$\mathbf{G}_l = \frac{e^2}{6\pi ac^2}\frac{\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m_t\mathbf{v}, \tag{70–15}$$

where m_t is the transverse mass of the electron.

If, now, the electron suffers a very small acceleration, we may still take (70–15) to represent the linear momentum of its field at every instant to a high degree of approximation, since the field in the immediate neighborhood of the electron, which is effectively that of a charge moving with constant velocity, contributes by far the larger part to the integral representing the electromagnetic momentum. Hence the derivative of (70–15) with respect to the time represents the force exerted on its field by the electron, and the negative of this derivative gives the force exerted on the electron by its field, in agreement with the result found in (57–14) by a more direct method.

As another example of the significance of electromagnetic linear momentum consider the infinite parallel plate vacuum condenser depicted in Fig. 65, which is placed in a uniform magnetic field H_0 parallel to the plates. Take the Y axis perpendicular to the plates with the Z axis in the direction of the magnetic field. We shall suppose that the plates have been given equal and opposite charges and then insulated. At the instant 0 we liberate a layer of ions of charge ρ_σ per unit area from the positive plate by suitable means. The electric field between the plates due to the unequal charges remaining on the plates we shall denote by E_0. The initial resultant

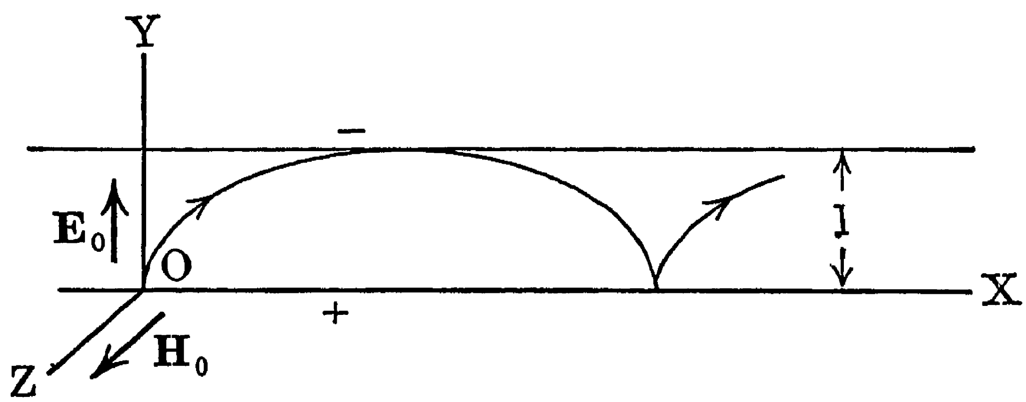

FIG. 65.

electric field is, then, $E_0 + \frac{1}{2}\rho_\sigma$, and, if the distance between the plates is l, the initial electromagnetic linear momentum, per unit area of the plates, is

$$G_{l0} = \frac{l}{c}\left\{E_0 H_0 + \tfrac{1}{2}\rho_\sigma H_0\right\} \tag{70–16}$$

in the X direction.

We shall assume that both E_0 and ρ_σ are very small compared with H_0. Then the velocities of the ions will at all times be small compared with c, and we can use the approximate formulas (66–21) for the ion paths, putting $-eH/mc$ for $k\Omega'$ in accord with (66–19). For an ion starting from rest at the origin O, $\epsilon' = \pi/2$ and

$$\left.\begin{aligned} x &= c\frac{E_0}{H_0}t - \frac{mc^2}{e}\frac{E_0}{H_0{}^2}\sin\frac{eH_0}{mc}t, \\ y &= \frac{mc^2}{e}\frac{E_0}{H_0{}^2}\left\{1 - \cos\frac{eH_0}{mc}t\right\}. \end{aligned}\right\} \tag{70–17}$$

We are going to compare the electromagnetic momentum when the ions are at the tops of their cycloidal paths with that existing

initially. For simplicity we shall suppose that the ions just fail to reach the negative plate of the condenser. Then

$$l = y_{\max} = 2\frac{mc^2}{e}\frac{E_0}{H_0{}^2}. \tag{70–18}$$

At the tops of their paths the ions have a velocity $v = 2cE_0/H_0$ in the X direction and therefore give rise to a magnetic field $\rho_\sigma E_0/H_0$ opposite to H_0 in the region between the plates. The resultant electric field between the plates is now $E_0 - \frac{1}{2}\rho_\sigma$. The electromagnetic linear momentum per unit area of the plates is then

$$G_l = \frac{l}{c}\left\{E_0 - \tfrac{1}{2}\rho_\sigma\right\}\left\{H_0 - \rho_\sigma\frac{E_0}{H_0}\right\}. \tag{70–19}$$

The loss in electromagnetic momentum per unit area is therefore

$$G_{l0} - G_l = \frac{l}{c}\rho_\sigma H_0$$

to our degree of approximation. Using (70–18) this becomes

$$G_{l0} - G_l = 2\frac{mc}{e}\rho_\sigma\frac{E_0}{H_0} = m\frac{\rho_\sigma}{e}v. \tag{70–20}$$

But, as ρ_σ/e represents the number of ions per unit area of the layer, this is just the mechanical momentum gained. We have here a conversion of electromagnetic momentum of the field into mechanical momentum of the ion stream, the *total* linear momentum being conserved.

The electromagnetic torque $\mathscr{L}$ about any fixed origin O on all the free charges in the portion of a homogeneous medium lying inside a fixed volume τ is $\int_\tau \mathbf{r} \times \mathscr{F} d\tau$, where $\mathbf{r}$ is the radius vector from O to the volume element $d\tau$. Using (70–2) for $\mathscr{F}$,

$$\mathscr{L} = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{r} \times (\mathbf{D} \times \mathbf{B})d\tau + \int_\tau \mathbf{r} \times \{\nabla\cdot(\mathbf{DE} + \mathbf{BF})\}d\tau$$
$$-\tfrac{1}{2}\int_\tau \mathbf{r} \times \nabla(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{F})d\tau.$$

Now, with the aid of (17–4),

$$\int_\tau \mathbf{r} \times (\nabla \cdot \overline{\mathbf{DE}}) d\tau = -\int_\tau \nabla \cdot \overline{\mathbf{DE} \times \mathbf{r}}\, d\tau - \int_\tau \mathbf{D} \cdot \nabla \mathbf{r} \times \mathbf{E} d\tau$$

$$= \int_\sigma \mathbf{r} \times \mathbf{ED} \cdot d\sigma + \int_\tau \mathbf{E} \times \mathbf{D} d\tau,$$

since

$$\nabla \mathbf{r} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (ix + jy + kz) = ii + jj + kk.$$

Also

$$\int_\tau \mathbf{r} \times \nabla(\mathbf{D} \cdot \mathbf{E}) d\tau = -\int_\tau \nabla \times \overline{(\mathbf{E} \cdot \mathbf{D})\mathbf{r}}\, d\tau + \int_\tau \mathbf{D} \cdot \mathbf{E} \nabla \times \mathbf{r} d\tau$$

$$= \int_\sigma \mathbf{E} \cdot \mathbf{D} \mathbf{r} \times d\sigma,$$

if we use (17–5) and note that $\nabla \times \mathbf{r}$ vanishes identically. Similar expressions hold for the corresponding integrals in **B** and **F**. Consequently

$$\mathscr{L} = -\frac{1}{c}\frac{d}{dt} \int_\tau \mathbf{r} \times (\mathbf{D} \times \mathbf{B}) d\tau + \int_\tau (\mathbf{E} \times \mathbf{D} + \mathbf{F} \times \mathbf{B}) d\tau$$

$$+ \int_\sigma \mathbf{r} \times (\mathbf{ED} + \mathbf{FB}) \cdot d\sigma - \tfrac{1}{2} \int_\sigma (\mathbf{E} \cdot \mathbf{D} + \mathbf{F} \cdot \mathbf{B}) \mathbf{r} \times d\sigma. \quad (70\text{–}21)$$

Comparing with (70–3), we notice that the integrand of each term is the vector moment about O of the integrand of the corresponding term in (70–3), with the exception of the second volume integral, to which no term in (70–3) corresponds. This volume integral, which fails to vanish only in an anisotropic medium, represents a couple independent of **r**. It is, however, exactly compensated by an equal and opposite couple due to the unsymmetrical stress system represented by the two surface integrals. For we can put the torque $\mathscr{L}_\sigma$ specified by these integrals in the form

$$\mathscr{L}_\sigma = \int_\sigma \mathbf{r} \times (d\sigma \cdot \mathbf{\Psi}) \quad (70\text{–}22)$$

where $\mathbf{\Psi}$ is the stress dyadic (70–8). Then (25–5) shows that the stresses give rise to a couple

$$\mathbf{\Psi}_* = \mathbf{D} \times \mathbf{E} + \mathbf{B} \times \mathbf{F} \quad (70\text{–}23)$$

per unit volume. If this compensation did not take place, each elementary free charge in an anisotropic medium would acquire an infinite angular acceleration in the presence of an impressed electromagnetic field.

To interpret (70–21) we arrange the equation in the form

$$\mathscr{L} + \frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{r} \times (\mathbf{D} \times \mathbf{B})d\tau = \int_\sigma \mathbf{r} \times (\mathbf{ED} + \mathbf{FB})\cdot d\boldsymbol{\sigma}$$
$$-\tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{B})\mathbf{r} \times d\boldsymbol{\sigma} + \int_\tau (\mathbf{E} \times \mathbf{D} + \mathbf{F} \times \mathbf{B})d\tau. \quad (70\text{–}24)$$

Since

$$\mathbf{g}_a \equiv \frac{1}{c}\mathbf{r} \times (\mathbf{D} \times \mathbf{B}) = \mathbf{r} \times \mathbf{g}_l \quad (70\text{–}25)$$

is the vector moment of the electromagnetic linear momentum, or the electromagnetic angular momentum, per unit volume, the equation states that the sum of the torque on the free charges in τ and the time rate of increase of the angular momentum of the field in τ, is equal to the torque due to the stresses on the surface σ bounding τ less, in the case of an anisotropic medium, the couple produced by the unsymmetrical stress system. Provided we attribute to each unit volume of the electromagnetic field the angular momentum specified by (70–25), the law of angular action and reaction and its corollary, the law of conservation of angular momentum, remain valid for charges in an electromagnetic field.

We can write (70–25) for an isotropic medium in the form

$$\mathbf{g}_a = \frac{\kappa\mu}{c}\mathbf{r} \times (\mathbf{E} \times \mathbf{F}) = \frac{\kappa\mu}{c^2}\mathbf{r} \times \mathbf{s} \quad (70\text{–}26)$$

and for empty space in the form

$$\mathbf{g}_a = \frac{1}{c}\mathbf{r} \times (\mathbf{E} \times \mathbf{H}) = \frac{1}{c^2}\mathbf{r} \times \mathbf{s}, \quad (70\text{–}27)$$

where $\mathbf{s}$ is the Poynting flux.

As an illustration of the concept of electromagnetic angular momentum we shall develop the theory of a famous experiment first performed by Trouton and Noble in an attempt to detect the motion of the earth relative to the ether. Consider a vacuum parallel plate

condenser (Fig. 66) which has a velocity $\mathbf{v}$ in a direction making an angle α with the normal $\mathbf{n}$ to the plates AB and CD relative to the inertial system S (the ether). Take the X axis parallel to $\mathbf{v}$ and the Y axis in the plane of $\mathbf{v}$ and $\mathbf{n}$. We shall calculate the electromagnetic momentum of the field between the condenser plates, treating the condenser as if the plates were infinite in extent.

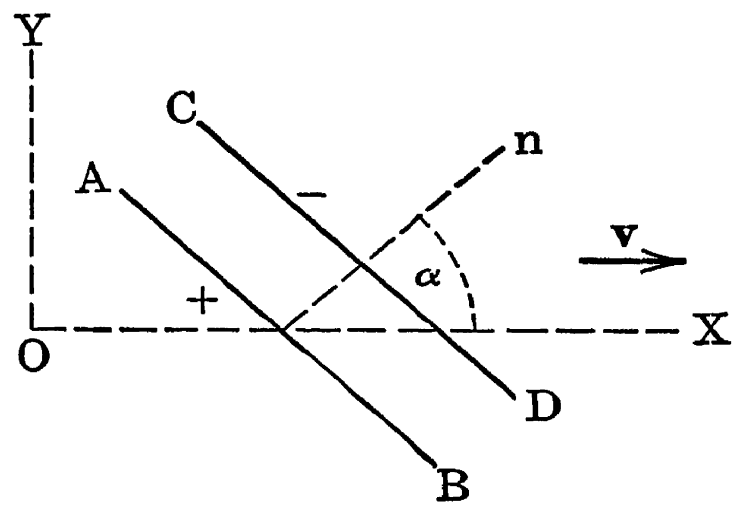

FIG. 66.

The first step is to find the field between the plates of the condenser. Evidently the angle α' which the normal to the plates makes with X' in the inertial system S' in which the condenser is at rest is given by $\tan \alpha' = \tan \alpha/\sqrt{1 - \beta^2}$. Also, if ρ_σ and ρ_σ' are the charges per unit area of the plates in S and S' respectively,

$$\rho_\sigma' = \rho_\sigma \frac{\cos \alpha'}{\cos \alpha} = \rho_\sigma \frac{\sqrt{1 - \beta^2}}{\sqrt{1 - \beta^2 \cos^2 \alpha}}.$$

In S', then,

$$E_x' = \rho_\sigma' \cos \alpha' = \rho_\sigma \frac{(1 - \beta^2) \cos \alpha}{1 - \beta^2 \cos^2 \alpha},$$

$$E_y' = \rho_\sigma' \sin \alpha' = \rho_\sigma \frac{\sqrt{1 - \beta^2} \sin \alpha}{1 - \beta^2 \cos^2 \alpha},$$

$$H_z' = 0.$$

Hence, by (47–5) and (47–6),

$$E_x = E_x' = \rho_\sigma \frac{(1 - \beta^2) \cos \alpha}{1 - \beta^2 \cos^2 \alpha},$$

$$E_y = kE_y' = \rho_\sigma \frac{\sin \alpha}{1 - \beta^2 \cos^2 \alpha},$$

$$H_z = \beta k E_y' = \rho_\sigma \frac{\beta \sin \alpha}{1 - \beta^2 \cos^2 \alpha},$$

in S. Consequently the electromagnetic linear momentum per unit volume is

$$\mathbf{g}_l = \frac{1}{c}(\mathbf{E}\times\mathbf{H}) = \frac{\rho_\sigma{}^2 \dfrac{\beta}{c}\sin\alpha}{(1-\beta^2\cos^2\alpha)^2}\{\mathbf{i}\sin\alpha - \mathbf{j}(1-\beta^2)\cos\alpha\}. \quad (70\text{–}28)$$

If τ is the volume of the condenser, the electromagnetic angular momentum about the point O is

$$\mathbf{G}_a = \int_\tau \mathbf{r}\times\mathbf{g}_l d\tau$$

and the time rate of decrease of this vector is

$$-\frac{d\mathbf{G}_a}{dt} = -\int_\tau \mathbf{v}\times\mathbf{g}_l\, d\tau = \mathbf{k}\rho_\sigma{}^2\beta^2\tau\sin\alpha\cos\alpha \qquad (70\text{–}29)$$

if we neglect terms in β^2 as compared with unity. Trouton and Noble concluded that a moving condenser would experience a torque of this magnitude tending to turn the plates parallel to the direction of motion. Experiment, however, supported the relativity principle in showing that no such torque exists.

How are we to explain this apparent paradox? It might be suspected that the torque (70–29) is balanced by an equal and opposite torque which our analysis has not revealed because we have neglected the edge effect in treating the condenser as if its plates were infinite in extent. That no significant error has been introduced by this approximation may be made evident, however, by considering an equivalent problem which we can solve exactly by elementary methods. The discussion of this problem will disclose the answer to the paradox.

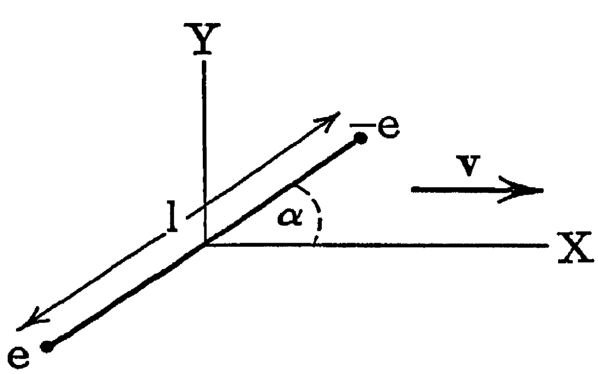

FIG. 67.

In this equivalent problem we consider two point charges e and $-e$ (Fig. 67) held apart by an insulating rod of length l which is moving with velocity $\mathbf{v}$ in a direction making an angle α with its length. Let the rod lie in the XY plane with the X axis in the direction of $\mathbf{v}$. From (54–3) and (54–4) we have for the electric field at $-e$ due to e

$$E = \frac{e(1-\beta^2)}{4\pi l^2(1-\beta^2\sin^2\alpha)^{3/2}}$$

parallel to the rod, and for the magnetic field

$$H = \frac{e(1 - \beta^2)\beta \sin \alpha}{4\pi l^2(1 - \beta^2 \sin^2 \alpha)^{3/2}}$$

along the Z axis. Therefore the force on the charge $-e$ is

$$\begin{aligned}\mathscr{F} &= -e\left\{\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right\} \\ &= \frac{e^2(1 - \beta^2)}{4\pi l^2(1 - \beta^2 \sin^2 \alpha)^{3/2}}\{-\boldsymbol{i} \cos \alpha - \boldsymbol{j}(1 - \beta^2) \sin \alpha\},\end{aligned}$$

and the force on e is equal and opposite to this. As, however, these forces do not act along the line joining the two charges, they give rise to a couple whose torque is

$$\mathscr{L} = k\frac{e^2}{4\pi l}\beta^2 \sin \alpha \cos \alpha \tag{70–30}$$

if we neglect terms in β^2 as compared with unity. This torque is of exactly the same form as (70–29), showing that our analysis of the Trouton-Noble experiment was not vitiated by our failure to include the edge effect.

If we view the mechanism of Fig. 67 from the inertial system S' it is evident that it cannot experience an angular acceleration. Hence we must conclude that the torque $\mathscr{L}$ given by (70–30) is balanced by an equal and opposite torque. The latter, obviously, is supplied by the forces exerted on the charges by the insulating rod which holds them apart, a pair of forces which we have hitherto neglected. The relativity principle requires the forces exerted by the rod to suffer the same aberration when the mechanism is in motion relative to the observer as do the electromagnetic forces which the charges exert on each other. If all forces are electromagnetic in character, this is quite in accord with expectations.

Our analysis of the Trouton-Noble experiment was incomplete, then, in that we failed to take into account the forces exerted on the plates by the insulating separators which hold the plates at a fixed distance from each other. These forces must give rise to a torque equal and opposite to that specified by (70–29).

As a final example of electromagnetic angular momentum we shall analyze the motion of an ion in the field of a particle fixed at the origin which has an electric charge q and a magnetic moment $\mathbf{p}_H$

parallel to the Z axis. To a first approximation this is the type of field existing near the surface of the earth. The electric and magnetic intensities due to the charged magnetic dipole at a point P whose position vector is $\mathbf{r}$ are respectively

$$\mathbf{E} = \frac{q}{4\pi r^3}\,\mathbf{r}, \quad \mathbf{H} = \frac{1}{4\pi r^5}\left\{2\mathbf{p}_H\cdot\mathbf{r}\mathbf{r} + (\mathbf{p}_H \times \mathbf{r}) \times \mathbf{r}\right\}, \qquad (70\text{–}31)$$

by (58–11). In this field there is a Poynting flux $\mathbf{s} = c\mathbf{E} \times \mathbf{H}$ directed along the circles of latitude about the Z axis, and an angular electromagnetic momentum $\mathbf{G}_a = (1/c^2)\int \mathbf{r} \times \mathbf{s}\,d\tau$ parallel to the Z axis. As the Poynting flux has no component along the radius vector, however, there is no flow of energy away from the dipole.

Now suppose an ion of charge e is present at P. We shall limit our discussion to velocities small compared with c, so that we can neglect the variation of the ion's mass with velocity and neglect its magnetic field as compared with its electric field, taking for the latter that of a point charge at rest. On account of the term in $\mathbf{V} \times \mathbf{H}$ in the force equation, there is a torque on the ion about the Z axis amounting to

$$\begin{aligned}\mathcal{L}_z &= \frac{e}{c}\,\mathbf{k}\cdot\left\{\mathbf{r} \times (\mathbf{V} \times \mathbf{H})\right\} = \frac{ep_H}{4\pi c}\left\{\frac{\dot{r}\sin^2\theta}{r^2} - 2\frac{\sin\theta\cos\theta\,\dot{\theta}}{r}\right\} \\ &= -\frac{ep_H}{4\pi c}\frac{d}{dt}\left(\frac{\sin^2\theta}{r}\right),\end{aligned} \qquad (70\text{–}32)$$

where θ is the angle which $\mathbf{r}$ makes with $\mathbf{p}_H$. This represents the time rate of increase of the angular momentum of the ion about the Z axis.

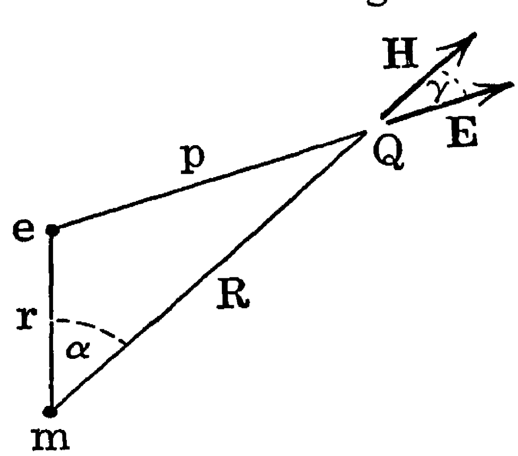

Fig. 68.

To simplify the computation of the electromagnetic angular momentum about the Z axis due to the electric field of the ion and the magnetic field of the dipole, we shall first calculate the angular momentum of the fields of an ion e (Fig. 68) and a single magnetic pole m a distance r apart. If p and R are the distances of the field-point Q from e and m respectively,

$$E = \frac{e}{4\pi p^2}, \qquad H = \frac{m}{4\pi R^2}.$$

It is evident that the entire electromagnetic angular momentum $\mathbf{G}_a$ is along the line joining m to e. If α is the angle between r and R and γ that between E and H,

$$G_a = -\frac{1}{c}\int_0^\infty \int_0^{2\pi} R \sin\alpha\, EH \sin\gamma\; 2\pi R^2 \sin\alpha\, d\alpha\, dR$$
$$= -\frac{mer}{8\pi c}\int_0^\infty \int_0^{2\pi} \frac{R\sin^3\alpha}{p^3}\, d\alpha\, dR$$

since $p \sin\gamma = r \sin\alpha$. Changing the variables of integration from α and R to p and R by means of the relation $p^2 = R^2 - 2Rr\cos\alpha + r^2$, we have

$$Rr\int_0^{2\pi} \frac{\sin^3\alpha\, d\alpha}{p^3} = \int \left\{1 - \left(\frac{r^2 + R^2 - p^2}{2Rr}\right)^2\right\}\frac{dp}{p^2}$$
$$= \frac{(r^2 - R^2)^2}{4R^2r^2}\frac{1}{p} + \frac{r^2 + R^2}{2R^2r^2}\, p - \frac{1}{12R^2r^2}\, p^3.$$

For $R < r$ the limits of p are $r - R$ and $r + R$, whereas for $R > r$ the limits are $R - r$ and $R + r$. Hence

$$G_a = -\frac{me}{6\pi r^2 c}\int_0^r R\,dR - \frac{mer}{6\pi c}\int_r^\infty \frac{dR}{R^2} = -\frac{me}{4\pi c} \qquad (70\text{–}33)$$

independent of the distance r of the charge from the pole.

Since we can form a magnetic dipole of moment $\mathbf{p}_H$ along the Z axis by placing a magnetic charge $-m$ at the origin and a magnetic charge m at the point $z = p_H/m$, the component of the electromagnetic angular momentum along the Z axis due to the electric field of the ion and the magnetic field of the dipole originally considered is

$$G_a = \frac{ep_H}{4\pi c}\frac{\partial}{\partial z}\cos\theta = \frac{ep_H}{4\pi c}\frac{\sin^2\theta}{r}. \qquad (70\text{–}34)$$

Comparing this with (70–32) we see that the angular momentum acquired by the ion is at the expense of the electromagnetic angular momentum of the field. If, for instance, a group of ions of opposite sign to that of the electric charge q on the dipole, which are initially at rest, approach the dipole under the electrical force of attraction, they will acquire angular momentum due to the torque exerted on them by the magnetic field. If, finally, they strike the dipole, this angular momentum will be transferred to the latter. This gain in

mechanical angular momentum, however, is just compensated by the loss of electromagnetic angular momentum of the field.

71. Stress and Momentum in a Non-Homogeneous Medium.—The application of the results obtained in the last article to material media is severely limited by the fact that, in order to pass from (70–1) to (70–2), we were forced to restrict ourselves to a homogeneous medium in which $\mathbf{D} = \mathbf{K}\cdot\mathbf{E}$ and $\mathbf{B} = \mathbf{M}\cdot\mathbf{F}$. Not only are the equations of article 70 inapplicable to a ferromagnetic medium, but we are not even warranted in using them to calculate the force on a charged conductor surrounded by a dielectric fluid, since the medium is not homogeneous throughout the volume over which we must integrate. These limitations are due to the fact that our analysis rested on the expression for the force on the free charges and currents alone. By calculating the force on all the charges and currents present—electric dipoles and Ampèrian currents as well as free charges—we shall now be able to deduce expressions of such broad applicability that no constitutive relations need be assumed, and no restrictions on the homogeneity of the medium need be made.

Distinguishing the forces per unit volume on the free charges, electric dipoles and Ampèrian currents, by $\mathscr{F}_F$, $\mathscr{F}_P$ and $\mathscr{F}_I$, respectively, we have from (70–1),

$$\mathscr{F}_F = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \nabla\cdot(\mathbf{DE}+\mathbf{BF}) - \nabla\mathbf{E}\cdot\mathbf{D} - \nabla\mathbf{F}\cdot\mathbf{B}. \qquad (71\text{–}1)$$

In calculating the electric force on a dipole of moment $\mathbf{p}_E$ which constitutes part of a material medium, we must make use of the mean electric intensity $\mathbf{E}$ less the contribution $\mathbf{E}_P$ to $\mathbf{E}$ made by the fields of the charges constituting the dipole. Thus the electric force on a single dipole is $\mathbf{p}_E\cdot\nabla(\mathbf{E}-\mathbf{E}_P)$ and the force per unit volume is $\mathbf{P}\cdot\nabla(\mathbf{E}-\mathbf{E}_P)$. Now whatever the value of $\mathbf{E}_P$ may be, it is certain that $\mathbf{E}_P$ in any portion of the medium must be proportional to the polarization existing there. So we can write $\mathbf{E}_P = \alpha\mathbf{P}$, where α is a constant. Furthermore, if the electric moment is changing with the time, there is an additional force $\dot{\mathbf{p}}_E\times(\mathbf{B}-\mathbf{B}_P)/c$ on each dipole due to the magnetic field, where $\mathbf{B}_P$ is the contribution to the mean magnetic field $\mathbf{B}$ made by the oscillating dipole itself. It is, however, quite clear from symmetry that $\mathbf{B}_P = 0$. Therefore the force per unit volume due to the magnetic field is $\dot{\mathbf{P}}\times\mathbf{B}/c$ and

$$\mathscr{F}_P = \mathbf{P}\cdot\nabla\mathbf{E} - \mathbf{P}\cdot\nabla\mathbf{E}_P + \frac{1}{c}\dot{\mathbf{P}}\times\mathbf{B}.$$

But $\nabla \times \mathbf{E}_P = 0$ since $\mathbf{B}_P$ remains always zero, and hence

$$0 = (\nabla \times \mathbf{E}_P) \times \mathbf{P} = \mathbf{P}\cdot\nabla\mathbf{E}_P - \nabla\mathbf{E}_P\cdot\mathbf{P} = \mathbf{P}\cdot\nabla\mathbf{E}_P - \frac{\alpha}{2}\nabla P^2.$$

Also, from (62–12c)

$$0 = -(\nabla \times \mathbf{E}) \times \mathbf{P} - \frac{1}{c}\dot{\mathbf{B}} \times \mathbf{P} = -\mathbf{P}\cdot\nabla\mathbf{E} + \nabla\mathbf{E}\cdot\mathbf{P} + \frac{1}{c}\mathbf{P} \times \dot{\mathbf{B}}.$$

Adding the last three equations

$$\mathscr{F}_P = \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{P} \times \mathbf{B}) + \nabla\mathbf{E}\cdot\mathbf{P} - \frac{\alpha}{2}\nabla P^2. \qquad (71\text{–}2)$$

No force is exerted on an Ampèrian current by the electric field, but a force $\nabla\mathbf{H}\cdot\mathbf{p}_H$ is exerted on a circuit of moment $\mathbf{p}_H$ by an external magnetic field $\mathbf{H}$ in accord with (64–19). In the case of an Ampèrian current constituting part of a material medium, we must use for $\mathbf{H}$ in this formula the true magnetic intensity $\mathbf{B}$ less the contribution $\mathbf{B}_I$ to $\mathbf{B}$ made by the magnetic field of the circuit itself. Evidently $\mathbf{B}_I$ must be proportional to the intensity of magnetization $\mathbf{I}$, so we can write $\mathbf{B}_I = (1 + \gamma)\,\mathbf{I}$ where γ is a constant. Hence, as $\mathbf{B} - \mathbf{I} = \mathbf{F}$,

$$\mathscr{F}_I = \nabla\mathbf{F}\cdot\mathbf{I} - \frac{\gamma}{2}\nabla I^2. \qquad (71\text{–}3)$$

Adding (71–1), (71–2) and (71–3) we find for the total electromagnetic force per unit volume

$$\mathscr{F}_E = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \nabla\cdot(\mathbf{DE} + \mathbf{BF}) - \tfrac{1}{2}\nabla(E^2 + F^2) - \tfrac{1}{2}\nabla(\alpha P^2 + \gamma I^2). \qquad (71\text{–}4)$$

From this we find for the electromagnetic force on the free charges *and the material medium* lying inside a fixed volume τ

$$\mathscr{K}_E = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{E} \times \mathbf{B}\,d\tau + \int_\sigma (\mathbf{ED} + \mathbf{FB})\cdot d\boldsymbol{\sigma} - \tfrac{1}{2}\int_\sigma (E^2 + F^2)\,d\boldsymbol{\sigma} - \tfrac{1}{2}\int_\sigma (\alpha P^2 + \gamma I^2)\,d\boldsymbol{\sigma}. \qquad (71\text{–}5)$$

This expression is valid for any kind of medium or body, no matter what constitutive relations may exist between $\mathbf{D}$ and $\mathbf{E}$ and between $\mathbf{B}$ and $\mathbf{F}$.

However, it must be borne in mind that (71–5) gives only the *electromagnetic* force on the matter located in the volume τ. Therefore it represents the total force only in the case that no mechanical forces, even though they be of ultimate electromagnetic origin, are acting. The total force on a body such as a permanent magnet, for instance, is obtained by evaluating (71–5) over the region enclosed in a surface lying just outside the body only when the exterior medium is empty space. Then, since $\mathbf{P} = \mathbf{I} = 0$ and $\mathbf{D} = \mathbf{E}, \mathbf{F} = \mathbf{B}$ everywhere on the surface σ,

$$\mathscr{K}_E = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{E} \times \mathbf{B} d\tau + \int_\sigma (\mathbf{EE} + \mathbf{BB})\cdot d\boldsymbol{\sigma} - \tfrac{1}{2}\int_\sigma (E^2 + B^2) d\boldsymbol{\sigma}. \tag{71–6}$$

If, on the other hand, the body is immersed in a fluid, the field exerts forces on the fluid elements which in turn are transmitted to the body in the form of mechanical stresses on its surface, and the force due to these stresses must be added to the electromagnetic force (71–5) to find the total force. To deal with problems of this kind we shall find the electromagnetic force $\mathscr{F}_{Ef}$ on a homogeneous region of an isotropic fluid of permittivity κ and permeability μ in which no free charge or current is located. As $\nabla\cdot\mathbf{D}$ and $\nabla\cdot\mathbf{B}$ both vanish, (71–4) gives for the force per unit volume

$$\mathscr{F}_{Ef} = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \kappa\mathbf{E}\cdot\nabla\mathbf{E} + \mu\mathbf{F}\cdot\nabla\mathbf{F} - \tfrac{1}{2}\nabla(E^2 + F^2) - \tfrac{1}{2}\nabla(\alpha P^2 + \gamma I^2). \tag{71–7}$$

But, since $\nabla\times\mathbf{E} + \dot{\mathbf{B}}/c = 0$ and $\nabla\times\mathbf{F} - \dot{\mathbf{D}}/c = 0$,

$$\mathbf{E}\cdot\nabla\mathbf{E} = \tfrac{1}{2}\nabla E^2 - \frac{1}{c}\dot{\mathbf{B}}\times\mathbf{E},$$

$$\mathbf{F}\cdot\nabla\mathbf{F} = \tfrac{1}{2}\nabla F^2 + \frac{1}{c}\dot{\mathbf{D}}\times\mathbf{F}.$$

Hence

$$\begin{aligned}\mathscr{F}_{Ef} &= \frac{\kappa - 1}{c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \frac{\kappa-1}{2}\nabla E^2 + \frac{\mu - 1}{2}\nabla F^2 - \tfrac{1}{2}\nabla(\alpha P^2 + \gamma I^2)\\ &= \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) - \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) + \tfrac{1}{2}\nabla(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{F}) - \tfrac{1}{2}\nabla(E^2 + F^2)\\ &\quad - \tfrac{1}{2}\nabla(\alpha P^2 + \gamma I^2),\end{aligned} \tag{71–8}$$

which vanishes, as it should, when $\kappa = \mu = 1$. If, now, the homogeneous portions of the fluid medium are at rest in the observer's inertial system, as we shall assume, this electromagnetic force must be balanced by a mechanical force $\mathscr{F}_M = -\mathscr{F}_{Ef}$. The total zero force per unit volume, then, may be expressed as the sum of $\mathscr{F}_M$ and $\mathscr{F}_E$. Using (71–4) for $\mathscr{F}_E$, we get by addition

$$0 = \mathscr{F}_E + \mathscr{F}_M = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) + \nabla\cdot(\mathbf{DE} + \mathbf{BF}) - \tfrac{1}{2}\nabla(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{F}). \quad (71\text{–}9)$$

This agrees with (70–2), which, since it represents the force per unit volume on the free charge in a homogeneous medium, must vanish in regions where no free charge is present. Not only does this agreement furnish a check on the reasoning in the present article, but it shows us that the total electromagnetic linear momentum per unit volume in the homogeneous portions of the fluid medium is given by the same expression (70–11) as was found in the case discussed in the last article.

The *mechanical* force on any portion τ of the homogeneous medium is obtained by integrating the negative of (71–8) over τ. It is

$$\begin{aligned}\mathscr{K}_M &= -\frac{\kappa - 1}{c}\frac{d}{dt}\int_\tau \mathbf{E} \times \mathbf{B}\,d\tau - \tfrac{1}{2}\int_\sigma \{(\kappa - 1)E^2 + (\mu - 1)F^2\}\,d\boldsymbol{\sigma} \\ &\qquad + \tfrac{1}{2}\int_\sigma (\alpha P^2 + \gamma I^2)\,d\boldsymbol{\sigma} \\ &= -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{D} \times \mathbf{B}\,d\tau + \frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{E} \times \mathbf{B}\,d\tau - \tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{B})\,d\boldsymbol{\sigma} \\ &\qquad + \tfrac{1}{2}\int_\sigma (E^2 + F^2)\,d\boldsymbol{\sigma} + \tfrac{1}{2}\int_\sigma (\alpha P^2 + \gamma I^2)\,d\boldsymbol{\sigma},\end{aligned}$$

the surface integrals representing a hydrostatic tension over the surface σ. This *mechanical* tension is, of course, transmitted through the boundary without diminution. Therefore the total force on any body immersed in a homogeneous isotropic fluid is obtained by adding to (71–5) the three surface integrals in the expression for $\mathscr{K}_M$, *provided the surface σ is located in the homogeneous fluid just outside the body*, so that $\mathbf{D} = \kappa\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{F}$, where κ and μ refer to the homo-

geneous portions of the fluid. This gives for the total force on the matter inside the surface σ

$$\mathcal{K} = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{E}\times\mathbf{B}d\tau + \int_\sigma (\mathbf{ED}+\mathbf{FB})\cdot d\boldsymbol{\sigma} - \tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D}+\mathbf{F}\cdot\mathbf{B})d\boldsymbol{\sigma}. \tag{71-10}$$

It should be noted that the surface integrals in (71–10) are identical with those in (70–3) and therefore that the stress system is specified by the same dyadic (70–8).

We are almost always interested in cases where the field inside the body on which we wish to calculate the force is either static or, if radiation is passing through the body, the mean value of $\mathbf{E}\times\mathbf{B}$ does not change with the time. In such cases (71–10) reduces to

$$\mathcal{K} = \int_\sigma (\mathbf{ED}+\mathbf{FB})\cdot d\boldsymbol{\sigma} - \tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D}+\mathbf{F}\cdot\mathbf{B})d\boldsymbol{\sigma}. \tag{71-11}$$

We shall now make a few applications of (71–11) to static fields. First consider a charge q_1 distributed through a small sphere in such a way that the charge density is a function of the distance from the center alone. We shall suppose this sphere to be immersed in a homogeneous dielectric fluid of permittivity κ of great extent. Then (62–12a) gives for the electric intensity at a point P outside the sphere at a distance R from its center,

$$E_0 = \frac{q_1}{4\pi\kappa R^2}.$$

We shall place at P another similar sphere of radius a very small compared with R, having a charge q_2. In the neighborhood of P we can consider that the field due to q_1, before q_2 is placed in position, is uniform and equal to E_0. Now consider the field in the dielectric near q_2 after q_2 has been placed in a cavity of radius a at P. As shown in article 63, the scalar potential Φ must be a solution of Laplace's equation. If r is the radius vector from the center of q_2 and θ the angle which it makes with the line drawn from q_1 to q_2, the appropriate solution of Laplace's equation is

$$\Phi = -E_0 r\cos\theta + \frac{q_2}{4\pi\kappa r},$$

provided the radius of q_1 is small compared with R, for then this potential yields for the electric intensity the superposition of the

field E_0 due to q_1 and the field $q_2/4\pi\kappa r^2$ due to q_2. For $r = a$ we get

$$E_r = -\left(\frac{\partial\Phi}{\partial r}\right)_{r=a} = E_0 \cos\theta + \frac{q_2}{4\pi\kappa a^2},$$

$$E_\theta = -\left(\frac{\partial\Phi}{r\partial\theta}\right)_{r=a} = -E_0 \sin\theta.$$

The radial tension on a surface σ lying just outside q_2 is, then, $\mathcal{S}_r = (\kappa/2)(E_r^2 - E_\theta^2)$, and the shear in the direction of increasing θ is $\mathcal{S}_\theta = \kappa E_r E_\theta$.

Therefore the repulsion exerted by q_1 on q_2 is

$$\mathcal{K} = \int_0^\pi (\mathcal{S}_r \cos\theta - \mathcal{S}_\theta \sin\theta) 2\pi a^2 \sin\theta \, d\theta = q_2 E_0$$

$$= \frac{q_1 q_2}{4\pi\kappa r^2}. \tag{71–12}$$

Our analysis, although only approximate for spheres of finite radius, approaches exactness as the radii of the two spheres become smaller and smaller compared with the distance R between their centers. Therefore we may conclude that the force between two *point* charges immersed in a homogeneous dielectric fluid is given exactly by (71–12). But the reader must be warned that *this formula does not, in general, lead to the correct force between two extended distributions of charge immersed in a homogeneous dielectric fluid*, as will be illustrated by a problem investigated later in this article.

Next consider a charged conductor immersed in a homogeneous dielectric fluid. As the surface of the conductor is an equipotential surface, **D** and **E** in the homogeneous medium just outside must be normal to the surface. Therefore the stress specified by (71–11) is the pure tension

$$\mathcal{S} = \tfrac{1}{2}DE = \frac{\kappa}{2}E^2. \tag{71–13}$$

Furthermore, the equation $\nabla\cdot\mathbf{D} = \rho$ requires that D be equal to the free charge ρ_σ per unit area on the surface of the conductor. Hence we may write (71–13) in the form

$$\mathcal{S} = \tfrac{1}{2}\rho_\sigma E = \frac{1}{2\kappa}\rho_\sigma^2. \tag{71–14}$$

Let us suppose that the charged conductor is a sphere of radius a lying

in a uniform external field $\mathbf{E}_0$. If r is the radius vector from the center of the sphere to a field-point in the medium, and θ the angle which it makes with $\mathbf{E}_0$, the appropriate solution of Laplace's equation to represent the potential outside the conductor is

$$\Phi = -E_0\left(1 - \frac{a^3}{r^3}\right) r \cos\theta + \frac{A}{r},$$

since this function is constant over the surface of the conductor and leads to the uniform field $\mathbf{E}_0$ at great distances. As the field just outside the conductor is

$$E_r = -\left(\frac{\partial \Phi}{\partial r}\right)_{r=a} = 3E_0 \cos\theta + \frac{A}{a^2}$$

we have, from the boundary condition (62–16), for the charge per unit area on the surface of the conductor,

$$\rho_\sigma = 3\kappa E_0 \cos\theta + \frac{\kappa A}{a^2},$$

from which it follows that the total charge on the sphere is $Q = 4\pi\kappa A$. The tension on the surface of the conductor is

$$\mathcal{S} = \frac{\kappa}{2}\left\{3E_0 \cos\theta + \frac{Q}{4\pi\kappa a^2}\right\}^2$$

from (71–13). From symmetry it is clear that the resultant force $\mathcal{K}$ is in the direction of $\mathbf{E}_0$. If $\boldsymbol{i}$ is a unit vector in this direction,

$$\mathcal{K} = \boldsymbol{i}\int_\sigma \mathcal{S} \cos\theta \, d\sigma = Q\mathbf{E}_0, \qquad (71\text{–}15)$$

which is the same as if the conductor were located in the field $\mathbf{E}_0$ in empty space.

Now take the case of a straight wire of radius a carrying a steady current i and immersed in a paramagnetic fluid in which there was, before the introduction of the wire, a uniform magnetic force $\mathbf{F}_0$ at right angles to the wire. In this case we look for a solution of (65–9) in the homogeneous fluid from which we can obtain $\mathbf{F}$ by (65–8). As, however, (65–8) and (65–9) do not hold in the wire, we must solve the problem in two steps, first finding the field in the medium as modified by the presence of the cylindrical cavity in which the wire lies, and then superposing the field produced by the current.

Evidently cylindrical coordinates r, θ, z are indicated, with the Z axis along the wire and θ measured from the direction of $\mathbf{F}_0$. We can satisfy the boundary conditions by taking the solution

$$\Phi_o = - F_0 r \cos\theta + \frac{A_o}{r}\cos\theta$$

of Laplace's equation for the magnetic potential in the medium, and

$$\Phi_i = A_i r \cos\theta$$

in the cavity. Determining A_o and A_i by (62–16) and (62–17) we have

$$\Phi_o = -\left(1 + \frac{\mu - 1}{\mu + 1}\frac{a^2}{r^2}\right) F_0 r \cos\theta,$$

$$\Phi_i = -\frac{2\mu}{\mu + 1} F_0 r \cos\theta.$$

Superposing the magnetic field of the current, we have for the two components of the magnetic force in the homogeneous medium just outside the wire

$$F_r = -\left(\frac{\partial\Phi_o}{\partial r}\right)_{r=a} = \frac{2}{\mu + 1} F_0 \cos\theta,$$

$$F_\theta = -\left(\frac{\partial\Phi_o}{r\partial\theta}\right)_{r=a} + \frac{i}{2\pi ac} = -\frac{2\mu}{\mu + 1} F_0 \sin\theta + \frac{i}{2\pi ac}.$$

Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be perpendicular unit vectors in a plane at right angles to the wire, $\boldsymbol{i}$ having the direction of $\mathbf{F}_0$. Then, if we apply (71–11) to obtain the resultant force on a unit length of the wire,

$$\begin{aligned}\mathcal{K} &= \mu\int_0^{2\pi}\{\boldsymbol{i}(F_r\cos\theta - F_\theta\sin\theta) + \boldsymbol{j}(F_r\sin\theta + F_\theta\cos\theta)\}F_r a\, d\theta \\ &\qquad - \tfrac{1}{2}\mu\int_0^{2\pi}(F_r^2 + F_\theta^2)(\boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta) a d\theta \\ &= \boldsymbol{j}\frac{i\mu F_0}{c} = \frac{1}{c}\mathbf{i}\times\mathbf{B}_0, \end{aligned} \qquad (71\text{–}16)$$

where $\mathbf{B}_0 = \mu\mathbf{F}_0$ is the magnetic induction in the homogeneous fluid before the introduction of the wire and $\mathbf{i} = \boldsymbol{k}i$ is the vector current. It is interesting to note that this is just the force the current would experience in the field $\mathbf{B}_0$ in empty space. Equation (71–16) is valid

whatever the angle between $\mathbf{i}$ and $\mathbf{B}_0$ may be, since a magnetic field parallel to the current exerts no force on the wire.

In the examples investigated so far, we have calculated the force on a body immersed in a homogeneous fluid by evaluating the surface integrals (71–11) over a closed surface just outside the body. When the force is due to radiation impinging on the body, it is generally more convenient to transform these surface integrals into a volume integral over the entire homogeneous fluid outside the body. To do this we integrate (71–9) over the volume τ (Fig. 69) between a surface σ with outward drawn positive normal lying in the homogeneous medium just outside the body B and a second surface Σ so far distant as to enclose the entire radiation field. Then, when we convert the last two terms in (71–9) into surface integrals, the integrals over Σ vanish as $\mathbf{E}$ and $\mathbf{F}$ are zero there, and, as the positive normal to σ is inward drawn with respect to τ, we get

$$0 = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{D}\times\mathbf{B}d\tau - \int_\sigma (\mathbf{ED}+\mathbf{FB})\cdot d\boldsymbol{\sigma} + \tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D}+\mathbf{F}\cdot\mathbf{B})d\boldsymbol{\sigma}.$$

Adding to (71–11), the force on B is found to be

$$\mathcal{K} = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{D}\times\mathbf{B}d\tau \tag{71–17}$$

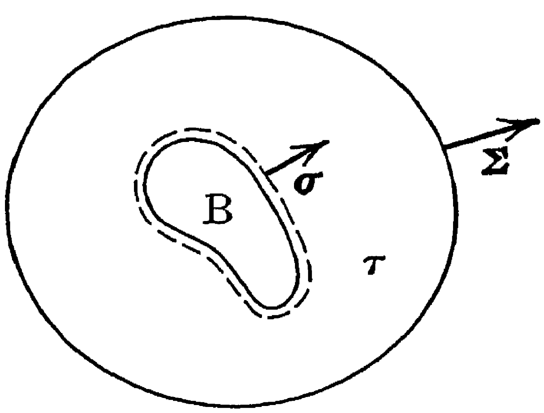

FIG. 69.

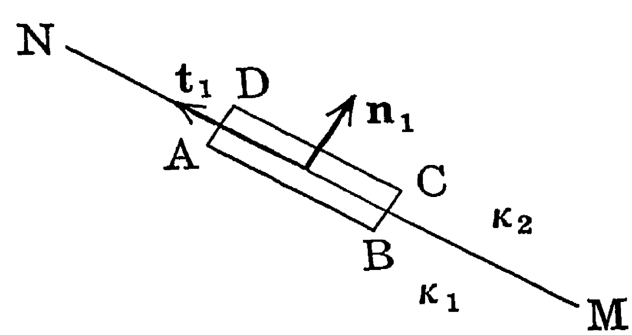

FIG. 70.

integrated through the entire homogeneous fluid lying outside B. As this equation is identical with (70–4), we can attribute the electromagnetic linear momentum (70–11) to the radiation field and employ the law of action and reaction to find the force on the body B.

The surface of discontinuity between two homogeneous isotropic fluids is not generally in equilibrium under the action of the stresses specified by (71–11). To illustrate this fact, let MN (Fig. 70) be the surface of discontinuity separating two fluid dielectrics of permittivi-

ties κ_1 and κ_2. To find the stress $\mathscr{S}$ on the surface we integrate (71–11) over the surface of the pill-box $ABCD$ of unit base. Denoting components in the directions of the unit vectors $\mathbf{n}_1$ and $\mathbf{t}_1$, parallel respectively to the normal and to the tangential components of the field, by the subscripts n and t,

$$\mathscr{S} = \mathbf{n}_1\{\tfrac{1}{2}(E_{2n}D_{2n} - E_{2t}D_{2t}) - \tfrac{1}{2}(E_{1n}D_{1n} - E_{1t}D_{1t})\} + \mathbf{t}_1\{E_{2t}D_{2n} - E_{1t}D_{1n}\}.$$

Using the boundary conditions $D_{2n} = D_{1n}$, $E_{2t} = E_{1t}$, we see that the tangential stress vanishes and the normal tension may be written

$$\mathscr{S}_n = \tfrac{1}{2}\kappa_2 E_{2n}^2\left(1 - \frac{\kappa_2}{\kappa_1}\right) + \tfrac{1}{2}\kappa_2 E_{2t}^2\left(\frac{\kappa_1}{\kappa_2} - 1\right). \qquad (71\text{–}18)$$

Therefore the surface tends to move from the dielectric of higher permittivity into that of lower permittivity. This phenomenon is known as *electrostriction*. If we put $\kappa_2 = 1$ we find that the tension on the free surface of a dielectric is $\tfrac{1}{2}E_2{}^2(1 - 1/\kappa_1)$ when the lines of force are normal to the surface, and $\tfrac{1}{2}E_2{}^2(\kappa_1 - 1)$ when the lines of force are tangential to the surface. The effect of these stresses on a solid dielectric is to urge it from the weaker toward the stronger parts of the field.

Now we shall calculate the electromagnetic torque on the free charges *and the material medium* lying inside any fixed volume τ. Following the analysis used in deriving (70–21) we get for the volume integral of the vector moment of the electromagnetic force (71–4) about any fixed origin O:

$$\int_\tau \mathbf{r}\times\mathscr{F}_E d\tau = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{r}\times(\mathbf{E}\times\mathbf{B})d\tau + \int_\tau(\mathbf{E}\times\mathbf{D}+\mathbf{F}\times\mathbf{B})d\tau$$
$$+\int_\sigma \mathbf{r}\times(\mathbf{ED}+\mathbf{FB})\cdot d\sigma - \tfrac{1}{2}\int_\sigma(E^2+F^2)\mathbf{r}\times d\sigma$$
$$-\tfrac{1}{2}\int_\sigma(\alpha P^2+\gamma I^2)\mathbf{r}\times d\sigma.$$

This does not, however, represent the total electromagnetic torque, for, in addition to the torque of the electromagnetic force, there is a couple $\mathbf{p}_E\times(\mathbf{E}-\mathbf{E}_P)$ on each electric dipole and a couple $\mathbf{p}_H\times(\mathbf{B}-\mathbf{B}_I)$ on each Ampèrian circuit. The first gives rise to a couple $\mathbf{P}\times(\mathbf{E}-\alpha\mathbf{P}) = \mathbf{D}\times\mathbf{E}$ per unit volume, and the second to a couple

$\mathbf{I} \times (\mathbf{F} - \gamma\mathbf{I}) = \mathbf{B} \times \mathbf{F}$ per unit volume. Therefore we must add $\int_\tau (\mathbf{D} \times \mathbf{E} + \mathbf{B} \times \mathbf{F}) d\tau$ to the torque of the electromagnetic force to obtain the total electromagnetic torque

$$\mathscr{L}_E = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d\tau + \int_\sigma \mathbf{r} \times (\mathbf{ED} + \mathbf{FB}) \cdot d\boldsymbol{\sigma}$$
$$-\tfrac{1}{2}\int_\sigma (E^2 + F^2)\mathbf{r} \times d\boldsymbol{\sigma} - \tfrac{1}{2}\int_\sigma (\alpha P^2 + \gamma I^2)\mathbf{r} \times d\boldsymbol{\sigma}. \quad (71\text{–}19)$$

Like (71–5) this expression is valid for any kind of medium or body, no matter what constitutive relations may exist between **D** and **E** and between **B** and **F**, but it gives only the torque of immediate electromagnetic origin.

To find the *mechanical* torque on any homogeneous portion τ of an isotropic fluid in equilibrium, we must integrate the vector moment of the negative of (71–8) over τ and add the negative of the couple $\int_\tau (\mathbf{D} \times \mathbf{E} + \mathbf{B} \times \mathbf{F}) d\tau$. This gives

$$\mathscr{L}_M = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{r} \times (\mathbf{D} \times \mathbf{B}) d\tau + \frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d\tau$$
$$+\int_\tau (\mathbf{E} \times \mathbf{D} + \mathbf{F} \times \mathbf{B}) d\tau - \tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{B})\mathbf{r} \times d\boldsymbol{\sigma}$$
$$+\tfrac{1}{2}\int_\sigma (E^2 + F^2)\mathbf{r} \times d\boldsymbol{\sigma} + \tfrac{1}{2}\int_\sigma (\alpha P^2 + \gamma I^2)\mathbf{r} \times d\boldsymbol{\sigma}.$$

As the portion of the mechanical torque represented by the surface integrals is transmitted undiminished through the boundary, the total torque on any body immersed in a homogeneous isotropic fluid is obtained by adding to (71–19) the three surface integrals in the expression for $\mathscr{L}_M$, *provided, as before, that the surface σ is located in the homogeneous fluid just outside the body.* Then the total torque on the matter inside σ is

$$\mathscr{L} = -\frac{1}{c}\frac{d}{dt}\int_\tau \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) d\tau + \int_\sigma \mathbf{r} \times (\mathbf{ED} + \mathbf{FB}) \cdot d\boldsymbol{\sigma}$$
$$-\tfrac{1}{2}\int_\sigma (\mathbf{E}\cdot\mathbf{D} + \mathbf{F}\cdot\mathbf{B})\mathbf{r} \times d\boldsymbol{\sigma}. \quad (71\text{–}20)$$

We are nearly always interested in cases where the field inside the body is either static or, if radiation is passing through the body, the mean value of $\mathbf{E} \times \mathbf{B}$ does not change with the time. Then (71–20) reduces to

$$\mathscr{L} = \int_\sigma \mathbf{r} \times (\mathbf{ED} + \mathbf{FB}) \cdot d\boldsymbol{\sigma} - \tfrac{1}{2} \int_\sigma (\mathbf{E} \cdot \mathbf{D} + \mathbf{F} \cdot \mathbf{B}) \mathbf{r} \times d\boldsymbol{\sigma}, \quad (71\text{–}21)$$

which is just the torque of the stress system defined by (71–11).

We shall give a couple of illustrations of the use of (71–21). First consider two permanent magnets M and M' (Fig. 71) in the form of uniformly magnetized spheres of radii a and a' respectively, which are immersed in a homogeneous fluid of permeability μ, the distance

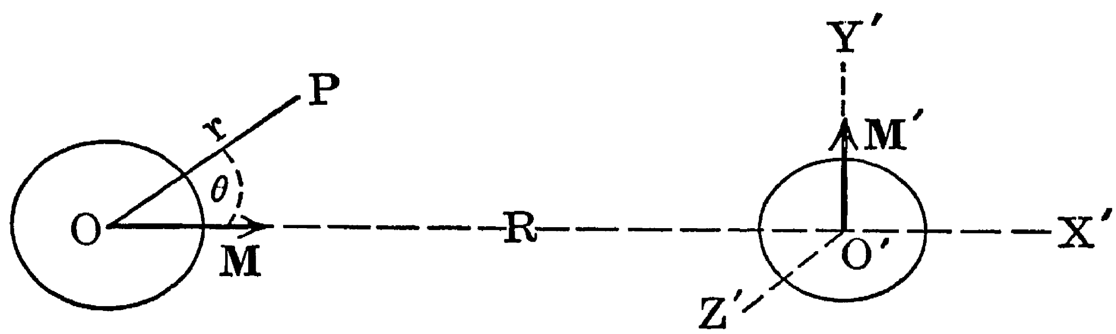

Fig. 71.

R between the centers of the magnets being large compared with the radius of either. We shall calculate the turning couple exerted by M on M' when the magnetic axes of the two magnets are perpendicular, as indicated in the figure.

First consider the field due to M at a point P in a surrounding homogeneous medium of great extent. If we use spherical coordinates r, θ, ϕ with origin at the center O of M, the boundary conditions at the surface of the magnet can be satisfied by taking the solution

$$\Phi_i = A_i r \cos\theta$$

of Laplace's equation for the magnetic potential inside the magnet, and the solution

$$\Phi_o = \frac{A_o}{r^2} \cos\theta$$

for the magnetic potential outside. If I is the intensity of mag-

netization of M the boundary conditions are

$$-\left(\frac{\partial \Phi_i}{\partial r}\right)_{r=a} + I\cos\theta = -\mu\left(\frac{\partial \Phi_o}{\partial r}\right)_{r=a},$$

$$-\left(\frac{\partial \Phi_i}{r\partial \theta}\right)_{r=a} = -\left(\frac{\partial \Phi_o}{r\partial \theta}\right)_{r=a},$$

from which we find the values of A_i and A_o, getting in the case of the latter, $A_o = a^3 I/(2\mu + 1)$. As $I = 3p_H/4\pi a^3$, where p_H is the moment of the magnet, this gives for the magnetic potential outside the magnet

$$\Phi_o = \frac{p_H}{4\pi\gamma r^2}\cos\theta, \quad \gamma \equiv \frac{2\mu + 1}{3}. \tag{71-22}$$

The magnetic force at the point O', where we are going to place the center of M', is therefore

$$F_0 = \left(\frac{\mu}{\gamma}\right)\frac{2p_H}{4\pi\mu R^3} \tag{71-23}$$

in the direction of the line $\overline{OO'}$. As R is large compared with a', we can consider that M' is placed in a *uniform* field of this magnitude.

Now we turn our attention to M', introducing a set of axes $X'Y'Z'$ with the X' axis parallel to $\overline{OO'}$ and the Y' axis to M'. With O' as origin and $\overline{O'Z'}$ as polar axis, we locate a field-point by the spherical coordinates r', θ', ϕ', the azimuth ϕ' being measured from the $X'Z'$ plane. The solutions of Laplace's equation appropriate for the magnetic potential Φ_i' inside and Φ_o' outside M' are

$$\Phi_i' = A'_{1i} r' \sin\theta' \sin\phi' + A'_{2i} r' \sin\theta' \cos\phi',$$

$$\Phi_o' = \frac{A'_{1o}}{r'^2}\sin\theta'\sin\phi' - F_0 r' \sin\theta' \cos\phi' + \frac{A'_{2o}}{r'^2}\sin\theta'\cos\phi',$$

the term in Φ_o' containing F_0 satisfying the boundary condition for $a' \ll r' \ll R$. If I' is the intensity of magnetization of M', the boundary conditions at the surface of the sphere are

$$-\left(\frac{\partial \Phi_i'}{\partial r'}\right)_{r'=a'} + I'\sin\theta'\sin\phi' = -\mu\left(\frac{\partial \Phi_o'}{\partial r'}\right)_{r'=a'},$$

$$-\left(\frac{\partial \Phi_i'}{r'\partial \theta'}\right)_{r'=a'} = -\left(\frac{\partial \Phi_o'}{r'\partial \theta'}\right)_{r'=a'},$$

$$-\left(\frac{\partial \Phi_i'}{r'\sin\theta'\partial \phi'}\right)_{r'=a'} = -\left(\frac{\partial \Phi_o'}{r'\sin\theta'\partial \phi'}\right)_{r'=a'}.$$

These give

$$A'_{1o} = \frac{a'^3 I'}{2\mu + 1} = \frac{p_H'}{4\pi\gamma}, \quad A'_{2o} = -\frac{\mu - 1}{2\mu + 1} a'^3 F_0,$$

where p_H' is the magnetic moment of M'. Consequently

$$\Phi_o' = \frac{p_H'}{4\pi\gamma r'^2} \sin\theta' \sin\phi' - \left\{1 + \frac{\mu - 1}{2\mu + 1} \frac{a'^3}{r'^3}\right\} F_0 r' \sin\theta' \cos\phi'. \quad (71\text{–}24)$$

The turning couple experienced by M' is clearly about the Z' axis. Hence we need only the shear $\mu F_{r'} F_{\phi'}$, and, as the lever arm is $r' \sin\theta'$, the torque is

$$\mathscr{L}_{z'} = \mu \int_0^{2\pi} \int_0^{\pi} \frac{\partial \Phi_o'}{\partial r'} \frac{\partial \Phi_o'}{r' \sin\theta' \partial\phi'} (r' \sin\theta')(r'^2 \sin\theta' \, d\theta' \, d\phi')$$

$$= -\frac{\mu}{\gamma} p_H' F_0.$$

Finally, putting in the value of F_0 given by (71–23) we have

$$\mathscr{L}_{z'} = -\left(\frac{\mu}{\gamma}\right)^2 \frac{2 p_H p_H'}{4\pi\mu R^3}, \quad \gamma \equiv \frac{2\mu + 1}{3}. \quad (71\text{–}25)$$

It may be noted that the factor (μ/γ) enters once in the expression (71–23) for the magnetic force F_0, and again in the expression for the torque in terms of F_0.* The coefficient γ is a shape coefficient in that the value which we have found for it is valid only for spherical magnets. In the cases of prolate or oblate spheroids different coefficients appear.

Last, consider the symmetrically magnetized permanent magnet M (Fig. 72) in the form of a solid of revolution with $\overline{OZ}$ as axis to which is rigidly attached the conducting side-arm $\overline{QP}$. A current i enters the system at P and leaves at a point R on the axis. The region outside the magnet is empty space. We shall calculate the torque about $\overline{OZ}$ on the magnet and side-arm.

To do this we calculate the torque due to the stresses on any closed surface of revolution passing through P and R, such as that

* Some texts give the expressions $qq'/\kappa r^2$ and $mm'/\mu r^2$ for the force between two charges q and q' immersed in a dielectric fluid and two poles m and m' immersed in a permeable fluid, respectively, as *fundamental* laws of electrostatics and of magnetostatics. While these expressions are valid for *point* charges and for *point* poles, the correct solution of the problem under consideration shows that they do not hold in general for extended distributions of charge or of magnetism, and therefore they should never be cited as fundamental laws.

represented in section by the broken curve. Let H_n be the normal component of the magnetic intensity at any point on this surface, and H_ϕ the tangential component in the direction of increasing azimuth ϕ measured about the Z axis. The shear in the direction of increasing ϕ is H_nH_ϕ and, if r is the distance from the Z axis, the moment of this shear is rH_nH_ϕ. If, then, $d\lambda$ is an element of the periphery of the longitudinal section of the surface of integration,

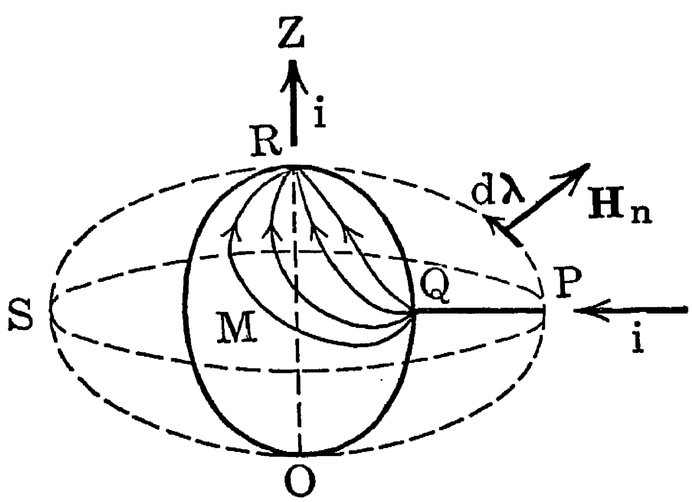

FIG. 72.

$$\mathscr{L}_z = \iint r^2 H_n H_\phi \, d\lambda \, d\phi$$

where ϕ is to be integrated from 0 to 2π and λ from O to R. Now we can express H_n as the sum of the field $H_n{}^m$ due to the magnet and the field $H_n{}^i$ due to the current, and similarly with H_ϕ. Hence

$$\mathscr{L}_z = \iint r^2 H_n{}^m H_\phi{}^m d\lambda \, d\phi + \iint r^2 H_n{}^i H_\phi{}^m d\lambda \, d\phi$$
$$+ \iint r^2 H_n{}^m H_\phi{}^i d\lambda \, d\phi + \iint r^2 H_n{}^i H_\phi{}^i d\lambda \, d\phi.$$

The first and second integrals vanish as $H_\phi{}^m = 0$, and the fourth is evidently inappreciable. Therefore, since $H_n{}^m$ is not a function of ϕ,

$$\mathscr{L}_z = \int_O^R H_n{}^m r d\lambda \int_0^{2\pi} H_\phi{}^i r d\phi.$$

From (62–12d)

$$\int_0^{2\pi} H_\phi{}^i r d\phi = \begin{cases} \dfrac{i}{c} \text{ for points on the periphery between } P \text{ and } R, \\ 0 \text{ for points on the periphery between } O \text{ and } P. \end{cases}$$

Consequently

$$\mathscr{L}_z = \frac{i}{2\pi c} \int_P^R H_n{}^m \, 2\pi r d\lambda.$$

The integral is equal to the magnetic flux through the portion PRS

of the surface, or, by virtue of the equation $\nabla \cdot \mathbf{B} = 0$, to the flux N of induction through the plane section PS at right angles to the Z axis. So, finally,

$$\mathscr{L}_z = \frac{iN}{2\pi c}. \tag{71–26}$$

This formula has been verified experimentally.[1] It may be obtained directly from the force equation by calculating the torque exerted on the current in the magnet and side-arm by the magnetic field. This torque is transmitted by the electrons constituting the current to the mechanical system. It should be noted that the torque on the side-arm is in the opposite sense to that on the magnet, and that the total torque approaches zero as the length of the side-arm increases without limit.

72. Electromagnetic Waves in Homogeneous Isotropic Media.— In a homogeneous isotropic medium for which $\mathbf{D} = \kappa\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{F}$ and in which no free charges or currents are present, the field equations (62–12a) to (62–12d) become

$$\left.\begin{aligned} &\nabla \cdot \mathbf{E} = 0, \quad (a) &&\nabla \cdot \mathbf{F} = 0, \quad (b) \\ &\nabla \times \mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{F}}, \ (c) &&\nabla \times \mathbf{F} = \frac{\kappa}{c}\dot{\mathbf{E}}. \ (d) \end{aligned}\right\} \tag{72–1}$$

To eliminate $\mathbf{F}$ combine the curl of (c) with the time derivative of (d) getting

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\kappa\mu}{c^2}\ddot{\mathbf{E}}.$$

Expanding the double curl and using (a), this becomes

$$\nabla \cdot \nabla \mathbf{E} = \frac{\kappa\mu}{c^2}\ddot{\mathbf{E}}, \tag{72–2}$$

and similarly, if we eliminate $\mathbf{E}$,

$$\nabla \cdot \nabla \mathbf{F} = \frac{\kappa\mu}{c^2}\ddot{\mathbf{F}}. \tag{72–3}$$

These are equations of a wave traveling with the phase velocity

$$v \equiv c/\sqrt{\kappa\mu}.$$

Equations (72–1a) and (72–1b) show that $\mathbf{E}$ and $\mathbf{F}$ are solenoidal

[1] Zeleny and Page, Phys. Rev. 24, p. 544 (1924).

vectors. Therefore any solenoidal solution of (72–2) represents the electric intensity in a possible electromagnetic wave, or any solenoid solution of (72–3) the magnetic force. These two vector functions are not independent, however, but must satisfy the first order differential equations (72–1*c*) and (72–1*d*). If one of these is satisfied, the other is satisfied automatically.

A convenient method of obtaining solenoidal solutions of (72–2) or (72–3) is to obtain first three solutions Φ_1, Φ_2, Φ_3 of the scalar wave equation

$$\nabla \cdot \nabla \Phi = \frac{1}{v^2}\ddot{\Phi}, \quad v \equiv \frac{c}{\sqrt{\kappa\mu}}. \tag{72–4}$$

Then the vector

$$\mathbf{V} = \boldsymbol{i}\Phi_1 + \boldsymbol{j}\Phi_2 + \boldsymbol{k}\Phi_3$$

is a solution of the vector wave equation (72–2) or (72–3). Although this solution may not be solenoidal, such vector functions as $\nabla \times \mathbf{V}$ and $\nabla \times \nabla \times \mathbf{V}$ formed from it are solutions of the vector wave equation which are necessarily solenoidal, and therefore may represent **E** or **F** in a possible electromagnetic wave.

If the wave is simple harmonic with frequency $\nu = \omega/2\pi$ we may write

$$\mathbf{V} = \mathbf{U}(x, y, z)e^{-i\omega t}$$

where $\mathbf{U}(x, y, z)$ may be complex, understanding that the physical quantity represented by **V** is given by the real or by the imaginary part of this complex vector function. As differentiation of **V** with respect to the time is equivalent to multiplication by $-i\omega$ and integration to multiplication by $1/(-i\omega) = i/\omega$, we see from (72–1*c*) that if we take $\nabla\times \mathbf{V}$ for the electric intensity we have

$$\mathbf{E} = \nabla \times \mathbf{V}, \qquad \mathbf{F} = -\frac{ic}{\mu\omega}\nabla \times \nabla \times \mathbf{V}, \tag{72–5}$$

and from (72–1*d*) that if we take $\nabla \times \nabla \times \mathbf{V}$ for the electric intensity

$$\mathbf{E} = \nabla \times \nabla \times \mathbf{V}, \quad \mathbf{F} = -\frac{i\kappa\omega}{c}\nabla \times \mathbf{V}. \tag{72–6}$$

Therefore, except for the constant factors, the one case is obtained from the other simply by interchanging the roles of **E** and **F**.

(I) *Plane Waves.* We shall apply the method outlined above to a plane wave advancing along the X axis. Then (72–4) becomes

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Phi}{\partial t^2} \tag{72–7}$$

of which the desired solution is $\Phi(x - vt)$ where Φ is an arbitrary function. Hence

$$\mathbf{V} = \boldsymbol{i}\Phi_1(x - vt) + \boldsymbol{j}\Phi_2(x - vt) + \boldsymbol{k}\Phi_3(x - vt),$$

and, if we take $\nabla \times \mathbf{V}$ for $\mathbf{E}$ to obtain a solenoidal solution of (72–2),

$$\mathbf{E} = \boldsymbol{j}f(x - vt) + \boldsymbol{k}g(x - vt), \tag{72–8}$$

where f and g are the derivatives of $-\Phi_3$ and Φ_2 with respect to their arguments, respectively. Similarly

$$\mathbf{F} = \boldsymbol{j}h(x - vt) + \boldsymbol{k}i(x - vt)$$

is a solenoidal solution of (72–3). But substitution in (72–1c) or (72–1d) shows that $h = -\sqrt{\kappa/\mu}\, g$ and $i = \sqrt{\kappa/\mu}\, f$, so that

$$\mathbf{F} = -\boldsymbol{j}\sqrt{\frac{\kappa}{\mu}}\, g(x - vt) + \boldsymbol{k}\sqrt{\frac{\kappa}{\mu}}\, f(x - vt). \tag{72–9}$$

Equations (72–8) and (72–9) show that $\mathbf{E}$ and $\mathbf{F}$ are perpendicular to the direction of propagation and at right angles to each other in such a sense that $\mathbf{E} \times \mathbf{F}$ has the direction of propagation. The electric and magnetic energies per unit volume are respectively

$$\left.\begin{aligned} u_E &= \frac{\kappa}{2}E^2 = \frac{\kappa}{2}(f^2 + g^2), \\ u_H &= \frac{\mu}{2}F^2 = \frac{\kappa}{2}(f^2 + g^2). \end{aligned}\right\} \tag{72–10}$$

We note that $u_E = u_H$. The flux of energy is

$$\mathbf{s} = c\mathbf{E} \times \mathbf{F} = \boldsymbol{i}c\sqrt{\frac{\kappa}{\mu}}(f^2 + g^2) = \boldsymbol{i}v\kappa(f^2 + g^2) = \boldsymbol{i}vu \tag{72–11}$$

per unit cross-section per unit time, where $u = u_E + u_H$ is the total energy per unit volume.

The two arbitrary functions f and g in (72–8) and (72–9) correspond to the two independent states of polarization of a solenoidal

wave. In each state the electric and magnetic vectors are in phase. We note that the relation between the electric and magnetic vectors implied in (72–8) and (72–9) can be expressed by either of the vector equations

$$\mathbf{F} = \frac{\mathbf{v}}{c} \times \mathbf{D}, \quad \mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{72–12}$$

where $\mathbf{v}$ is the vector velocity of propagation.

To express the argument $x - vt$ of $\mathbf{E}$ or $\mathbf{F}$ in a form independent of the orientation of the axes we introduce the *wave-slowness* $\mathbf{S}$, a vector having the direction of propagation and a magnitude equal to the reciprocal of the phase velocity. Then, if $\mathbf{r}$ is the position vector of the field-point x, y, z, we have

$$x - vt = v(Sx - t) = v(\mathbf{S}\cdot\mathbf{r} - t).$$

As the two states of polarization are independent, we need consider only one, and may write

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}_0 f(\mathbf{S}\cdot\mathbf{r} - t), \\ \mathbf{F} &= \mathbf{F}_0 f(\mathbf{S}\cdot\mathbf{r} - t), \quad \mathbf{F}_0 \equiv \frac{c}{\mu}\mathbf{S} \times \mathbf{E}_0, \end{aligned}\right\} \tag{72–13}$$

where $\mathbf{E}_0$ is a constant vector perpendicular to $\mathbf{S}$. If the wave is simple harmonic the electric and magnetic vectors are usually most conveniently represented as the real parts, or the imaginary parts, of the complex quantities

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i\omega(\mathbf{S}\cdot\mathbf{r} - t)}, \\ \mathbf{F} &= \mathbf{F}_0 e^{i\omega(\mathbf{S}\cdot\mathbf{r} - t)}, \end{aligned}\right\} \tag{72–14}$$

respectively, where the amplitudes $\mathbf{E}_0$ and $\mathbf{F}_0$ may be real or complex, and the angular frequency ω is equal to 2π multiplied by the true frequency ν.

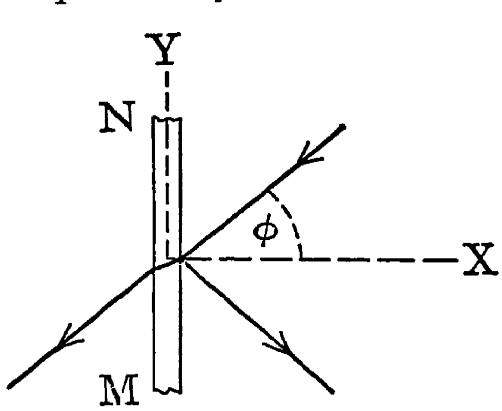

FIG. 73.

Let us calculate the radiation pressure on a slab MN (Fig. 73) with parallel faces, immersed in a homogeneous isotropic fluid, on which radiation in the form of plane waves impinges at the angle of incidence ϕ. Let ρ_r be the fraction of the incident energy which is reflected, and ρ_t the fraction transmitted. If u_i is the energy density

of the incident beam, the electromagnetic momentum in this beam is $vu_i\kappa\mu/c^2 = u_i/v$ per unit volume from (70–13) and (72–11). Similarly the electromagnetic momenta in the reflected and transmitted beams are respectively $\rho_r u_i/v$ and $\rho_t u_i/v$ per unit volume. Hence the components of the force per unit area on MN are

$$\left.\begin{aligned}\mathscr{S}_x &= -\ (1 + \rho_r - \rho_t)u_i \cos^2 \phi,\\ \mathscr{S}_y &= -\ (1 - \rho_r - \rho_t)u_i \sin \phi \cos \phi,\\ \mathscr{S}_z &= 0.\end{aligned}\right\} \qquad (72\text{–}15)$$

The negative sign in the first of these indicates that the normal force $\mathscr{S}_x$ is a pressure. Its magnitude is greatest when all the impinging energy is reflected and the incidence is normal. Since $1 - \rho_r - \rho_t$ is the fraction of the energy absorbed, the tangential force $\mathscr{S}_y$ is proportional to the absorption.

If radiation of the same intensity is incident at all angles on a perfectly reflecting surface, the energy du_i incident at angles between ϕ and $\phi + d\phi$ is

$$du_i = \tfrac{1}{2}u \sin \phi \, d\phi,$$

where u represents the total energy density. In this case

$$\left.\begin{aligned}\mathscr{S}_x &= -\ u\int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi = -\ \tfrac{1}{3}u,\\ \mathscr{S}_y &= 0,\\ \mathscr{S}_z &= 0.\end{aligned}\right\} \qquad (72\text{–}16)$$

(II) *Spherical Waves.* We look now for a scalar solution of the wave equation which is a function of the radius vector r alone. In this case (72–4) becomes

$$\frac{\partial^2}{\partial r^2}(r\Phi) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\Phi) \qquad (72\text{–}17)$$

from (19–12). The general solution is

$$\Phi = \frac{1}{r}f_1(r - vt) + \frac{1}{r}f_2(r + vt), \qquad (72\text{–}18)$$

the first term representing a wave diverging from the origin and the second a wave converging on the origin. We shall confine our discussion to a simple harmonic wave diverging from the origin, taking

$$\Phi = \frac{A}{r}\, e^{i(\epsilon r - \omega t)}, \qquad (72\text{–}19)$$

where ω is the angular frequency, and $\epsilon \equiv \omega/v = 2\pi/\lambda$, where λ is the wave-length.

Putting $\mathbf{V} = \mathbf{i}\Phi$ we may take either $\nabla \times \mathbf{V}$ or $\nabla \times \nabla \times \mathbf{V}$ for the electric intensity in a possible spherical electromagnetic wave. We shall discuss each case, treating first the wave described by (72–6), as it is the more important. As $\mathbf{V}$ satisfies the wave equation $\nabla \cdot \nabla \mathbf{V} = (1/v^2)\ddot{\mathbf{V}}$,

$$\begin{aligned}\mathbf{E} &= \nabla \times \nabla \times \mathbf{V} = \nabla\nabla \cdot \mathbf{V} - \frac{1}{v^2}\frac{\partial^2 \mathbf{V}}{\partial t^2}\\ &= \nabla \frac{\partial \Phi}{\partial x} - \mathbf{i}\frac{1}{v^2}\frac{\partial^2 \Phi}{\partial t^2}.\end{aligned}$$

Using spherical coordinates r, θ, ϕ, where θ is the angle which the radius vector makes with the X axis,

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial r}\cos\theta = A\cos\theta\left\{-\frac{1}{r^2} + \frac{i\epsilon}{r}\right\}e^{i(\epsilon r - \omega t)}$$

and the components of $\mathbf{E}$ in the directions of increasing r, θ, ϕ are

$$\left.\begin{aligned}E_r &= A\cos\theta\left\{\frac{2}{r^3} - \frac{2i\epsilon}{r^2}\right\}e^{i(\epsilon r - \omega t)}\\ &= A\cos\theta\left\{\frac{2}{r^3}\cos(\epsilon r - \omega t) + \frac{2\epsilon}{r^2}\sin(\epsilon r - \omega t)\right\},\\ E_\theta &= A\sin\theta\left\{\frac{1}{r^3} - \frac{i\epsilon}{r^2} - \frac{\epsilon^2}{r}\right\}e^{i(\epsilon r - \omega t)}\\ &= A\sin\theta\left\{\left(\frac{1}{r^3} - \frac{\epsilon^2}{r}\right)\cos(\epsilon r - \omega t) + \frac{\epsilon}{r^2}\sin(\epsilon r - \omega t)\right\},\\ E_\phi &= 0,\end{aligned}\right\} \quad (72\text{–}20)$$

where, in the final expressions for E_r, E_θ, E_ϕ we have written explicitly the real parts of the complex values.

At a distance from the origin very small compared with the wave-length $\lambda = 2\pi/\epsilon$,

$$\begin{aligned}E_r &= \frac{2A\cos\omega t}{r^3}\cos\theta,\\ E_\theta &= \frac{A\cos\omega t}{r^3}\sin\theta,\\ E_\phi &= 0.\end{aligned}$$

Comparing these equations with (58–11) we see that the source may be a charged spherical shell with oscillating electric moment

$$p_E = p_{0E} \cos \omega t, \quad p_{0E} \equiv 4\pi\gamma A, \tag{72–21}$$

along the X axis, where $\gamma \equiv (2\kappa + 1)/3$ by analogy with (71–22).

Using (19–11) for the curl in spherical coordinates, we get from (72–6) for the components of $\mathbf{F}$

$$\left.\begin{aligned} F_r &= 0, \\ F_\theta &= 0, \\ F_\phi &= -\frac{i\kappa\omega}{cr}\left\{\frac{\partial}{\partial r}(-r\Phi \sin \theta) - \frac{\partial}{\partial \theta}(\Phi \cos \theta)\right\} \\ &= \frac{i\kappa\omega A}{c} \sin \theta \left\{-\frac{1}{r^2} + \frac{i\epsilon}{r}\right\} e^{i(\epsilon r - \omega t)} \\ &= \sqrt{\frac{\kappa}{\mu}} A \sin \theta \left\{\frac{\epsilon}{r^2} \sin(\epsilon r - \omega t) - \frac{\epsilon^2}{r} \cos(\epsilon r - \omega t)\right\}. \end{aligned}\right\} \tag{72–22}$$

At a great distance from the source the components of $\mathbf{E}$ and $\mathbf{F}$ of dominant magnitude are

$$\left.\begin{aligned} E_\theta &= -\frac{p_{0E}\epsilon^2}{4\pi\gamma r} \sin \theta \cos(\epsilon r - \omega t), \\ F_\phi &= -\sqrt{\frac{\kappa}{\mu}} \frac{p_{0E}\epsilon^2}{4\pi\gamma r} \sin \theta \cos(\epsilon r - \omega t). \end{aligned}\right\} \tag{72–23}$$

These components constitute the *radiation wave*. It should be noted that the wave is polarized with the electric vector in the plane determined by the direction of oscillation of the source and the radius vector r, and with the magnetic vector perpendicular to this plane. The amplitudes of both vectors are proportional to $\sin \theta$, showing that the radiation emitted in the direction of oscillation is zero, and at right angles a maximum. On account of the factor ϵ^2, the amplitude of each vector varies inversely with the square of the wave-length for a given p_{0E}.

The time rate of radiation is obtained by integrating the Poynting flux $cE_\theta F_\phi$ over the surface of a sphere of radius r. It is

$$\frac{c}{6\pi\gamma^2} \sqrt{\frac{\kappa}{\mu}} p_{0E}^2 \epsilon^4 \cos^2(\epsilon r - \omega t). \tag{72–24}$$

To treat the wave described by (72–5) we take the three expressions (72–20) for the three components of **F**. Then, in accord with (72–5) and (72–6), the three components of **E** are given by the three expressions (72–22) multiplied by $-\mu/\kappa$. The field close to the origin, therefore, is that of a uniformly magnetized sphere with oscillating magnetic moment

$$p_H = p_{0H} \cos \omega t, \quad p_{0H} \equiv 4\pi\gamma A, \tag{72–25}$$

where $\gamma \equiv (2\mu + 1)/3$. The radiation field is

$$\left.\begin{aligned} E_\phi &= \sqrt{\frac{\mu}{\kappa}} \frac{p_{0H}\,\epsilon^2}{4\pi\gamma r} \sin\theta \cos(\epsilon r - \omega t), \\ F_\theta &= -\frac{p_{0H}\,\epsilon^2}{4\pi\gamma r} \sin\theta \cos(\epsilon r - \omega t), \end{aligned}\right\} \tag{72–26}$$

and the time rate of radiation is

$$\frac{c}{6\pi\gamma^2}\sqrt{\frac{\mu}{\kappa}}\, p_{0H}^2\, \epsilon^4 \cos^2(\epsilon r - \omega t). \tag{72–27}$$

The field (72–23) is typical of that radiated from an oscillating spark gap or a straight antenna, whereas (72–26) represents the radiation from a closed circuit or loop of oscillating magnetic moment p_H.

73. **Waves Guided by Perfect Conductors.** — As the electric intensity vanishes everywhere inside a perfect conductor, electric charge can reside only on its surface, and, as the tangential component of the electric intensity is continuous at the surface, the resultant electric intensity just outside the conductor must be normal to the surface. We shall limit ourselves to simple harmonic waves guided by straight conductors immersed in a homogeneous isotropic medium for which $\mathbf{D} = \kappa\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{F}$.

(I) *Parallel Planes*. First we shall consider plane waves traveling in the X direction in a homogeneous isotropic non-conducting medium filling the space between two perfectly conducting planes $y = 0$ and $y = a$. In this case Φ is not a function of z and the wave equation (72–4) becomes

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2}, \quad v \equiv \frac{c}{\sqrt{\kappa\mu}}. \tag{73–1}$$

The type of wave in which we are interested is of the form

$\Phi = Y(y)\,e^{i(lx-wt)}$. Substituting in (73–1) we find that $Y(y)$ must satisfy the equation

$$\frac{d^2Y}{dy^2} + m^2Y = 0, \tag{73–2}$$

where $l^2 + m^2 = \omega^2/v^2$.

First consider the solution for which $m = 0$. Then $l^2 = \omega^2/v^2$, giving for the phase velocity $\omega/l = v$, and we have the simple plane wave discussed in the last article. Corresponding to the two independent states of polarization we may have

$$\mathbf{E} = \mathbf{k}Ae^{i(lx-wt)} = \mathbf{k}A\cos(lx - \omega t), \tag{73–3}$$

or

$$\mathbf{E} = \mathbf{j}Ae^{i(lx-wt)} = \mathbf{j}A\cos(lx - \omega t). \tag{73–4}$$

As the first solution does not satisfy the boundary conditions imposed by the presence of the two conducting planes, only the second can exist.

But other solutions of (73–2) may be found. If m is not zero, we may have $Y = \cos(my - \delta)$, and

$$\Phi = A\cos(my - \delta)\,e^{i(lx-\omega t)},$$

which represents a wave traveling in the X direction with phase velocity

$$\frac{\omega}{l} = \frac{v}{\sqrt{1 - \dfrac{m^2v^2}{\omega^2}}} \tag{73–5}$$

greater than v. Putting $\mathbf{V} = \mathbf{i}\Phi$ as before we can get a solution satisfying the boundary conditions from both (72–5) and (72–6). The first gives

$$\mathbf{E} = \mathbf{k}Am\sin(my-\delta)\,e^{i(lx-\omega t)} = \mathbf{k}Am\sin(my-\delta)\cos(lx-\omega t). \tag{73–6}$$

If we make $\delta = 0$ and $m = k\pi/a$, where k is an integer, $\mathbf{E}$ vanishes for both $y = 0$ and $y = a$, satisfying the boundary conditions. Similarly the second gives

$$\begin{aligned}\mathbf{E} &= Am\{\mathbf{i}m\cos(my - \delta) - \mathbf{j}\mathrm{i}l\sin(my - \delta)\}e^{i(lx-\omega t)}\\ &= Am\{\mathbf{i}m\cos(my - \delta)\cos(lx - \omega t)\\ &\qquad + \mathbf{j}l\sin(my - \delta)\sin(lx - \omega t)\},\end{aligned} \tag{73–7}$$

which satisfies the boundary conditions provided $\delta = \pi/2$ and $m = k\pi/a$.

As $l = 2\pi/\lambda$, where λ is the wave-length, we have in the case of either wave the relation

$$\left(\frac{2\pi}{\lambda}\right)^2 = \frac{\omega^2}{c^2}\kappa\mu - \left(\frac{k\pi}{a}\right)^2$$

from $l^2 + m^2 = \omega^2/v^2$. Since λ must be real, and the smallest value of k is unity, this gives for the minimum frequency necessary to produce the wave

$$\nu = \frac{\omega}{2\pi} = \frac{c}{2a\sqrt{\kappa\mu}}. \tag{73-8}$$

As this minimum frequency is approached the phase velocity (73–5) increases without limit.

We can give a simple physical interpretation of these waves. Since $\delta = 0$ in (73–6) we can write this equation in the form

$$\mathbf{E} = \mathbf{k}\tfrac{1}{2}Am \sin(lx + my - \omega t) - \mathbf{k}\tfrac{1}{2}Am \sin(lx - my - \omega t), \tag{73-9}$$

showing that the wave motion is due to the superposition of two trains of plane waves traveling with the normal phase velocity v at angles with the X axis whose tangents are m/l and $-m/l$ respectively. The first is reflected as the second at the plane $y = a$, and *vice versa* at the plane $y = 0$. The abnormally high phase velocity (73–5) of the resultant wave is due to the fact that it represents the reciprocal of the component in the X direction of the normal wave-slowness $S = 1/v$ of each of the two component waves. Similarly (73–7) can be represented as the superposition of two trains of plane waves traveling with the normal phase velocity v by writing this equation in the form

$$\mathbf{E} = \tfrac{1}{2}Am\{(\mathbf{i}m - \mathbf{j}l)\sin(lx + my - \omega t) - (\mathbf{i}m + \mathbf{j}l)\sin(lx - my - \omega t)\}. \tag{73-10}$$

It should be noted that the electric intensity in each component wave is normal to the direction of propagation. In fact, the waves (73–9) and (73–10) differ only in their state of polarization. The first is plane polarized with the electric vector at right angles to the XY plane in which the directions of propagation of the two component elementary waves lie, and the second is polarized with the electric vector in this plane.

Now let us introduce in addition to the pair of conducting planes $y = 0, y = a$ an additional pair $z = 0, z = b$, and investigate electromagnetic waves traveling in the X direction inside the tube of rectangular cross-section so formed. The boundary conditions on the components of the electric intensity are: $E_x = 0$ for $y = 0, a$ and $z = 0, b$; $E_y = 0$ for $z = 0$, b; $E_z = 0$ for $y = 0, a$. Therefore neither of the waves (73–3) and (73–4) can be propagated along the tube, and of the waves (73–6) and (73–7) only the first can exist, that is the wave

$$\mathbf{E} = \mathbf{k}Am \sin my \cos (lx - \omega t), \tag{73–11}$$

where $m = k\pi/a$. We may, however, have the analogous wave

$$\mathbf{E} = \mathbf{j}An \sin nz \cos (lx - \omega t), \tag{73–12}$$

where $n = k\pi/b$. Here $l^2 + n^2 = \omega^2/v^2$.

Next we look for waves of the form $\Phi = Y(y)Z(z)\, e^{i(lx-\omega t)}$ represented by solutions of the complete wave equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2}. \tag{73–13}$$

Substituting, and separating the variables, we find

$$\frac{d^2Y}{dy^2} + m^2Y = 0,$$

$$\frac{d^2Z}{dz^2} + n^2Z = 0,$$

where $l^2 + m^2 + n^2 = \omega^2/v^2$. Therefore we may have

$$\Phi = A \cos (my - \delta) \cos (nz - \epsilon)\, e^{i(lx-\omega t)}.$$

This wave function gives rise to two electromagnetic waves satisfying the boundary conditions. These are

$$\mathbf{E} = A\{-\mathbf{j}n \cos my \sin nz + \mathbf{k}m \sin my \cos nz\} \cos (lx - \omega t), \tag{73–14}$$

and

$$\begin{aligned}\mathbf{E} = A\{&\mathbf{i}(m^2 + n^2) \sin my \sin nz \cos (lx - \omega t)\\ &- \mathbf{j}lm \cos my \sin nz \sin (lx - \omega t)\\ &- \mathbf{k}nl \sin my \cos nz \sin (lx - \omega t)\},\end{aligned} \tag{73–15}$$

in each of which $m = k_1\pi/a$ and $n = k_2\pi/b$, where k_1 and k_2 are integers. The minimum frequency necessary to excite these waves is

$$\nu = \frac{c}{2\sqrt{\kappa\mu}} \frac{\sqrt{a^2 + b^2}}{ab}. \tag{73-16}$$

Each of these waves can be represented as the superposition of four plane waves traveling with the normal phase velocity v.

(II) *Circular Cylinders.* First we shall consider a cable consisting of two coaxial cylindrical conductors, the space between being filled with a homogeneous isotropic non-conducting medium. Taking the X axis along the common axis of the two conductors, and using cylindrical coordinates r, ϕ, x, the wave equation (72–4) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Phi}{\partial t^2} \tag{73-17}$$

by (19–12), if the wave function Φ does not contain the azimuth ϕ, as we shall suppose to be the case.

We are interested in wave functions of the form $R(r)\, e^{i(lx - \omega t)}$. Substituting in (73–17) we find that $R(r)$ must satisfy the equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + k^2R = 0, \tag{73-18}$$

where $l^2 + k^2 = \omega^2/v^2$. The solution applicable to the problem under consideration is that for which $k = 0$ and therefore the phase velocity $\omega/l = v$. We get then

$$\Phi = A \log r\, e^{i(lx - \omega t)}.$$

Next we put $\mathbf{V} = i\Phi$ and use (72–6) for $\mathbf{E}$ and $\mathbf{F}$. This gives for the non-vanishing components

$$\left.\begin{aligned} E_r &= i\frac{lA}{r}e^{i(lx-\omega t)} = -\frac{lA}{r}\sin(lx - \omega t), \\ F_\phi &= i\sqrt{\frac{\kappa}{\mu}}\frac{lA}{r}e^{i(lx-\omega t)} = -\sqrt{\frac{\kappa}{\mu}}\frac{lA}{r}\sin(lx - \omega t). \end{aligned}\right\} \tag{73-19}$$

The lines of electric force are seen to extend radially from the axis of the cable, whereas the lines of magnetic force are circles in planes at right angles to the axis with centers lying upon it. The charge per unit length on the inner conductor is

$$\rho_\lambda = -2\pi\kappa l A \sin(lx - \omega t) \tag{73-20}$$

from (62–16), and that upon the outer conductor is equal in magnitude but opposite in sign. The current, which has the opposite sense but the same magnitude in the two conductors, is equal to $\rho_\lambda v$.

Equation (73–18) has, however, other solutions of much physical interest. If we put $\rho \equiv kr$ in (73–18) this equation becomes

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + R = 0, \tag{73–21}$$

which is a special case of Bessel's equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(1 - \frac{n^2}{\rho^2}\right)R = 0. \tag{73–22}$$

We shall be concerned only with cases where n is a positive integer, including zero. Assuming a power series solution of this equation of the form $R = \sum_p a_p \rho^p$ we find the recurrence formula $(p^2 - n^2)a_p + a_{p-2} = 0$, which gives a solution

$$R = J_n(\rho) \equiv \frac{\rho^n}{2^n n!}\left\{1 - \frac{\rho^2}{2(2n+2)} + \frac{\rho^4}{2.4(2n+2)(2n+4)} - \cdots\right\} \tag{73–23}$$

known as a *Bessel function of the first kind of order n*. The solution of (73–21), then, is

$$R = J_0(\rho) \equiv 1 - \frac{\rho^2}{2^2} + \frac{\rho^4}{2^2 \cdot 4^2} - \cdots. \tag{73–24}$$

Evidently

$$J_0'(\rho) \equiv \frac{dJ_0(\rho)}{d\rho} = -J_1(\rho). \tag{73–25}$$

Consequently we obtain as another solution of (73–17),

$$\Phi = AJ_0(kr)\,e^{i(lx-\omega t)}, \tag{73–26}$$

where $l^2 + k^2 = \omega^2/v^2$. The phase velocity along the X axis is given by

$$\frac{\omega}{l} = \frac{v}{\sqrt{1 - \frac{k^2v^2}{\omega^2}}}, \tag{73–27}$$

which is greater than v for real values of k other than zero.

Putting $\mathbf{V} = i\Phi$ we shall discuss the two cases (72–5) and (72–6). For the latter

$$\left.\begin{aligned} E_r &= \mathrm{i}AlkJ_0'(kr)\,\mathrm{e}^{\mathrm{i}(lx-\omega t)} = -AlkJ_0'(kr)\sin(lx-\omega t),\\ E_\phi &= 0,\\ E_x &= Ak^2J_0(kr)\,\mathrm{e}^{\mathrm{i}(lx-\omega t)} = Ak^2J_0(kr)\cos(lx-\omega t), \end{aligned}\right\} \quad (73\text{–}28)$$

and

$$\left.\begin{aligned} F_r &= 0,\\ F_\phi &= \mathrm{i}Ak\frac{\kappa\omega}{c}J_0'(kr)\,\mathrm{e}^{\mathrm{i}(lx-\omega t)} = -Ak\frac{\kappa\omega}{c}J_0'(kr)\sin(lx-\omega t),\\ F_x &= 0. \end{aligned}\right\} \quad (73\text{–}29)$$

By a consideration of the boundary conditions we shall now show that this wave can be propagated inside a perfectly conducting tube of circular cross-section without the presence of an axial wire. As $J_0'(kr) = -kr/2$ for very small r, the electric flux through a cylindrical surface of small radius r coaxial with the X axis is proportional to r^2, and therefore vanishes as r approaches zero. Consequently no charge is required along the X axis to satisfy the divergence equation (62–12a). In fact, as E_x does not vanish for $r = 0$, the presence of an axial wire is precluded. However, $J_0(\rho)$, and therefore E_x, vanishes for an infinite number of values of ρ of which the first three are 2.405, 5.520, 8.654. Hence we can terminate the electric field by means of a perfectly conducting cylindrical tube coaxial with the X axis of a radius a specified by any one of the zeros of $J_0(\rho)$.* If ρ_0 gives such a zero, the wave-length $\lambda = 2\pi/l$ is related to the frequency $\nu = \omega/2\pi$ by the equation

$$\left(\frac{2\pi}{\lambda}\right)^2 = \frac{\kappa\mu}{c^2}(2\pi\nu)^2 - \frac{\rho_0^{\,2}}{a^2}$$

obtained from the relation $l^2 + k^2 = \omega^2/v^2$. As the smallest value of the argument which makes $J_0(\rho)$ vanish is 2.405, the frequency must be at least as great as

$$\nu = \frac{2.405c}{2\pi a\sqrt{\kappa\mu}} = \frac{1.15(10)^{10}}{a\sqrt{\kappa\mu}}\ \text{cm/sec} \qquad (73\text{–}30)$$

* Of course, the wave may also be confined between coaxial cylinders of radii specified by any two of these zeros.

for the wave to be propagated along the tube. Hence for tubes of moderate radius very high frequencies must be employed. It follows from (73–27) that the phase velocity increases without limit as the minimum frequency is approached. The type of wave under discussion is known as an " E_0 " wave.

As in the discussion of spherical waves in article 72, the wave described by (72–5) is obtained, except for a constant factor in one of the coefficients, by interchanging the components of **E** and **F** in (73–28) and (73–29). In this case, then, the boundary condition at the surface of the tube requires that $J_0'(\rho)$ be zero. As the first zero of $J_0'(\rho)$, other than the value zero, comes at $\rho = 3.832$, the minimum frequency for propagation of the wave is

$$\nu = \frac{3.832c}{2\pi a\sqrt{\kappa\mu}} = \frac{1.83(10)^{10}}{a\sqrt{\kappa\mu}}\text{ cm/sec.} \qquad (73\text{–}31)$$

This type of wave is known as an " H_0 " wave.

The lines of electric force in the " E_0 " wave are shown in Fig. 74*a* and the lines of magnetic force in the " H_0 " wave in Fig. 74*b*. The

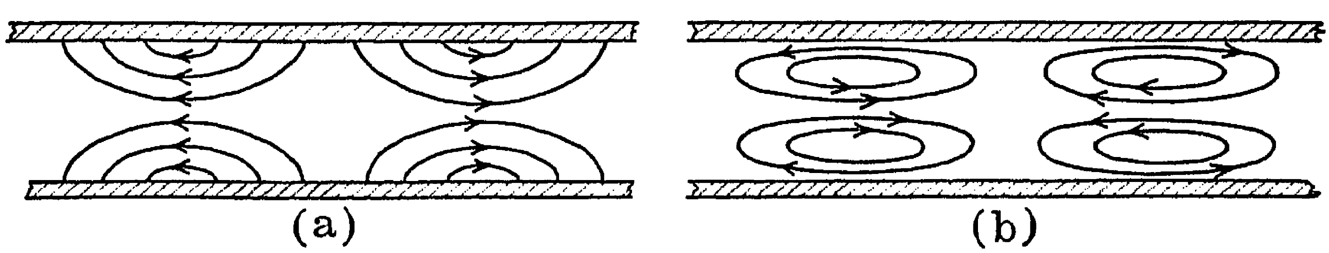

FIG. 74.

lines of force of the other field, in each case, are circles in planes at right angles to the axis of the tube.

We have discussed merely the two simplest types of waves which may be propagated inside a conducting tube. The theory of these waves has been developed by Carson, Mead and Schelkunoff,[2] and by Barrow,[3] and confirmed experimentally by Southworth [4] for empty tubes and for tubes filled with water ($\kappa = 81$).

The fact that the phase velocity ω/l of these waves is greater than v indicates, as in the case of the waves (73–6) and (73–7), that the resultant wave is due to the superposition of elementary plane waves traveling with velocity v at an angle with the axis of the tube

[2] Carson, Mead and Schelkunoff, Bell Sys. Tech. Jour. 15, p. 310 (1936).
[3] W. L. Barrow, Proc. I. R. E., 24, p. 1298 (1936).
[4] G. C. Southworth, Bell Sys. Tech. Jour. 15, p. 284 (1936).

and reflected back and forth from one side to the other. That this is the case has been shown by Page and Adams.[5] Since such elementary waves evidently must occur in pairs, we may conveniently take (73–6) and (73–7) as primary waves. Let Fig. 75 represent a cross-section of the tube. At a point P on the fixed Y_0 axis distant r from the axis O of the tube the electric intensity due to the wave (73–7) with $\delta = 0$ and k replacing m is

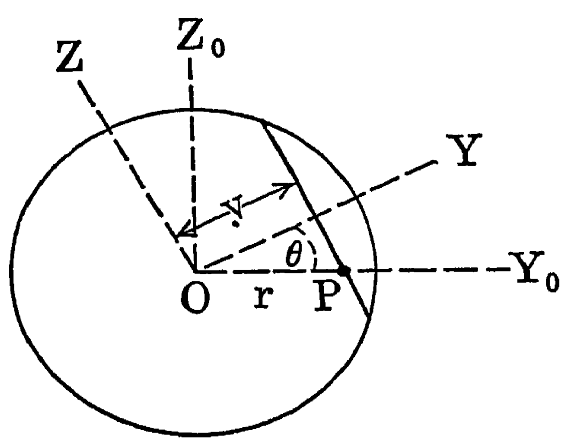

FIG. 75.

$$\mathbf{E} = Ak\{\mathbf{i}k \cos (kr \cos \theta) \cos (lx - \omega t) + \mathbf{j}l \sin (kr \cos \theta) \sin (lx - \omega t)\},$$

since $y = r \cos \theta$, where θ is the angle between the Y and Y_0 axes. Now suppose we have primary waves of this type of equal amplitude for all values of θ. Then, if we resolve along X, Y_0, Z_0, the components of the resultant electric intensity at P are

$$E_x = Ak^2 \left\{\int_0^{2\pi} \cos (kr \cos \theta)\, d\theta\right\} \cos (lx - \omega t),$$

$$E_{y_0} = Alk \left\{\int_0^{2\pi} \sin (kr \cos \theta) \cos \theta\, d\theta\right\} \sin (lx - \omega t),$$

$$E_{z_0} = Alk \left\{\int_0^{2\pi} \sin (kr \cos \theta) \sin \theta\, d\theta\right\} \sin (lx - \omega t).$$

But [6]

$$\sin (kr \cos \theta) = 2J_1(kr) \cos \theta - 2J_3(kr) \cos 3\theta + 2J_5(kr) \cos 5\theta - \cdots,$$

$$\cos (kr \cos \theta) = J_0(kr) - 2J_2(kr) \cos 2\theta + 2J_4(kr) \cos 4\theta - \cdots.$$

If we carry out the integration, replacing $J_1(kr)$ by $-J_0'(kr)$, we get exactly the " E_0 " wave (73–28), since our present E_{z_0} corresponds to the former E_ϕ, and E_{y_0} to E_r. Similarly, if we use (73–6) with $\delta = 0$ and k replacing m,

$$\mathbf{E} = \mathbf{k}Ak \sin (kr \cos \theta) \cos (lx - \omega t),$$

[5] Page and Adams, Phys. Rev. 52, p. 647 (1937).
[6] Gray, Mathews and Macrobert, *Bessel Functions*, p. 32.

which gives, under the same conditions,

$$E_x = 0,$$

$$E_{y_0} = -Ak\left\{\int_0^{2\pi} \sin(kr\cos\theta)\sin\theta\, d\theta\right\}\cos(lx - \omega t),$$

$$E_{z_0} = Ak\left\{\int_0^{2\pi} \sin(kr\cos\theta)\cos\theta\, d\theta\right\}\cos(lx - \omega t),$$

leading to the " H_0 " wave in the tube. Therefore the " E_0 " and " H_0 " waves differ only in the state of polarization of the elementary plane waves to whose superposition they are due.

(III) *Parallel Wires.* A simple harmonic plane wave advancing in the X direction with phase velocity $v = c/\sqrt{\kappa\mu}$ can be represented as the real or the imaginary part of the complex function

$$\Phi = h(y, z)\, e^{i(\epsilon x - \omega t)} \tag{73-32}$$

where $\epsilon \equiv \omega/v$. Substituting in the wave equation (72–4) we find that $h(y, z)$ is a solution of the two-dimensional Laplace's equation

$$\frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0. \tag{73-33}$$

Now either the real or the imaginary part of any function $F(y + iz)$ of the complex variable $y + iz$ is a solution of (73–33), since

$$\frac{\partial^2 F}{\partial y^2} = F'', \quad \frac{\partial^2 F}{dz^2} = i^2 F'' = -F''.$$

Therefore, if $F(y + iz) = f(y, z) + ig(y, z)$, where f and g are real functions, we may take for Φ

$$\Phi = \begin{Bmatrix} f(y, z) \\ g(y, z) \end{Bmatrix} e^{i(\epsilon x - \omega t)}. \tag{73-34}$$

In this case we can take for $\mathbf{E}$ or $\mathbf{F}$ in the wave either $-\nabla f\, e^{i(\epsilon x - \omega t)}$ or $-\nabla g\, e^{i(\epsilon x - \omega t)}$, since each is a solenoidal solution of the wave equation on account of the fact that f and g each satisfy Laplace's equation. Let us represent $\mathbf{E}$ by the first solution. Then, from (72–1c),

$$\mathbf{F} = \sqrt{\frac{\kappa}{\mu}}\left\{\boldsymbol{j}\frac{\partial f}{\partial z} - \boldsymbol{k}\frac{\partial f}{\partial y}\right\} e^{i(\epsilon x - \omega t)}.$$

But, as

$$F' = \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + i\frac{\partial g}{\partial y}, \quad iF' = \frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + i\frac{\partial g}{\partial z},$$

it follows that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial z}, \quad \frac{\partial f}{\partial z} = -\frac{\partial g}{\partial y}, \tag{73-35}$$

and consequently we may write

$$\mathbf{F} = -\sqrt{\frac{\kappa}{\mu}}\left\{j\frac{\partial g}{\partial y} + k\frac{\partial g}{\partial z}\right\} e^{i(\epsilon x - \omega t)}.$$

Hence we have the pair of related solutions

$$\mathbf{E} = -\nabla f\, e^{i(\epsilon x - \omega t)}, \quad \mathbf{F} = -\sqrt{\frac{\kappa}{\mu}}\nabla g\, e^{i(\epsilon x - \omega t)}, \tag{73-36}$$

and similarly the pair

$$\mathbf{E} = -\nabla g\, e^{i(\epsilon x - \omega t)}, \quad \mathbf{F} = \sqrt{\frac{\kappa}{\mu}}\nabla f\, e^{i(\epsilon x - \omega t)}. \tag{73-37}$$

Effectively we have reduced the problem of finding the amplitudes of $\mathbf{E}$ and $\mathbf{F}$ to a two-dimensional electrostatic problem; the amplitudes of $\mathbf{E}$ and $\mathbf{F}$ in (73-36) being the negative gradients of the potential functions $f(y, z)$ and $\sqrt{\kappa/\mu}\, g(y, z)$ respectively, and correspondingly in (73-37). As $\nabla g \cdot \nabla f = 0$ by virtue of (73-35), the lines of electric force for the case (73-36), since they intersect the curves $f(y, z) =$ Constant orthogonally, are defined by the equations $g(y, z) =$ Constant, and the lines of magnetic force, since they intersect the curves $g(y, z) =$ Constant orthogonally, are defined by the equations $f(y, z) =$ Constant. In the case (73-37) the lines of electric force and the lines of magnetic force are interchanged, the former being defined by the equations $f(y, z) =$ Constant and the latter by $g(y, z) =$ Constant.

The appropriate function for the discussion of waves along a pair of perfectly conducting parallel wires of circular cross-section is

$$F(y + iz) = \log\frac{y + iz + b}{y + iz - b} = f(y, z) + ig(y, z), \tag{73-38}$$

from which we find that

$$f(y, z) = \tfrac{1}{2}\log\frac{(y + b)^2 + z^2}{(y - b)^2 + z^2}, \tag{73-39}$$

$$g(y, z) = -\tan^{-1}\frac{2bz}{y^2 + z^2 - b^2}. \tag{73-40}$$

If we define **E** and **F** by (73–36), the lines of electric force are specified by the family of circles

$$y^2 + (z + b \cot g)^2 = b^2 \csc^2 g, \quad 0 < g < \pi, \qquad (73\text{–}41)$$

shown by broken lines in Fig. 76, and the lines of magnetic force by the second family of circles

$$(y - b \coth f)^2 + z^2 = b^2 \operatorname{csch}^2 f, \quad -\infty < f < \infty, \qquad (73\text{–}42)$$

shown by solid lines. By taking any pair of solid circles for the traces of the conducting wires we can satisfy the boundary conditions.

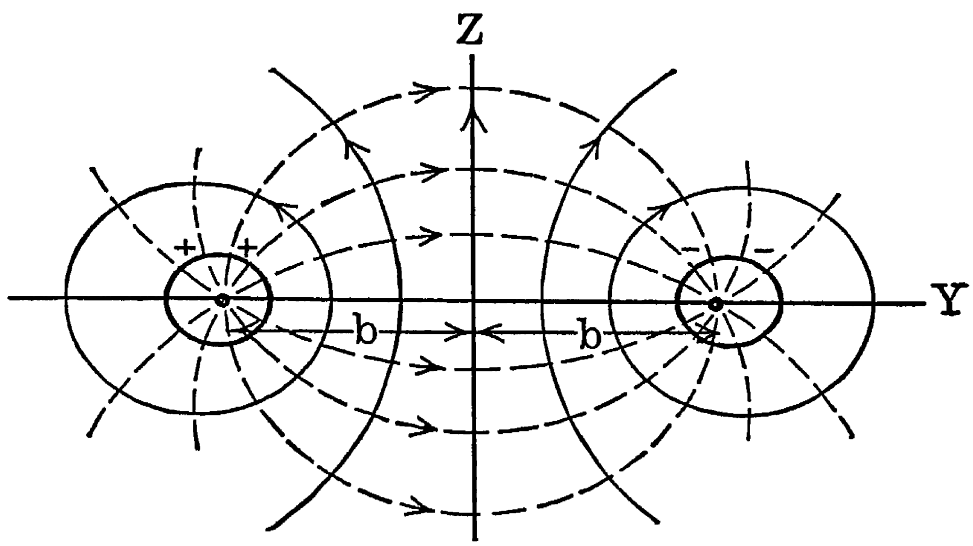

FIG. 76.

If the two wires have the same radius a, the distance between their axes is $d = 2\sqrt{b^2 + a^2}$ from (73–42).

If $\mathbf{E}_0$ and $\mathbf{F}_0$ represent the amplitudes of **E** and **F** respectively,

$$\mathbf{E}_0 = -\nabla f = -\mathbf{j}\frac{\partial f}{\partial y} - \mathbf{k}\frac{\partial f}{\partial z} \qquad (73\text{–}43)$$

from (73–36), and

$$\mathbf{F}_0 = -\sqrt{\frac{\kappa}{\mu}}\nabla g = \sqrt{\frac{\kappa}{\mu}}\left(\mathbf{j}\frac{\partial f}{\partial z} - \mathbf{k}\frac{\partial f}{\partial y}\right) \qquad (73\text{–}44)$$

from (73–36) and (73–35). Therefore $F_0 = \sqrt{\kappa/\mu}\, E_0$ everywhere, and the boundary conditions (62–16) and (62–17) at the surface of the wire make the current density on the surface equal to the charge density multiplied by the phase velocity $v = c/\sqrt{\kappa\mu}$ of the waves.

74. Waves Guided by Imperfect Conductors. — In this article we shall consider simple harmonic waves guided by conductors pos-

sessing a finite conductivity σ immersed in a homogeneous isotropic medium for which $\mathbf{D} = \kappa\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{F}$. Outside the conductors the field equations (72–1) still hold, but inside we must use the more general equations (62–12). However, as $\rho\mathbf{V} = \sigma\mathbf{E}$ and $\mathbf{E}$ is a simple harmonic function of the time, we can put $\rho\mathbf{V} = (\mathrm{i}\sigma/\omega)\dot{\mathbf{E}}$. Also from the equation of continuity (62–1) we have $\rho = -\nabla\cdot(\mathrm{i}\sigma/\omega)\mathbf{E}$. Hence (62–12*a*) becomes $\nabla\cdot(\kappa + \mathrm{i}\sigma/\omega)\mathbf{E} = 0$ where κ is the permittivity of the conductor. If, as we shall assume, the conductor is homogeneous, κ and σ are not functions of the coordinates, and this field equation reduces to $\nabla\cdot\mathbf{E} = 0$. So, altogether, we have for the field equations in the homogeneous parts of the metal,

$$\left.\begin{aligned} &\nabla\cdot\mathbf{E} = 0, \quad (a) \qquad \nabla\cdot\mathbf{F} = 0, \quad (b) \\ &\nabla\times\mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{F}}, \ (c) \quad \nabla\times\mathbf{F} = \frac{1}{c}\left(\kappa + \mathrm{i}\frac{\sigma}{\omega}\right)\dot{\mathbf{E}}, \ (d) \end{aligned}\right\} \qquad (74\text{–}1)$$

where μ is the permeability of the conductor. These differ in form from (72–1) only in that κ is replaced by the complex quantity $\kappa + \mathrm{i}\sigma/\omega$. Therefore the wave equations (72–2) and (72–3) become

$$\nabla\cdot\nabla\mathbf{E} = \left(\kappa + \mathrm{i}\frac{\sigma}{\omega}\right)\frac{\mu}{c^2}\ddot{\mathbf{E}}, \qquad (74\text{–}2)$$

and

$$\nabla\cdot\nabla\mathbf{F} = \left(\kappa + \mathrm{i}\frac{\sigma}{\omega}\right)\frac{\mu}{c^2}\ddot{\mathbf{F}}, \qquad (74\text{–}3)$$

yielding the complex wave-slowness $S = \sqrt{(\kappa + \mathrm{i}\sigma/\omega)\mu}/c$.

The physical significance of a complex wave-slowness is easily revealed by considering plane waves traveling in the interior of the homogeneous conductor. If the vector complex wave-slowness is written in the form $\mathbf{S} = \mathbf{S}' + \mathrm{i}\mathbf{S}''$, where $\mathbf{S}'$ and $\mathbf{S}''$ are real, we find, by taking the real parts of (72–14) for $\mathbf{E}$ and $\mathbf{F}$,

$$\mathbf{E} = \mathbf{E}_0 e^{-\omega\mathbf{S}''\cdot\mathbf{r}} \cos\omega(\mathbf{S}'\cdot\mathbf{r} - t), \qquad (74\text{–}4)$$

$$\mathbf{F} = \mathbf{F}_0 e^{-\omega\mathbf{S}''\cdot\mathbf{r}} \cos\omega(\mathbf{S}'\cdot\mathbf{r} - t), \qquad (74\text{–}5)$$

which show that S' is the reciprocal of the actual velocity of propagation and that $\omega S''$ measures the attenuation of the wave as it progresses into the medium.

It will be convenient to write down the boundary conditions (62–16) and (62–17) at the surface separating two homogeneous

isotropic media which have conductivity as well as permittivity and permeability, for the special case where the field is simple harmonic. As the effective permittivity is $\kappa + i\sigma/\omega$, these equations become

$$\left(\kappa_2 + i\frac{\sigma_2}{\omega}\right)E_{2n} = \left(\kappa_1 + i\frac{\sigma_1}{\omega}\right)E_{1n}, \quad \mu_2 F_{2n} = \mu_1 F_{1n}, \tag{74-6}$$

$$E_{2t} = E_{1t}, \qquad F_{2t} = F_{1t}, \tag{74-7}$$

where the numerical subscripts distinguish the two media, and the subscripts n and t refer respectively to the normal and tangential components of **E** and **F**.

We can greatly simplify our notation by putting

$$\psi \equiv \frac{\omega}{c}\left(\kappa + i\frac{\sigma}{\omega}\right), \quad \chi \equiv \frac{\omega}{c}\mu. \tag{74-8}$$

Then the wave equation (72–4) becomes

$$\nabla\cdot\nabla\Phi + \psi\chi\Phi = 0, \tag{74-9}$$

the pairs of equations (72–5) and (72–6) take the form

$$\mathbf{E} = \nabla\times\mathbf{V}, \qquad \mathbf{F} = -\frac{i}{\chi}\nabla\times\nabla\times\mathbf{V}, \tag{74-10}$$

$$\mathbf{E} = \nabla\times\nabla\times\mathbf{V}, \qquad \mathbf{F} = -i\psi\nabla\times\mathbf{V}, \tag{74-11}$$

and the boundary condition (74–6) is simply

$$\psi_2 E_{2n} = \psi_1 E_{1n}, \qquad \chi_2 F_{2n} = \chi_1 F_{1n}. \tag{74-12}$$

(I) *Plane Conductor*. First we shall consider waves propagated in the X direction in a non-conducting medium (1) situated above the plane $y = 0$, the region below this plane being occupied by a conductor (2). To a fair approximation this is the situation existing when radio waves travel horizontally near the surface of the earth.

The solution of the wave equation (74–9) required to satisfy the boundary conditions is

$$\Phi = a\,e^{(\alpha x + \beta y - i\omega t)}, \tag{74-13}$$

where

$$\alpha^2 + \beta^2 + \psi\chi = 0, \tag{74-14}$$

the parameters α, β, a being, in general, complex.

We are interested in the wave defined by (74–11). Putting $\mathbf{V} = \boldsymbol{i}\Phi$ we find

$$\mathbf{E} = (-\boldsymbol{i}\beta^2 + \boldsymbol{j}\alpha\beta)\Phi, \quad \mathbf{F} = \boldsymbol{k}i\psi\beta\Phi. \tag{74-15}$$

Evidently the boundary conditions at the plane $y = 0$ can be satisfied only if α as well as ω has the same value in the two media. Then (74–12) and (74–7) give only the two independent relations:

$$a_2\beta_2^{\,2} = a_1\beta_1^{\,2}, \tag{74–16}$$

$$a_2\psi_2\beta_2 = a_1\psi_1\beta_1, \tag{74–17}$$

to which we must add the relations

$$\alpha^2 + \beta_1^{\,2} + \psi_1\chi_1 = 0, \tag{74–18}$$

$$\alpha^2 + \beta_2^{\,2} + \psi_2\chi_2 = 0, \tag{74–19}$$

obtained from (74–14). Solving these four equations for α, β_1, β_2 and a_2/a_1 we get

$$\left.\begin{aligned} \alpha^2 &= \psi_1\psi_2\frac{\psi_1\chi_2 - \psi_2\chi_1}{\psi_2^{\,2} - \psi_1^{\,2}}, \\ \beta_1^{\,2} &= \psi_1^{\,2}\frac{\psi_1\chi_1 - \psi_2\chi_2}{\psi_2^{\,2} - \psi_1^{\,2}}, \\ \beta_2^{\,2} &= \psi_2^{\,2}\frac{\psi_1\chi_1 - \psi_2\chi_2}{\psi_2^{\,2} - \psi_1^{\,2}}, \\ \frac{a_2}{a_1} &= \frac{\psi_1^{\,2}}{\psi_2^{\,2}}. \end{aligned}\right\} \tag{74–20}$$

As ψ_2 is complex, each of these expressions is also complex. They can be somewhat simplified for the usual case where $\mu_2 = \mu_1 = 1$ and consequently $\chi_2 = \chi_1$.

The resistivity of common metals is of the order of $(10)^{-5}$ ohm cm, and therefore the conductivity σ is of the order of $(10)^{18}$ h.l.u. Consequently, even for frequencies as high as $(10)^6$/sec, the ratio σ/ω is very large compared with κ, which is rarely greater than 10. Hence, if we write the expression for ψ_2 in the form

$$\psi_2 = i\frac{\sigma_2}{c}\left(1 - i\frac{\omega\kappa_2}{\sigma_2}\right), \tag{74–21}$$

the second term in the parentheses is extremely small compared with the first for metallic conductors even at the highest radio frequencies. In calculating α we shall neglect the square and higher powers of ω/σ_2 as compared with unity, and in calculating β_1, β_2 and a_2/a_1 we shall neglect all powers of this small quantity. This means the

neglect of ψ_1^2 as compared with ψ_2^2 in all cases. Putting $\epsilon_1 = \omega/v_1$, where $v_1 = c/\sqrt{\kappa_1\mu_1}$ is the normal phase velocity of a wave in medium (1), we find

$$\left.\begin{aligned}
\alpha &\equiv -\alpha' + i\epsilon_1 = -\tfrac{1}{2}\epsilon_1 \frac{\omega\kappa_1}{\sigma_2}\frac{\mu_2}{\mu_1} + i\epsilon_1,\\
\beta_1 &\equiv -\beta_1' - i\beta_1'' = -\frac{1}{\sqrt{2}}(1+i)\frac{\omega\kappa_1}{c}\sqrt{\frac{\omega\mu_2}{\sigma_2}}\\
&= -\frac{\omega\kappa_1}{c}\sqrt{\frac{\omega\mu_2}{\sigma_2}}\,e^{i\frac{\pi}{4}},\\
\beta_2 &\equiv \beta_2' - i\beta_2'' = \frac{1}{\sqrt{2}}(1-i)\frac{\omega\mu_2}{c}\sqrt{\frac{\sigma_2}{\omega\mu_2}}\\
&= \frac{\omega\mu_2}{c}\sqrt{\frac{\sigma_2}{\omega\mu_2}}\,e^{-i\frac{\pi}{4}},\\
\frac{a_2}{a_1} &= -\frac{\omega^2\kappa_1^2}{\sigma_2^2}.
\end{aligned}\right\} \quad (74\text{–}22)$$

As E_{1y} is the dominant component of the wave we shall put $A \equiv \alpha\beta_1 a_1$. Then (74–15) gives

$$\left.\begin{aligned}
\mathbf{E}_1 &= iA\sqrt{\frac{\omega\mu_2}{\sigma_2}\frac{\kappa_1}{\mu_1}}\,e^{-\alpha'x-\beta_1'y}\,e^{i(\epsilon_1 x-\beta_1''y-\omega t-\pi/4)}\\
&\quad + jAe^{-\alpha'x-\beta_1'y}\,e^{i(\epsilon_1 x-\beta_1''y-\omega t)},\\
\mathbf{F}_1 &= kA\sqrt{\frac{\kappa_1}{\mu_1}}\,e^{-\alpha'x-\beta_1'y}\,e^{i(\epsilon_1 x-\beta_1''y-\omega t)},
\end{aligned}\right\} \quad (74\text{–}23)$$

for the wave in the non-conducting medium, and

$$\left.\begin{aligned}
\mathbf{E}_2 &= iA\sqrt{\frac{\omega\mu_2}{\sigma_2}\frac{\kappa_1}{\mu_1}}\,e^{-\alpha'x+\beta_2'y}\,e^{i(\epsilon_1 x-\beta_2''y-\omega t-\pi/4)}\\
&\quad + jA\frac{\omega\kappa_1}{\sigma_2}\,e^{-\alpha'x+\beta_2'y}\,e^{i(\epsilon_1 x-\beta_2''y-\omega t-\pi/2)},\\
\mathbf{F}_2 &= kA\sqrt{\frac{\kappa_1}{\mu_1}}\,e^{-\alpha'x+\beta_2'y}\,e^{i(\epsilon_1 x-\beta_2''y-\omega t)},
\end{aligned}\right\} \quad (74\text{–}24)$$

for the wave in the conductor. For the purpose of discussion let us

refer to 1, $(\omega/\sigma_2)^{1/2}$ and ω/σ_2 as quantities of zero, first and second order, respectively. For ordinary metals these three numbers are in the ratio $1 : (10)^{-6} : (10)^{-12}$ for frequencies as high as $(10)^6$/sec.

First consider the wave in the non-conducting medium. While the amplitudes of the components E_{1y} and F_{1z} are of zero order and bear the same ratio as if the conductor were perfect, the amplitude of of E_{1x} is of the first order and therefore very much smaller. The attenuation constants α' and β_1' are of the second and first orders respectively. Consequently the amplitude falls off more rapidly with increase in y than with increase in x. It is interesting to note that α' can be decreased by making the permeability μ_1 larger. Through terms in the second order the velocity of propagation, ω/ϵ_1, is the same as if the conductor were perfect. Hence, for common metals and frequencies not higher than $(10)^6$/sec, we have made errors of only one part in a million, or less, in basing the theory of guided waves developed in the last article on the assumption of perfect conductivity.

Inside the conductor the amplitude of only the magnetic field is of zero order, the amplitudes of E_{2x} and E_{2y} being of first and second orders respectively. As the attenuation constant β_2' is very large for high frequencies, the wave has an appreciable magnitude only in the region very close to the surface. Charge exists only on the surface of the conductor. As E_{2y} is negligible compared with E_{1y} the charge per unit area is equal to $\kappa_1 E_{1y}$ at the surface. The current density at any depth y below the surface is equal to $\sigma_2(\mathbf{i}E_{2x} + \mathbf{j}E_{2y})$. As E_{2y} is an order smaller than E_{2x}, the current normal to the surface is negligible compared with that in the direction of propagation of the wave. Furthermore, as β_2' is proportional to $\sqrt{\omega}$, the distance from the surface at which the current density falls to 1/e times its value at the surface becomes smaller the higher the frequency. This phenomenon is known as the *skin effect*. At sufficiently high frequencies we can consider the current to be confined entirely to the surface, as with the perfect conductors treated in the last article.

The fact that the longitudinal current at the surface is not in phase with the magnetic field may occasion some surprise. It must be remembered, however, that, on account of the term $-i\beta_2''y$ in the exponential factor of E_{2x}, the phase of the longitudinal current changes with the depth below the surface. The integrated current

$$\int_{-\infty}^{0} \sigma_2 E_{2x} dy = cA\sqrt{\frac{\kappa_1}{\mu_1}}\, e^{-\alpha' x}\, e^{i(\epsilon_1 x - \omega t)}$$

is in phase with F at the surface and, indeed, satisfies the condition that it should be equal to $c(F)_{y=0}$.

The mean flux of energy into the conductor per unit time per unit area is given by the time average of the component $cE_{1x}F_{1z}$ of the Poynting flux. It is equal to the Joule heat $\int_{-\infty}^{0}\sigma_2 E_{2x}^2 dy$ developed per unit time per unit area, each being given by

$$\frac{\sqrt{2}}{4}A^2 c\,\frac{\kappa_1}{\mu_1}\sqrt{\frac{\omega\mu_2}{\sigma_2}}\,e^{-2\alpha' x}.$$

(II) *Cable*. We shall now reconsider the problem of the cable which we treated in article 73 on the assumption that the two coaxial cylindrical conductors had infinite conductivity. Here we shall attribute to the inner conductor of radius a a permeability μ_2 and a finite but very large conductivity σ_2, neglecting entirely its permittivity κ_2 as compared with σ_2/ω. The permittivity and permeability of the intervening non-conducting medium we shall designate by κ_1 and μ_1, and we shall suppose the outer conductor of radius b to be a perfect conductor of infinite conductivity. As the example just investigated suffices to convince us that the wave in the non-conducting medium is described to a very high degree of approximation by the solution obtained in article 73, we shall confine our attention to the wave inside the inner conductor, with special reference to the calculation of the effective resistance and effective self-inductance per unit length of the cable.

Using cylindrical coordinates as in article 73, we look for a solution of the wave equation (74–9) of the form

$$\Phi = R(r)\,e^{(\alpha x - i\omega t)}. \qquad (74\text{–}25)$$

Following the method of the previous article we find that R must satisfy the modified Bessel's equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - R = 0, \qquad (74\text{–}26)$$

where $\rho \equiv kr$ and

$$\alpha^2 + k^2 + \psi\chi = 0. \qquad (74\text{–}27)$$

We are interested in the case (74–11). Putting $\mathbf{V} = i\Phi$ we have for the non-vanishing components of **E** and **F**

$$\left.\begin{aligned} E_r &= \alpha k \frac{dR}{d\rho}\, e^{(\alpha x - i\omega t)}, \\ E_x &= -\, k^2 R\, e^{(\alpha x - i\omega t)}, \\ F_\phi &= i\psi k \frac{dR}{d\rho}\, e^{(\alpha x - i\omega t)}. \end{aligned}\right\} \tag{74–28}$$

Evidently $J_0(i\rho)$ is a solution of (74–26) which remains finite on the axis of the wire. The independent solution $K_0(i\rho)$, contains the logarithm of ρ and becomes infinite for $\rho = 0$. Therefore the solution of (74–26) applicable to the inner conductor is

$$\Phi = CJ_0(ikr)\, e^{i(\epsilon x - \omega t)}$$

where we have neglected the small attenuation in the direction of propagation. Here $\epsilon \equiv \epsilon_1 = \omega/v_1$ where $v_1 = c/\sqrt{\kappa_1\mu_1}$ and, as $\psi \equiv \psi_2 = i\sigma_2/c$ and $\chi \equiv \chi_2 = \omega\mu_2/c$, it follows from (74–27) that

$$k^2 = \frac{\omega^2}{{v_1}^2}\left\{1 - i\frac{\sigma_2}{\omega}\frac{\mu_2}{\kappa_1\mu_1}\right\}.$$

Since, however, σ_2/ω is of the order of $(10)^{12}$ for common metals and frequencies not greater than $(10)^6$/sec, the first term in this expression is negligible compared with the second, and we may write

$$k^2 = -\, i\frac{\omega\sigma_2\mu_2}{c^2}. \tag{74–29}$$

The axial electric intensity inside the wire is therefore

$$E_x = AJ_0(ikr)\, e^{i(\epsilon x - \omega t)} \tag{74–30}$$

from (74–28). The axial current density ρV_x is equal to $\sigma_2 E_x$, that is,

$$\rho V_x = \sigma_2 AJ_0(ikr)\, e^{i(\epsilon x - \omega t)}. \tag{74–31}$$

Also the magnetic force inside the wire is

$$\begin{aligned} F_\phi &= \frac{i\psi}{-k^2} A \frac{dJ_0(ikr)}{dr}\, e^{i(\epsilon x - \omega t)} \\ &= \frac{icA}{\omega\mu_2}\frac{dJ_0(ikr)}{dr}\, e^{i(\epsilon x - \omega t)}, \end{aligned} \tag{74–32}$$

and the time rate of increase of magnetic induction is

$$\dot{B}_\phi = \mu_2 \dot{F}_\phi = cA\frac{dJ_0(\mathrm{i}kr)}{dr}\,\mathrm{e}^{\mathrm{i}(\epsilon x-\omega t)}.$$

Therefore the excess of the electromotive force on a current filament at a distance r from the axis over one at a distance a due to changing magnetic flux is, per unit length of the cable,

$$\Delta\mathcal{E} = -\frac{1}{c}\int_r^a \dot{B}_\phi\,dr = -A\{J_0(\mathrm{i}ka) - J_0(\mathrm{i}kr)\}\mathrm{e}^{\mathrm{i}(\epsilon x-\omega t)}. \quad (74\text{–}33)$$

Let $\mathcal{E}$ be the electromotive force per unit length due to the time rate of decrease of the magnetic flux outside the wire. Evidently $\mathcal{E}$ is the same for all current filaments. Then Ohm's law gives for the applied electromotive force $\mathcal{E}_0$ per unit length of the cable

$$\begin{aligned}\mathcal{E}_0 &= \frac{1}{\sigma_2}\rho V_x - \mathcal{E} - \Delta\mathcal{E}\\ &= -\mathcal{E} + AJ_0(\mathrm{i}ka)\,\mathrm{e}^{\mathrm{i}(\epsilon x-\omega t)},\end{aligned}$$

which is independent of r, as it should be. The total impedance per unit length of the cable may be written $R_e - \mathrm{i}\omega L_e$ where R_e is the effective resistance per unit length of the cable and L_e the effective self-inductance, the negative sign being due to the fact that $d/dt = -\mathrm{i}\omega$ in our notation instead of $\mathrm{i}\omega$ as in the usual circuit theory notation. Now the total impedance is the quotient of the applied electromotive force $\mathcal{E}_0$ by the current i. The current is

$$i = 2\pi\sigma_2 A\int_0^a J_0(\mathrm{i}kr)r\,dr\,\mathrm{e}^{\mathrm{i}(\epsilon x-\omega t)} \quad (74\text{–}34)$$

from (74–31). But, if L_s is the self-inductance of the surface of the wire, $\mathcal{E}/i = \mathrm{i}\omega L_s$, where

$$L_s = \frac{\mu_1}{2\pi c^2}\log\frac{b}{a} \quad (74\text{–}35)$$

from (69–12). Consequently

$$R_e - \mathrm{i}\omega L_e = -\mathrm{i}\omega L_s + \frac{1}{2\pi\sigma_2}\frac{J_0(\mathrm{i}ka)}{\displaystyle\int_0^a J_0(\mathrm{i}kr)r\,dr}.$$

By equating the real and imaginary parts of this equation we can find R_e and L_e.

Now, from (73–24),

$$\int_0^a J_0(\mathrm{i}kr)r\,dr = \int_0^a \left\{1 + \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2\cdot 4^2} + \cdots\right\} r\,dr$$
$$= \frac{a^2}{2}\left\{1 + \frac{k^2a^2}{2\cdot 4} + \frac{k^4a^4}{2\cdot 4\cdot 4\cdot 6} + \cdots\right\} = \frac{a}{\mathrm{i}k} J_1(\mathrm{i}ka),$$

and hence

$$R_e - \mathrm{i}\omega L_e = -\,\mathrm{i}\omega L_s + \frac{\mathrm{i}k}{2\pi a\sigma_2}\frac{J_0(\mathrm{i}ka)}{J_1(\mathrm{i}ka)}. \qquad (74\text{–}36)$$

If the frequency is low, so that ka is small, the series (73–23) for J_n converges rapidly. Using these series we find

$$\frac{J_0(\mathrm{i}ka)}{J_1(\mathrm{i}ka)} = \frac{2}{\mathrm{i}ka}\left\{1 + \frac{k^2a^2}{8} - \frac{k^4a^4}{192} + \frac{k^6a^6}{3072} - \frac{k^8a^8}{46{,}080} + \frac{13k^{10}a^{10}}{8{,}847{,}360} - \cdots\right\},$$

from which it follows, if we use the value of k^2 given by (74–29), that the effective resistance R_e per unit length is given by

$$R_e = R_0\left\{1 + \frac{\omega^2{\sigma_2}^2{\mu_2}^2a^4}{192c^4} - \frac{\omega^4{\sigma_2}^4{\mu_2}^4a^8}{46{,}080c^8} + \cdots\right\}, \qquad (74\text{–}37)$$

where $R_0 \equiv 1/(\pi a^2\sigma_2)$ is the resistance for zero frequency, and the effective self-inductance per unit length by

$$L_e = L_s + \frac{\mu_2}{8\pi c^2} - \frac{\omega^2{\sigma_2}^2{\mu_2}^3a^4}{3072\pi c^6} + \frac{13\omega^4{\sigma_2}^4{\mu_2}^5a^8}{8{,}847{,}360\pi c^{10}} - \cdots. \qquad (74\text{–}38)$$

We note from the latter that the self-inductance for zero frequency is not L_s but

$$L_0 = L_s + \frac{\mu_2}{8\pi c^2} = \frac{\mu_1}{2\pi c^2}\log\frac{b}{a} + \frac{\mu_2}{8\pi c^2},$$

the second term of which is large if the permeability μ_2 of the wire is great. In electromagnetic units (74–37) and (74–38) are

$$\left.\begin{aligned} R_{em} &= R_{0m}\left\{1 + \frac{1}{12}\pi^2\omega^2{\sigma_m}^2{\mu_2}^2a^4 - \frac{1}{180}\pi^4\omega^4{\sigma_m}^4{\mu_2}^4a^8 + \cdots\right\},\\ L_{em} &= 2\mu_1\log\frac{b}{a} + \frac{1}{2}\mu_2 - \frac{1}{48}\pi^2\omega^2{\sigma_m}^2{\mu_2}^3a^4\\ &\qquad + \frac{13}{8640}\pi^4\omega^4{\sigma_m}^4{\mu_2}^5a^8 - \cdots. \end{aligned}\right\} \qquad (74\text{–}39)$$

It should be noted that the resistance increases with increasing frequency due to the concentration of the current in the outer portion of the wire, whereas the self-inductance decreases.

The series just obtained for R_e and L_e are useful only for moderate frequencies. For very large values of the argument [7]

$$J_n(ika) = i^n \frac{e^{ka}}{\sqrt{2\pi ka}}. \tag{74-40}$$

Hence, for very high frequencies

$$\frac{J_0(ika)}{J_1(ika)} = -i.$$

Under these circumstances (74–36) gives

$$R_e = \frac{1}{2\sqrt{2\pi}ac}\sqrt{\frac{\omega\mu_2}{\sigma_2}}, \tag{74-41}$$

and

$$L_e = L_s + \frac{1}{2\sqrt{2\pi}ac}\sqrt{\frac{\mu_2}{\omega\sigma_2}}, \tag{74-42}$$

or, in electromagnetic units,

$$\left.\begin{aligned} R_{em} &= \sqrt{\frac{\omega\mu_2}{2\pi a^2\sigma_m}}, \\ L_{em} &= 2\mu_1 \log\frac{b}{a} + \sqrt{\frac{\mu_2}{2\pi a^2\omega\sigma_m}}. \end{aligned}\right\} \tag{74-43}$$

The resistance continues to increase indefinitely and the self-inductance to decrease to the limiting value L_s with increasing frequency. The reader must be warned, however, that the two expressions contained in (74–43) are incomplete if the outer conductor has a conductivity of the same order of magnitude as that of the wire.

[7] Gray, Mathews and Macrobert, *Bessel Functions*, p. 58.

CHAPTER 7

RADIATION AND RADIATING SYSTEMS

75. Radiation from a Point Charge. — In article 48 we found retarded expressions for the electric and magnetic intensities of a point charge moving with an arbitrary velocity **V** and acceleration **f** relative to the observer's inertial system S. These expressions contain both a term proportional to the inverse square of the radius vector $[r]$ and a term proportional to the inverse first power of $[r]$. In this chapter we shall be concerned only with the field at great distances from the charge producing it. Therefore we can neglect the first term as compared with the second in (48–7) and (48–8) and write for **E** and **H**:

$$\mathbf{E} = \left[\frac{e}{4\pi r c^4\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\{\mathbf{f}\times(\mathbf{c}-\mathbf{V})\}\times\mathbf{c}\right], \tag{75–1}$$

$$\mathbf{H} = \left[\frac{e}{4\pi r c^4\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3}\frac{1}{c}\mathbf{c}\times(\{\mathbf{f}\times(\mathbf{c}-\mathbf{V})\}\times\mathbf{c})\right]. \tag{75–2}$$

This distant field is known as the *radiation field* of the charge e. Not only are **E** and **H** perpendicular to each other but both are perpendicular to the direction **c** of propagation of the field.

Since $\mathbf{H} = (1/c)\mathbf{c}\times\mathbf{E}$ and $\mathbf{E}\cdot\mathbf{c} = 0$, the Poynting flux at a great distance from the charge is

$$\mathbf{s} = c(\mathbf{E}\times\mathbf{H}) = E^2\mathbf{c}. \tag{75–3}$$

Therefore the flow of energy in the radiation field is in the direction of propagation of the field.

We wish to calculate the time rate $\mathscr{R}$ of radiation of energy from the point charge. To do this as simply as possible we make use of the flux of energy through the surface of a fixed sphere of very large radius r with center at the position of the charge at time t. As the field is propagated with the velocity of light, energy emitted in any

direction from the charge at time t will pass through the surface of the sphere at time $t + r/c$. On account of the velocity of the charge, however, energy emitted at the later time $t + dt$ in a direction $\mathbf{c}$ has to travel a distance $r - (\mathbf{c}\cdot\mathbf{V}\,dt)/c$ to reach the surface of the sphere, where it will arrive at time $t + dt + r/c - (\mathbf{c}\cdot\mathbf{V}\,dt)/c^2$. Hence the energy radiated from the charge in the direction $\mathbf{c}$ in the time dt will pass through the surface of the sphere in a time $(1 - \mathbf{c}\cdot\mathbf{V}/c^2)dt$. So it follows from (75–3) and (75–1) that the rate of radiation of energy by the charge at time t per unit solid angle Ω is

$$\frac{d\mathscr{R}}{d\Omega} = \frac{e^2}{16\pi^2 c^7\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^5} \left| \{\mathbf{f} \times (\mathbf{c} - \mathbf{V})\} \times \mathbf{c} \right|^2$$

$$= \frac{e^2}{16\pi^2 c^3}\left\{\frac{f^2}{\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^3} + \frac{2\dfrac{\mathbf{f}\cdot\mathbf{V}}{c}\dfrac{\mathbf{f}\cdot\mathbf{c}}{c}}{\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^4} - \frac{\left(1 - \dfrac{V^2}{c^2}\right)\left(\dfrac{\mathbf{f}\cdot\mathbf{c}}{c}\right)^2}{\left(1 - \dfrac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)^5}\right\}. \tag{75–4}$$

This is a simultaneous expression, giving the rate of radiation at the instant when the charge has the velocity $\mathbf{V}$ and acceleration $\mathbf{f}$. If V is negligible compared with c,

$$\frac{d\mathscr{R}}{d\Omega} = \frac{e^2}{16\pi^2 c^5}\left|\mathbf{f} \times \mathbf{c}\right|^2, \tag{75–5}$$

which indicates, in agreement with the theory of the electric oscillator discussed in article 72, that the maximum radiation of energy occurs in directions at right angles to $\mathbf{f}$ and that no radiation is emitted in the direction of $\mathbf{f}$.

To obtain the total rate of radiation we must integrate (75–4) over all directions. If we introduce spherical coordinates with polar axis in the direction of $\mathbf{V}$ and azimuth ϕ measured from the plane of $\mathbf{V}$ and $\mathbf{f}$, we have to evaluate the integrals

$$J_1 \equiv \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta\;d\theta\;d\phi}{(1 - \mathrm{B}\cos\theta)^3} = \frac{4\pi}{(1 - \mathrm{B}^2)^2},$$

$$J_2 \equiv \int_0^{2\pi}\int_0^{\pi} \frac{\cos\theta\sin\theta\;d\theta\;d\phi}{(1 - \mathrm{B}\cos\theta)^4} = \frac{4\pi}{(1 - \mathrm{B}^2)^3}\tfrac{4}{3}\mathrm{B},$$

$$J_3 \equiv \int_0^{2\pi}\int_0^{\pi} \frac{\cos^2\theta \sin\theta \, d\theta \, d\phi}{(1 - B\cos\theta)^5} = \frac{4\pi}{(1 - B^2)^4}\left(\tfrac{1}{3} + \tfrac{5}{3}B^2\right),$$

$$J_4 \equiv \int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta \, d\theta \, d\phi}{(1 - B\cos\theta)^5} = \frac{4\pi}{(1 - B^2)^4}(1 + B^2),$$

where $B \equiv V/c$. In terms of these integrals

$$\begin{aligned}\mathscr{R} &= \frac{e^2}{16\pi^2 c^3}\Big\{J_1 f^2 + \{2BJ_2 - (1 - B^2)J_3\}f_x^2 \\ &\qquad\qquad - \tfrac{1}{2}(1 - B^2)(J_4 - J_3)(f_y^2 + f_z^2)\Big\} \\ &= \frac{e^2}{6\pi c^3}\left\{\frac{f_x^2}{(1 - B^2)^3} + \frac{f_y^2 + f_z^2}{(1 - B^2)^2}\right\},\end{aligned} \tag{75-6}$$

where f_x is the component of $\mathbf{f}$ in the direction of $\mathbf{V}$, f_y and f_z being at right angles. If $\mathbf{f}'$ is the acceleration of the charge relative to the inertial system in which it is momentarily at rest, we find from (43–7) that

$$\mathscr{R} = \frac{e^2 f'^2}{6\pi c^3}. \tag{75-7}$$

Hence the rate of radiation is an invariant of the Lorentz transformation.

We see from (75–6) and (75–7) that a point charge radiates energy whenever it is accelerated. In so far as we can consider the Lorentz electron as a point charge we may conclude that it must radiate whenever it is accelerated and that the rate of radiation is independent of the structure of the electron. Certainly a Lorentz electron subject to an inverse square force of attraction cannot describe orbits of linear dimensions large compared with its radius without radiating energy. On the other hand Schott's[1] work on the rigid electron model indicates that a Lorentz electron may describe orbits of linear dimensions less than its diameter without radiating.

It should be noted that the rate of radiation (75–7) from a Lorentz electron is identical with the rate of dissipation (57–21) calculated previously from the equation of motion.

It is instructive to trace the formation of the lines of electric force in the field of an oscillating charge. For simplicity we shall confine

[1] G. A. Schott, Proc. Roy. Soc. **159**, p. 570 (1937).

ourselves to a point charge which is executing simple harmonic vibrations of amplitude a along the X axis. Then we may put

$$x = -a\cos\omega t,$$
$$v = a\omega\sin\omega t,$$

for the displacement and velocity of the charge, respectively.

On account of symmetry we lose no generality by confining our attention to lines of force in the XY plane. Consider a moving-element emitted by the charge when it is at rest at time 0 at an angle α_0 with the X axis. Moving-elements forming constituents of the same line of force which are emitted at a later time move in a direction making an angle α with the X axis, where

$$\tan\alpha = \frac{c_y}{c_x} = \frac{c_y'\sqrt{1-\beta^2}}{c_x' + v} = \frac{\sqrt{1-\beta^2}\sin\alpha_0}{\cos\alpha_0 + \beta}$$

by (43–1). We see at once that no radiation is emitted in the X direction ($\alpha_0 = 0$) and that the maximum radiation is emitted in

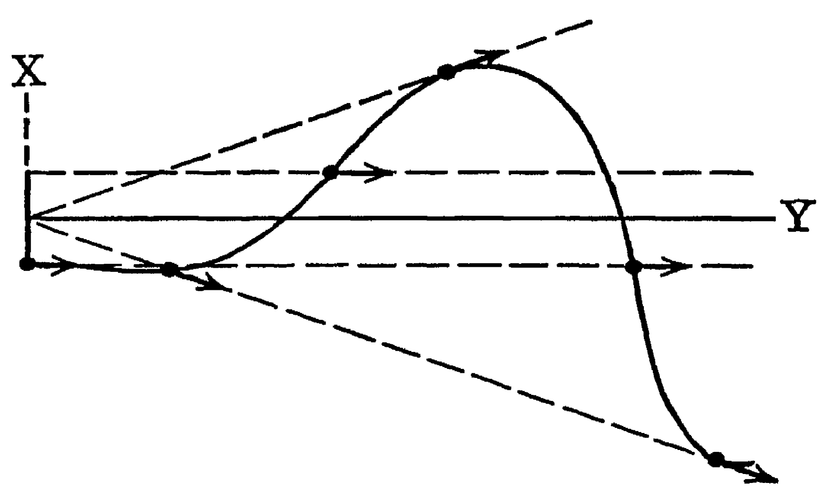

FIG. 77.

the Y direction ($\alpha_0 = \pi/2$). A line of force extending in the latter direction is plotted in Fig. 77, the broken lines representing the paths of the moving-elements of which it is the locus. Of course, this graph represents the *exact* electric field, both near to and far from the source. It should be noted that the ratio of the transverse to the longitudinal component of the field increases with the distance from the oscillating charge. In (75–1) we have ignored the latter as compared with the former.

76. Radiation Field of a Group of Point Charges.—In the last article we calculated the energy radiated by an isolated point charge. If a number of point charges are near together, the resultant radiation

field due to the group has an electric intensity given by the vector sum of as many expressions of the form (75–1) as there are charges. Since, however, the effective positions of the various charges are reached at different times, it is necessary, in order to calculate the radiation from the group, to refer the configuration of effective positions to the actual configuration at a specified time. In other words, we must take account of the interference with one another of the electromagnetic waves proceeding from the individual charges. To do this, we shall take an origin O (Fig. 78) somewhere in the group of charges at time t and develop an expression for the resultant electric intensity $\mathbf{E}$ at a field-point P at a great distance r at time $t + r/c$ in terms of the position vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_s, \cdots$ of the charges relative to O at time t, and their time derivatives.

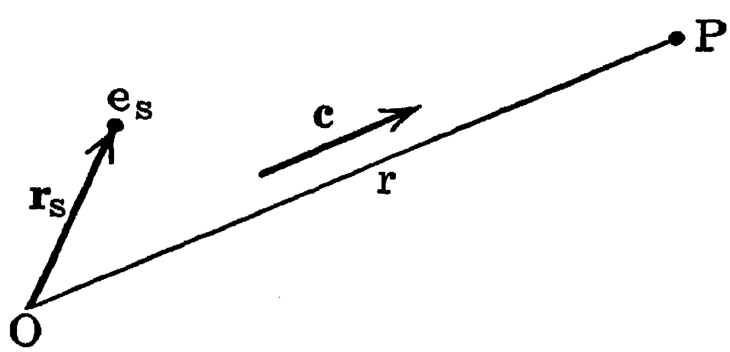

FIG. 78.

Since P is at a great distance from O the fields at P of the individual charges will all have effectively the same direction of propagation, namely that of the line $\overline{OP}$. Hence $\mathbf{c}$ for each elementary field is parallel to $\overline{OP}$. If, then, $[\mathbf{r}_s]$ is the position vector of the effective position of the charge e_s for the field-point P at time $t + r/c$, this charge occupies its effective position at the time

$$[t_s] = t + \left[\frac{\mathbf{c}\cdot\mathbf{r}_s}{c^2}\right]. \tag{76–1}$$

Since $[\mathbf{V}_s] = [d\mathbf{r}_s/dt_s]$, we get by differentiation

$$\frac{1}{\left[1 - \dfrac{\mathbf{c}\cdot\mathbf{V}_s}{c^2}\right]}\frac{d}{d[t_s]} = \frac{d}{dt}. \tag{76–2}$$

Now, since

$$\begin{aligned}\{\mathbf{f}\times(\mathbf{c}-\mathbf{V})\}\times\mathbf{c} &= \mathbf{f}\cdot\mathbf{c}(\mathbf{c}-\mathbf{V}) - c^2\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)\mathbf{f}\\ &= \left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)(\mathbf{f}\cdot\mathbf{cc} - c^2\mathbf{f}) + \frac{\mathbf{f}\cdot\mathbf{c}}{c^2}(\mathbf{V}\cdot\mathbf{cc} - c^2\mathbf{V})\\ &= \left\{\left\{\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)\mathbf{f} + \frac{\mathbf{f}\cdot\mathbf{c}}{c^2}\mathbf{V}\right\}\times\mathbf{c}\right\}\times\mathbf{c},\end{aligned}$$

we can write the electric intensity (75–1) in the radiation field of the charge e_s in the form

$$E_s = \frac{e_s}{4\pi rc^4}\left\{\left[\frac{\mathbf{f}_s}{\left(1-\frac{\mathbf{c}\cdot\mathbf{V}_s}{c^2}\right)^2} + \frac{\mathbf{f}_s\cdot\mathbf{c}}{c^2}\frac{\mathbf{V}_s}{\left(1-\frac{\mathbf{c}\cdot\mathbf{V}_s}{c^2}\right)^3}\right]\times\mathbf{c}\right\}\times\mathbf{c}. \quad (76\text{–}3)$$

In this expression we have replaced the distance of P from e_s in the coefficient by the distance r of P from O, since the ratio of the one to the other approaches unity as r increases without limit.

Combining the two terms in (76–3) and using (76–2),

$$\begin{aligned}\mathbf{E}_s &= \frac{e_s}{4\pi rc^4}\left\{\left[\frac{1}{\left(1-\frac{\mathbf{c}\cdot\mathbf{V}_s}{c^2}\right)}\frac{d}{dt_s}\left\{\frac{\mathbf{V}_s}{\left(1-\frac{\mathbf{c}\cdot\mathbf{V}_s}{c^2}\right)}\right\}\right]\times\mathbf{c}\right\}\times\mathbf{c}\\
&= \frac{e_s}{4\pi rc^4}\left\{\left\{\frac{d}{dt}\left[\frac{\mathbf{V}_s}{\left(1-\frac{\mathbf{c}\cdot\mathbf{V}_s}{c^2}\right)}\right]\right\}\times\mathbf{c}\right\}\times\mathbf{c}\\
&= \frac{e_s}{4\pi rc^4}\left\{\left\{\frac{d}{dt}\left[\frac{1}{\left(1-\frac{\mathbf{c}\cdot\mathbf{V}_s}{c^2}\right)}\frac{d\mathbf{r}_s}{dt_s}\right]\right\}\times\mathbf{c}\right\}\times\mathbf{c}\\
&= \frac{e_s}{4\pi rc^4}\left\{\frac{d^2[\mathbf{r}_s]}{dt^2}\times\mathbf{c}\right\}\times\mathbf{c}. \qquad (76\text{–}4)\end{aligned}$$

We can find a simultaneous series in powers of $1/c$ for $[\mathbf{r}_s]$ from Lagrange's expansion (55–9). Comparing (76–1) with (55–1) we see that $\alpha = 1/c, f([t]) = [\mathbf{c}\cdot\mathbf{r}_s/c]$ for the present case. Hence

$$[\mathbf{r}_s] = \mathbf{r}_s + \frac{1}{1!c}\frac{\mathbf{c}\cdot\mathbf{r}_s}{c}\mathbf{V}_s + \frac{1}{2!c^2}\frac{d}{dt}\left\{\left(\frac{\mathbf{c}\cdot\mathbf{r}_s}{c}\right)^2\mathbf{V}_s\right\} + \cdots,$$

and the resultant electric intensity $\mathbf{E} = \sum_s \mathbf{E}_s$ in the radiation field is

$$\begin{aligned}\mathbf{E} = \frac{1}{4\pi rc^4}\Bigg(\Bigg\{&\frac{d}{dt}\left(\sum_s e_s\mathbf{V}_s\right) + \frac{1}{1!c}\frac{d^2}{dt^2}\left(\sum_s\frac{\mathbf{c}\cdot\mathbf{r}_s}{c}e_s\mathbf{V}_s\right)\\
&+\frac{1}{2!c^2}\frac{d^3}{dt^3}\left(\sum_s\left(\frac{\mathbf{c}\cdot\mathbf{r}_s}{c}\right)^2 e_s\mathbf{V}_s\right) + \cdots\\
&+\frac{1}{(n-1)!c^{n-1}}\frac{d^n}{dt^n}\left(\sum_s\left(\frac{\mathbf{c}\cdot\mathbf{r}_s}{c}\right)^{n-1}e_s\mathbf{V}_s\right)+\cdots\Bigg\}\times\mathbf{c}\Bigg)\times\mathbf{c}. \quad (76\text{–}5)\end{aligned}$$

The successive terms in this series will be referred to as the first, second, third, $\cdots$ nth $\cdots$ order radiation fields. The Poynting flux is obtained immediately from (76–5) by means of (75–3). Since the Poynting flux is proportional to E^2, the total radiation from the group of particles can vanish only when $\mathbf{E} = 0$ for every direction of $\mathbf{c}$.

The first order term in the electric intensity in the radiation field of an individual charge is proportional to its acceleration. This term vanishes only when $\sum_s e_s\mathbf{f}_s = 0$ or $\sum_s e_s\mathbf{V}_s =$ Constant. In fact, if the charges remain in a group of small dimensions and do not become infinitely separated as t becomes infinite, the nth order term in (76–5) vanishes everywhere on the sphere of radius r only when

$$\sum_s \left(\frac{\mathbf{c}\cdot\mathbf{r}_s}{c}\right)^{n-1} e_s\mathbf{V}_s = \text{Constant}$$

for all directions of $\mathbf{c}$. If $\mathbf{c} = c(\mathbf{i}l + \mathbf{j}m + \mathbf{k}n)$ and $\mathbf{r}_s = \mathbf{i}x_s + \mathbf{j}y_s + \mathbf{k}z_s$, this means that

$$\left.\begin{aligned} \sum_s e_s(lx_s + my_s + nz_s)^{n-1}\dot{x}_s &= \text{Constant},\\ \sum_s e_s(lx_s + my_s + nz_s)^{n-1}\dot{y}_s &= \text{Constant},\\ \sum_s e_s(lx_s + my_s + nz_s)^{n-1}\dot{z}_s &= \text{Constant},\end{aligned}\right\} \qquad (76\text{–}6)$$

for all l, m, n. Therefore $\frac{3}{2}n(n+1)$ scalar relations involving the coordinates and velocity components of the charged particles must be satisfied in order that the nth order radiation field shall vanish.

Consider two Lorentz electrons at opposite ends of a diameter, revolving with constant angular velocity ω in a fixed circular orbit of radius a very large compared with the linear dimensions of an electron. Evidently the first order term in (76–5) vanishes and therefore the main part of the Poynting flux is due to the second order term. We shall calculate the second order radiation specified by this term.

Take the XY plane as that of the orbit with origin at the center. Then the angle which the diameter connecting the two electrons makes with the X axis may be denoted by ωt. If we put $\mathbf{c} = c\{\mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta\}$,

$$\begin{aligned}\sum_s \frac{\mathbf{c}\cdot\mathbf{r}_s}{c} e_s\mathbf{V}_s &= 2e\omega a^2 \sin\theta\,(\cos\phi\cos\omega t \\ &\qquad + \sin\phi\sin\omega t)(-\mathbf{i}\sin\omega t + \mathbf{j}\cos\omega t)\\ &= e\omega a^2\sin\theta\,\{\mathbf{i}(\sin(\phi - 2\omega t) - \sin\phi)\\ &\qquad + \mathbf{j}(\cos(\phi - 2\omega t) + \cos\phi)\}\end{aligned}$$

and

$$\mathbf{J} \equiv \frac{d^2}{dt^2}\left(\sum_s \frac{\mathbf{c}\cdot\mathbf{r}_s}{c} e_s\mathbf{V}_s\right) = -4e\omega^3a^2 \sin\theta \left\{\mathbf{i} \sin(\phi - 2\omega t) + \mathbf{j}\cos(\phi - 2\omega t)\right\}.$$

Now the Poynting flux is

$$\mathbf{s} = E^2\mathbf{c} = \frac{1}{16\pi^2r^2c^8}(c^2J^2 - \overline{\mathbf{c}\cdot\mathbf{J}}^2)\mathbf{c}$$

$$= \frac{e^2f^2V^2}{\pi^2r^2c^6}\left\{\sin^2\theta - \sin^4\theta\sin^2 2(\phi - \omega t)\right\}\mathbf{c}$$

and the total rate of radiation from the pair of electrons is

$$\mathscr{R} = \int_0^{2\pi}\int_0^{\pi} sr^2 \sin\theta\, d\theta\, d\phi = \frac{8e^2f^2}{5\pi c^3}\left(\frac{V^2}{c^2}\right). \tag{76-7}$$

Comparing with (75–7) we note that the second order radiation from a pair of electrons may not be negligible compared with the first order radiation from a single electron if V^2 is comparable with c^2.

77. Radiation Field of the Spinning Electron. — In this article we shall investigate the radiation field of the spinning electron with

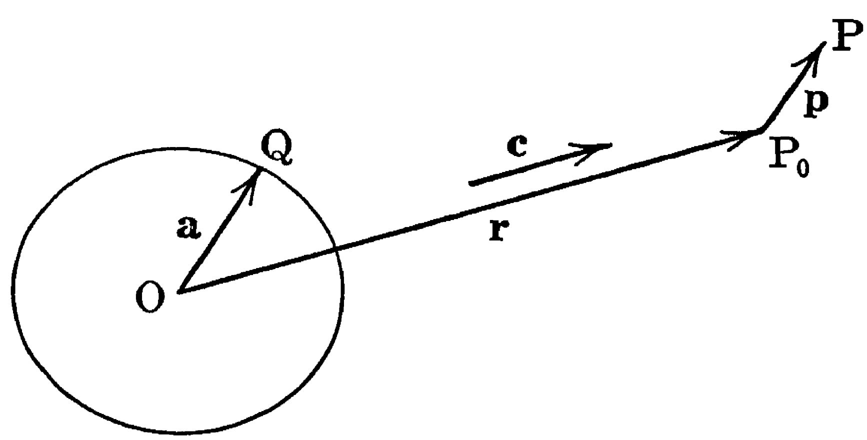

FIG. 79.

electric moment $\mathbf{p}_E$ and magnetic moment $\mathbf{p}_H$, the kinetic reaction of which was computed in article 59. Specifically we shall consider an electron whose center is momentarily at rest at the origin O at time 0 and has an acceleration $\mathbf{f}$ and rate of change of acceleration $\dot{\mathbf{f}}$, and we shall calculate the electric and magnetic intensities of its field at a point P_0 (Fig. 79) at a very great distance r from O at a time r/c. To do this we shall use the retarded expressions (50–9) and (50–10) for the scalar and vector potentials. In our expansions we shall retain all terms which are linear in $\mathbf{f}$ and $\dot{\mathbf{f}}$ and in which the coefficients are

constants or linear in one of the moments. Also we shall include terms which are linear in the moments and their first and second time derivatives and in which the coefficients are constants.

Let **a** be the position vector of a point Q on the surface of the electron at time 0. Then the effective position of the charge at Q for calculation of the field at P at a time $r/c + t$, where t is small, is occupied at the time

$$[t] = t + \left[\frac{\mathbf{c}\cdot\mathbf{a}}{c^2}\right] - \frac{\mathbf{c}\cdot\mathbf{p}}{c^2}, \tag{77-1}$$

where **p** is the position vector of the point P relative to the nearby point P_0, and $[\mathbf{a}]$ is the position vector of the effective position of Q relative to O. Since

$$[\mathbf{a}] = \mathbf{a} + \tfrac{1}{2}\mathbf{f}[t]^2 + \tfrac{1}{6}\dot{\mathbf{f}}[t]^3 + \cdots$$

we find

$$[t] = t + \frac{\mathbf{c}\cdot\mathbf{a}}{c^2} - \frac{\mathbf{c}\cdot\mathbf{p}}{c^2} + \frac{1}{2}\frac{\mathbf{f}\cdot\mathbf{c}}{c^2}\left(t + \frac{\mathbf{c}\cdot\mathbf{a}}{c^2} - \frac{\mathbf{c}\cdot\mathbf{p}}{c^2}\right)^2$$
$$+ \frac{1}{6}\frac{\dot{\mathbf{f}}\cdot\mathbf{c}}{c^2}\left(t + \frac{\mathbf{c}\cdot\mathbf{a}}{c^2} - \frac{\mathbf{c}\cdot\mathbf{p}}{c^2}\right)^3 + \cdots. \tag{77-2}$$

If x, y, z are the coordinates of P and x_0, y_0, z_0 those of P_0, $\mathbf{c}\cdot\mathbf{p} = c_x(x - x_0) + c_y(y - y_0) + c_z(z - z_0)$ and $\nabla(\mathbf{c}\cdot\mathbf{p}) = \mathbf{c}$. Hence for $t = 0$ and $\mathbf{p} = 0$,

$$\nabla[t] = -\frac{\mathbf{c}}{c^2}\left\{1 + \frac{\mathbf{f}\cdot\mathbf{c}}{c^2}\frac{\mathbf{c}\cdot\mathbf{a}}{c^2} + \frac{1}{2}\frac{\dot{\mathbf{f}}\cdot\mathbf{c}}{c^2}\left(\frac{\mathbf{c}\cdot\mathbf{a}}{c^2}\right)^2\right\}, \tag{77-3}$$

$$\frac{\partial[t]}{\partial t} = 1 + \frac{\mathbf{f}\cdot\mathbf{c}}{c^2}\frac{\mathbf{c}\cdot\mathbf{a}}{c^2} + \frac{1}{2}\frac{\dot{\mathbf{f}}\cdot\mathbf{c}}{c^2}\left(\frac{\mathbf{c}\cdot\mathbf{a}}{c^2}\right)^2. \tag{77-4}$$

As the velocity of the center of the electron in its effective position is

$$[\mathbf{v}] = \mathbf{f}[t] + \tfrac{1}{2}\dot{\mathbf{f}}[t]^2 + \cdots,$$

$$\frac{1}{\left[1 - \dfrac{\mathbf{c}\cdot\mathbf{v}}{c^2}\right]} = 1 + \frac{\mathbf{f}\cdot\mathbf{c}}{c^2}[t] + \frac{1}{2}\frac{\dot{\mathbf{f}}\cdot\mathbf{c}}{c^2}[t]^2 + \cdots \tag{77-5}$$

to our degree of approximation.

The expressions (59–4) and (59–5) for the charge and current elements due to spin must be carried to higher order terms. As in

article 59 we find from the equation of continuity that the time rate of change of angular velocity of a point Q on the surface of the electron is

$$\dot{\boldsymbol{\omega}}_Q = \dot{\boldsymbol{\omega}}\left(1 - 2\frac{\mathbf{f}\cdot\mathbf{a}}{c^2}\right) - \boldsymbol{\omega}\frac{\dot{\mathbf{f}}\cdot\mathbf{a}}{c^2}$$

in terms of the mean angular velocity $\boldsymbol{\omega}$ and the mean rate of change of angular velocity $\dot{\boldsymbol{\omega}}$. Hence, as we are neglecting terms in $\mathbf{f}\cdot\mathbf{a}\dot{\boldsymbol{\omega}}$,

$$[de_1] = \frac{e}{4\pi}\left\{1 + \frac{\boldsymbol{\omega}\times\mathbf{a}\cdot\mathbf{f}}{c^2}[t] + \frac{1}{2}\frac{\boldsymbol{\omega}\times\mathbf{a}\cdot\dot{\mathbf{f}}}{c^2}[t]^2 + \cdots\right\}d\Omega, \qquad (77\text{–}6)$$

$$[\mathbf{V}_1 de_1] = \frac{e}{4\pi}\left\{\boldsymbol{\omega}\times\mathbf{a}\left(1 - \frac{\mathbf{f}\cdot\mathbf{a}}{c^2}\right) + \left(\dot{\boldsymbol{\omega}}\times\mathbf{a} - \boldsymbol{\omega}\times\mathbf{a}\frac{\dot{\mathbf{f}}\cdot\mathbf{a}}{c^2}\right)[t]\right.$$
$$\left. + \tfrac{1}{2}\ddot{\boldsymbol{\omega}}\times\mathbf{a}[t]^2 + \mathbf{f}[t] + \tfrac{1}{2}\dot{\mathbf{f}}[t]^2 + \cdots\right\}d\Omega. \qquad (77\text{–}7)$$

Also, adding needed higher order terms to (59–6) and (59–7), we have for the charge and current elements due to the electric moment,

$$[de_2] = \frac{3}{4\pi a^2}\{\mathbf{p}_E\cdot\mathbf{a} + \dot{\mathbf{p}}_E\cdot\mathbf{a}[t] + \tfrac{1}{2}\ddot{\mathbf{p}}_E\cdot\mathbf{a}[t]^2 + \cdots\}d\Omega, \qquad (77\text{–}8)$$

$$[\mathbf{V}_2 de_2] = \frac{3}{4\pi a^2}\{\tfrac{1}{2}(\mathbf{a}\times\dot{\mathbf{p}}_E)\times\mathbf{a} + \tfrac{1}{2}(\mathbf{a}\times\ddot{\mathbf{p}}_E)\times\mathbf{a}[t] + \mathbf{p}_E\cdot\mathbf{a}\mathbf{f}[t]$$
$$+ \tfrac{1}{2}\mathbf{p}_E\cdot\mathbf{a}\dot{\mathbf{f}}[t]^2 + \cdots\}d\Omega. \qquad (77\text{–}9)$$

Substituting in (59–8) and (59–9), remembering that $[r] = r$,

$$\Phi = \frac{e}{16\pi^2 r}\int\left\{1 + \frac{\mathbf{f}\cdot\mathbf{c}}{c^2}[t] + \frac{1}{2}\frac{\dot{\mathbf{f}}\cdot\mathbf{c}}{c^2}[t]^2 + \frac{\boldsymbol{\omega}\times\mathbf{a}\cdot\mathbf{f}}{c^2}[t]\right.$$
$$\left. + \frac{1}{2}\frac{\boldsymbol{\omega}\times\mathbf{a}\cdot\dot{\mathbf{f}}}{c^2}[t]^2 + \cdots\right\}d\Omega$$

$$+ \frac{3}{16\pi^2 a^2 r}\int\left\{\mathbf{p}_E\cdot\mathbf{a} + \frac{\mathbf{p}_E\cdot\mathbf{a}\,\mathbf{f}\cdot\mathbf{c}}{c^2}[t] + \frac{1}{2}\frac{\mathbf{p}_E\cdot\mathbf{a}\,\dot{\mathbf{f}}\cdot\mathbf{c}}{c^2}[t]^2 + \dot{\mathbf{p}}_E\cdot\mathbf{a}[t]\right.$$
$$\left. + \tfrac{1}{2}\ddot{\mathbf{p}}_E\cdot\mathbf{a}[t]^2 + \cdots\right\}d\Omega, \qquad (77\text{–}10)$$

and

$$\mathbf{A} = \frac{e}{16\pi^2 rc}\int\left\{\boldsymbol{\omega}\times\mathbf{a}\left(1-\frac{\mathbf{f}\cdot\mathbf{a}}{c^2}\right)+\frac{\boldsymbol{\omega}\times\mathbf{a}\,\mathbf{f}\cdot\mathbf{c}}{c^2}[t]+\frac{1}{2}\frac{\boldsymbol{\omega}\times\mathbf{a}\,\dot{\mathbf{f}}\cdot\mathbf{c}}{c^2}[t]^2\right.$$

$$\left.+\dot{\boldsymbol{\omega}}\times\mathbf{a}[t]-\frac{\boldsymbol{\omega}\times\mathbf{a}\,\dot{\mathbf{f}}\cdot\mathbf{a}}{c^2}[t]+\tfrac{1}{2}\ddot{\boldsymbol{\omega}}\times\mathbf{a}[t]^2+\mathbf{f}[t]+\tfrac{1}{2}\dot{\mathbf{f}}[t]^2+\cdots\right\}d\Omega$$

$$+\frac{3}{16\pi^2 a^2 rc}\int\{\tfrac{1}{2}(\mathbf{a}\times\dot{\mathbf{p}}_E)\times\mathbf{a}+\tfrac{1}{2}(\mathbf{a}\times\ddot{\mathbf{p}}_E)\times\mathbf{a}[t]+\mathbf{p}_E\cdot\mathbf{a}\mathbf{f}[t]$$

$$+\tfrac{1}{2}\mathbf{p}_E\cdot\mathbf{a}\dot{\mathbf{f}}[t]^2+\cdots\}d\Omega. \quad (77\text{–}11)$$

To find the component $-\nabla\Phi$ of the electric intensity at P_0 we make use of (77–3) and, to find the component $-\dfrac{1}{c}\dfrac{\partial\mathbf{A}}{\partial t}$, of (77–4). Omitting terms which obviously vanish on integration we get, on making $t = 0$ and $\mathbf{p} = 0$,

$$-\nabla\Phi = \frac{e}{16\pi^2 r}\frac{\mathbf{c}}{c^2}\left\{\frac{4\pi\mathbf{f}\cdot\mathbf{c}}{c^2}-\frac{1}{c^4}\int\boldsymbol{\omega}\times\dot{\mathbf{f}}\cdot\mathbf{a}\,\mathbf{c}\cdot\mathbf{a}d\Omega\right\}$$

$$+\frac{3}{16\pi^2 a^2 r}\frac{\mathbf{c}}{c^2}\left\{\frac{\dot{\mathbf{f}}\cdot\mathbf{c}}{c^4}\int\mathbf{p}_E\cdot\mathbf{a}\,\mathbf{c}\cdot\mathbf{a}d\Omega+\frac{1}{c^2}\int\ddot{\mathbf{p}}_E\cdot\mathbf{a}\,\mathbf{c}\cdot\mathbf{a}d\Omega\right\},$$

$$-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} = -\frac{e}{16\pi^2 rc^2}\left\{4\pi\mathbf{f}+\frac{\dot{\mathbf{f}}\cdot\mathbf{c}}{c^4}\int\boldsymbol{\omega}\times\mathbf{a}\,\mathbf{c}\cdot\mathbf{a}d\Omega\right.$$

$$\left.-\frac{1}{c^2}\int\dot{\mathbf{f}}\cdot\mathbf{a}\,\boldsymbol{\omega}\times\mathbf{a}d\Omega+\frac{1}{c^2}\int\ddot{\boldsymbol{\omega}}\times\mathbf{a}\,\mathbf{c}\cdot\mathbf{a}d\Omega\right\}$$

$$-\frac{3}{16\pi^2 a^2 rc^2}\left\{\frac{\dot{\mathbf{f}}}{c^2}\int\mathbf{p}_E\cdot\mathbf{a}\,\mathbf{c}\cdot\mathbf{a}d\Omega+\tfrac{1}{2}\int(\mathbf{a}\times\ddot{\mathbf{p}}_E)\times\mathbf{a}d\Omega\right\}.$$

The integration over the solid angle is easily carried out. Expressing $\boldsymbol{\omega}$ in terms of $\mathbf{p}_H$ by (58–9), we have for the electric intensity in the radiation field of the spinning electron

$$\mathbf{E} = \frac{e}{4\pi rc^4}(\mathbf{f}\times\mathbf{c})\times\mathbf{c}+\frac{1}{4\pi rc^5}\{(\dot{\mathbf{f}}\times\mathbf{c})\times\mathbf{p}_H\}\times\mathbf{c}-\frac{1}{4\pi rc^3}\ddot{\mathbf{p}}_H\times\mathbf{c}$$

$$-\frac{1}{4\pi rc^6}(\{(\dot{\mathbf{f}}\times\mathbf{c})\times\mathbf{p}_E\}\times\mathbf{c})\times\mathbf{c}+\frac{1}{4\pi rc^4}(\ddot{\mathbf{p}}_E\times\mathbf{c})\times\mathbf{c}. \quad (77\text{–}12)$$

It should be noted that the terms in $\ddot{\mathbf{p}}_E$ and $\ddot{\mathbf{p}}_H$ agree with those found

in (72–23) and (72–26) for the case where $\kappa = \mu = 1$. To find $\mathbf{H}$ it is not necessary to compute the curl of the vector potential since we know that $\mathbf{H} = (1/c)\mathbf{c} \times \mathbf{E}$.

Finally we shall calculate the rate of radiation, omitting the terms in $\ddot{\mathbf{p}}_E$ and $\ddot{\mathbf{p}}_H$ in (77–12). As the electric intensity is of the form $\mathbf{E} = \mathbf{P} \times \mathbf{c}$, the Poynting flux is given by (75–3) with $E^2 = c^2P^2 - \overline{\mathbf{c}\cdot\mathbf{P}}^2$. Summing over all directions, the energy radiated by the spinning electron per unit time is found to be

$$\mathscr{R} = \frac{e^2f^2}{6\pi c^3} + \frac{e\mathbf{p}_H\cdot\mathbf{f}\times\dot{\mathbf{f}}}{6\pi c^4} + \frac{p_H{}^2\dot{\mathbf{f}}\cdot\dot{\mathbf{f}}}{15\pi c^5} - \frac{\overline{\mathbf{p}_H\cdot\dot{\mathbf{f}}}^2}{30\pi c^5} + \frac{p_E{}^2\dot{\mathbf{f}}\cdot\dot{\mathbf{f}}}{15\pi c^5} - \frac{\overline{\mathbf{p}_E\cdot\dot{\mathbf{f}}}^2}{30\pi c^5} \quad (77\text{–}13)$$

with neglect of terms in the time rates of change of the moments $\mathbf{p}_E$ and $\mathbf{p}_H$.

If a spinning electron of radius a with magnetic moment (60–1) is describing a circular orbit of radius r with linear velocity V, the ratio of the second term in (77–13) to the first is of the order of magnitude of

$$\frac{a}{r}\frac{V}{c}\left(\frac{hc}{e^2}\right),$$

whereas the ratio of the third term to the first is of the order of magnitude of the square of this small quantity. Therefore the radiation from an extra-nuclear electron in an atom due to its magnetic and electric moments is quite negligible compared with that due to the acceleration of the resultant charge e, and the rate of radiation from the spinning electron is effectively the same as that given by (75–6) for the non-spinning Lorentz electron.

78. **Axially Symmetrical Waves.** — Consider a perfect conductor which possesses symmetry of revolution about an axis, such as a spheroid or a straight wire of circular cross-section. Let u and v be orthogonal curvilinear coordinates in a plane through the axis and ϕ the azimuth measured around the axis. If electrical oscillations along the axis are set up in the conductor, it is clear from symmetry that the only non-vanishing components of $\mathbf{E}$ and $\mathbf{F}$ in the field outside the conductor are E_u, E_v and F_ϕ, and that none of these is a function of ϕ. Whatever form the oscillations may take, we can always represent them as the sum of a number of simple harmonic oscillations. Therefore we lose no generality if we limit our discussion to oscillations in which the components of $\mathbf{E}$ and $\mathbf{F}$ are of the form $f(u, v)e^{-i\omega t}$. In this

event, the field equations (72–1*c*) and (72–1*d*) applicable to the region outside the conductor become respectively

$$\left.\begin{aligned} \frac{1}{a_v\rho}\frac{\partial}{\partial v}(\rho F_\phi) &= -\frac{i\omega\kappa}{c}E_u, \\ \frac{1}{a_u\rho}\frac{\partial}{\partial u}(\rho F_\phi) &= \frac{i\omega\kappa}{c}E_v, \end{aligned}\right\} \qquad (78\text{–}1)$$

and

$$\frac{1}{a_u a_v}\left\{\frac{\partial}{\partial u}(a_v E_v) - \frac{\partial}{\partial v}(a_u E_u)\right\} = \frac{i\omega\mu}{c}F_\phi, \qquad (78\text{–}2)$$

in terms of the notation of article 19, where $a_w = \rho$ is the perpendicular distance of the field-point from the axis. If we eliminate E_u and E_v from these equations, and write A for ρF_ϕ, we obtain the homogeneous linear partial differential equation of the second order

$$\frac{\partial}{\partial u}\left\{\frac{a_v}{a_u\rho}\frac{\partial A}{\partial u}\right\} + \frac{\partial}{\partial v}\left\{\frac{a_u}{a_v\rho}\frac{\partial A}{\partial v}\right\} + \frac{\omega^2}{v^2}\frac{a_u a_v}{\rho}A = 0, \qquad (78\text{–}3)$$

where $v \equiv c/\sqrt{\kappa\mu}$ as heretofore.

In terms of the solution A of (78–3)

$$\left.\begin{aligned} E_u &= i\frac{v}{\omega}\sqrt{\frac{\mu}{\kappa}}\frac{1}{a_v\rho}\frac{\partial A}{\partial v}, \\ E_v &= -i\frac{v}{\omega}\sqrt{\frac{\mu}{\kappa}}\frac{1}{a_u\rho}\frac{\partial A}{\partial u}, \\ F_\phi &= \frac{A}{\rho}, \end{aligned}\right\} \qquad (78\text{–}4)$$

from (78–1) and the definition of A. That the vector functions **E** and **F** so determined are solenoidal, as demanded by (72–1*a*) and (72–1*b*), is evident from the fact that each is proportional to the curl of the other in accord with (72–1*c*) and (72–1*d*).

The boundary conditions for free electrical oscillations are (*a*) that the tangential component of **E** vanishes at the surface of the conductor, and (*b*) that **E** and **F** both vanish at infinity. Therefore it is necessary that we should choose coordinates u and v so that one coordinate surface, such as $u(x, y, z) = \text{Const.}$, coincides with the surface of the conductor. Then $E_v = 0$ at every point on this sur-

face. From Ampère's law (62–12d) it follows that the total current i through any cross-section of the conductor is

$$i = 2\pi c \rho_0 (F_\phi)_0 = 2\pi c A_0, \tag{78–5}$$

where ρ_0 is the radius of the section and $(F_\phi)_0$ is the magnetic force at the surface.

We shall solve (78–3) both (I) for the sphere and (II) for the prolate spheroid. The limiting case of the latter as the eccentricity approaches unity is of special interest since it affords a close approximation to the straight wire antenna of finite length.

(I) *Spherical Conductor.* In spherical coordinates the square of the linear element $d\lambda$ is

$$d\lambda^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{78–6}$$

as shown in (19–14), and, therefore, if we identify r with u and θ with v, we have $a_u = 1$, $a_v = r$. Consequently, as $\rho = r \sin \theta$, (78–3) becomes

$$r^2 \left\{ \frac{\partial^2 A}{\partial r^2} + \epsilon^2 A \right\} + \sin \theta \frac{\partial}{\partial \theta} \left\{ \frac{1}{\sin \theta} \frac{\partial A}{\partial \theta} \right\} = 0, \tag{78–7}$$

if we put ϵ for ω/v. We see that we can separate the variables, obtaining a solution of the form $A = R(r)\ \Theta(\theta)$ where R and Θ satisfy the ordinary equations

$$\left.\begin{aligned} &\frac{d^2R}{dr^2} + \left(\epsilon^2 - \frac{\alpha}{r^2}\right) R = 0, \\ &\frac{d^2\Theta}{d\mu^2} + \frac{\alpha}{1 - \mu^2}\,\Theta = 0, \end{aligned}\right\} \tag{78–8}$$

in the second of which μ has been put for $\cos \theta$.

(II) *Prolate Spheroidal Conductor.* If the origin is taken at the center of the spheroidal conductor and the X axis coincides with the axis of symmetry, the appropriate coordinate surfaces for this case are the confocal prolate spheroids

$$\frac{x^2}{f^2\eta^2} + \frac{\rho^2}{f^2(\eta^2 - 1)} = 1, \quad 1 < \eta < \infty, \tag{78–9}$$

and the orthogonal confocal hyperboloids of two sheets

$$\frac{x^2}{f^2\xi^2} - \frac{\rho^2}{f^2(1 - \xi^2)} = 1\,, \qquad -1 \leqslant \xi \leqslant 1\,, \tag{78–10}$$

the common foci of the two families of surfaces lying on the X axis at distances f and $-f$ from the origin. The eccentricity of each prolate spheroid is the reciprocal of η. Hence, as η increases, the spheroids

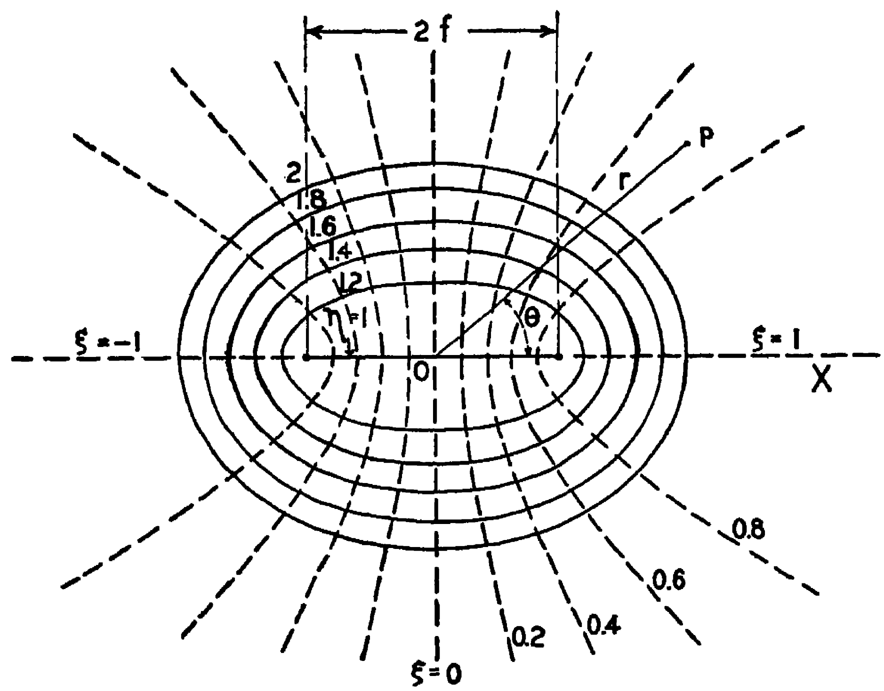

FIG. 80.

become more and more nearly spherical. A section of the coordinate surfaces cut by a plane through the X axis is shown in Fig. 80.

Solving for x and ρ,

$$x = f\xi\eta, \quad \rho = f\sqrt{(1 - \xi^2)(\eta^2 - 1)}. \tag{78–11}$$

If r is the radius vector from the origin, and θ the angle which it makes with the X axis,

$$r^2 = f^2(\eta^2 + \xi^2 - 1),$$

$$\cos\theta = \frac{x}{r} = \frac{\xi\eta}{\sqrt{\eta^2 + \xi^2 - 1}}.$$

So, far from the conductor, where η is large compared with unity, $r = f\eta$ and $\cos\theta = \xi$.

Differentiating equations (78–11) we obtain for the square of the linear element

$$d\lambda^2 = dx^2 + d\rho^2 + \rho^2 d\phi^2$$
$$= f^2\left\{\frac{\eta^2 - \xi^2}{1 - \xi^2}\, d\xi^2 + \frac{\eta^2 - \xi^2}{\eta^2 - 1}\, d\eta^2 + (1 - \xi^2)(\eta^2 - 1)d\phi^2\right\}. \tag{78–12}$$

As the unit vectors $\boldsymbol{\xi}_1$, $\boldsymbol{\eta}_1$, $\boldsymbol{\phi}_1$, in the directions of increasing ξ, η, ϕ respectively, form a right-handed set in the order stated, we identify ξ with u and η with v. Hence $a_u = f\sqrt{(\eta^2 - \xi^2)/(1 - \xi^2)}$, $a_v = f\sqrt{(\eta^2 - \xi^2)/(\eta^2 - 1)}$ and (78–3) becomes

$$(1 - \xi^2)\frac{\partial^2 A}{\partial \xi^2} - \gamma^2\xi^2 A + (\eta^2 - 1)\frac{\partial^2 A}{\partial \eta^2} + \gamma^2\eta^2 A = 0, \quad (78\text{–}13)$$

where $\gamma \equiv (\omega/v)f$. As the variables can be separated, there is a solution of the form $A = X(\xi)Y(\eta)$ where X and Y satisfy the ordinary differential equations

$$\left.\begin{aligned} (1 - \xi^2)\frac{d^2X}{d\xi^2} + (\alpha - \gamma^2\xi^2)X = 0, \\ (\eta^2 - 1)\frac{d^2Y}{d\eta^2} + (-\alpha + \gamma^2\eta^2)Y = 0. \end{aligned}\right\} \quad (78\text{–}14)$$

If we put $y \equiv \gamma\eta$, then, for large values of η, $y = (\omega/v)r$. In terms of y, the second equation above becomes

$$(y^2 - \gamma^2)\frac{d^2Y}{dy^2} + (-\alpha + y^2)Y = 0. \quad (78\text{–}15)$$

If, now, we go over to the case of the sphere by letting the semi-focal separation f, and therefore γ, approach zero, (78–15) and the first of the pair (78–14) reduce to the two equations (78–8), as they should.

79. Oscillations of Spherical Conductor. — In this article we shall consider first the axially symmetrical radiation field due to the free oscillations of a perfectly conducting sphere, and second the forced oscillations and consequent radiation from such a sphere when simple harmonic plane waves of length long compared with the diameter of the sphere pass over it. In connection with the second problem we shall also compute the radiation resistance of the sphere.

The first step is to solve the differential equations (78–8). Starting with the equation in $\mu = \cos\theta$ we make the substitution $\Theta(\mu) = \sqrt{1 - \mu^2}\, u(\mu)$ which gives for u the equation

$$\frac{d}{d\mu}\left\{(1 - \mu^2)\frac{du}{d\mu}\right\} - \frac{u}{1 - \mu^2} + \alpha u = 0.$$

The only solutions of this equation which have physical significance in the problem under consideration are those which remain finite over

the whole range $-1 \leqslant \mu \leqslant 1$. As is shown in treatises on spherical harmonics such solutions exist only when the constant of separation α is equal to $l(l+1)$, where l is a positive integer equal to or greater than unity. These discrete values of α are known as the *characteristic values* of the parameter, and to them correspond the various harmonic oscillations of which the sphere is capable, the fundamental or first harmonic being given by $l = 1$, the second harmonic by $l = 2$, and so on. The differential equation for u which we have to consider is, then,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\} - \frac{u}{1-\mu^2} + l(l+1)u = 0, \quad l = 1, 2, 3 \cdots. \qquad (79\text{–}1)$$

This equation is a special case of the more general equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\} - \frac{m^2 u}{1-\mu^2} + l(l+1)u = 0, \qquad (79\text{–}2)$$

where m is an integer equal to or less than l. The solutions of (79–2) are the associated Legendrian functions

$$u = P_{lm}(\mu) = (1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_l(\mu), \qquad (79\text{–}3)$$

where

$$P_l(\mu) = (-1)^l \frac{1}{2^l l!}\frac{d^l}{d\mu^l}(1-\mu^2)^l \qquad (79\text{–}4)$$

are the Legendrian polynomials of degree l. The solutions of (79–1) are therefore

$$P_{l1}(\mu) = (-1)^l \frac{1}{2^l l!}\sqrt{1-\mu^2}\,\frac{d^{l+1}}{d\mu^{l+1}}(1-\mu^2)^l. \qquad (79\text{–}5)$$

The first three of these functions are

$$\left.\begin{aligned} P_{11} &= \sqrt{1-\mu^2}, \\ P_{21} &= 3\mu\sqrt{1-\mu^2}, \\ P_{31} &= \tfrac{3}{2}(5\mu^2-1)\sqrt{1-\mu^2}. \end{aligned}\right\} \qquad (79\text{–}6)$$

Now we are ready to consider the first of the pair of equations (78–8). Replacing α by $l(l+1)$ and putting $R(r) = \sqrt{r}\,v(r)$ we find that v satisfies the equation

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \left\{\epsilon^2 - \frac{(l+\frac{1}{2})^2}{r^2}\right\}v = 0, \qquad (79\text{–}7)$$

and, if we put $x \equiv \epsilon r$, this equation becomes

$$\frac{d^2 v}{dx^2} + \frac{1}{x}\frac{dv}{dx} + \left\{1 - \frac{(l + \frac{1}{2})^2}{x^2}\right\} v = 0, \tag{79-8}$$

which is Bessel's equation (73–22). The complete solution is

$$v = C_n J_n(x) + C_{-n} J_{-n}(x), \tag{79-9}$$

where n is the half-integer $l + \frac{1}{2}$. The first three pairs of Bessel's functions of half-integral order, starting with $n = \pm \frac{3}{2}$, are

$$J_{3/2} = \frac{2}{\sqrt{\pi x}}\left\{\frac{\sin x}{x} - \cos x\right\},$$

$$J_{-3/2} = \frac{2}{\sqrt{\pi x}}\left\{-\frac{\cos x}{x} - \sin x\right\};$$

$$J_{5/2} = \frac{2}{\sqrt{\pi x}}\left\{\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right\},$$

$$J_{-5/2} = \frac{2}{\sqrt{\pi x}}\left\{\left(\frac{3}{x^2} - 1\right)\cos x + \frac{3}{x}\sin x\right\};$$

$$J_{7/2} = \frac{2}{\sqrt{\pi x}}\left\{\left(\frac{15}{x^3} - \frac{6}{x}\right)\sin x - \left(\frac{15}{x^2} - 1\right)\cos x\right\},$$

$$J_{-7/2} = \frac{2}{\sqrt{\pi x}}\left\{-\left(\frac{15}{x^3} - \frac{6}{x}\right)\cos x - \left(\frac{15}{x^2} - 1\right)\sin x\right\}.$$

Since we are interested in solutions representing diverging waves and hence containing the factor e^{ix}, we need solutions of the form $C\{J_n + i(-1)^{n+1/2}J_{-n}\}$, that is,

$$\left.\begin{aligned} v_1 &= \frac{1}{\sqrt{x}}\left\{1 + \frac{i}{x}\right\} e^{ix}, \\ v_2 &= \frac{1}{\sqrt{x}}\left\{1 + \frac{3i}{x} - \frac{3}{x^2}\right\} e^{ix}, \\ v_3 &= \frac{1}{\sqrt{x}}\left\{1 + \frac{6i}{x} - \frac{15}{x^2} - \frac{15i}{x^3}\right\} e^{ix}, \end{aligned}\right\} \tag{79-10}$$

where the subscript on v represents the integer l.

Consequently the *characteristic solutions* of (78–7) corresponding to the first three characteristic values (l = 1, 2, 3) are

$$A_1 = C_1 \sin^2\theta \left\{1 + \frac{i}{\epsilon r}\right\} e^{i(\epsilon r - \omega t)}, \tag{79–11}$$

$$A_2 = C_2 \sin^2\theta \cos\theta \left\{1 + \frac{3i}{\epsilon r} - \frac{3}{\epsilon^2 r^2}\right\} e^{i(\epsilon r - \omega t)}, \tag{79–12}$$

$$A_3 = C_3 \sin^2\theta\,(5\cos^2\theta - 1)\left\{1 + \frac{6i}{\epsilon r} - \frac{15}{\epsilon^2 r^2} - \frac{15i}{\epsilon^3 r^3}\right\} e^{i(\epsilon r - \omega t)}. \tag{79–13}$$

The boundary condition at the surface of the sphere requires that E_θ, and consequently $\dfrac{\partial A}{\partial r}$, shall vanish for all values of θ when r is equal to the radius a of the sphere. This condition determines the possible values of ϵ, and therefore of the wave-length λ and logarithmic decrement δ of the free oscillations. In fact, since $\epsilon = \omega/v$ must be complex for damped oscillations, we have

$$\epsilon = \frac{\omega}{v} = -\,i\frac{\delta}{\lambda} \pm \frac{2\pi}{\lambda}, \tag{79–14}$$

where the sign in front of the second term must be chosen so as to make λ positive.

Let us consider first the fundamental represented by (79–11). The boundary condition $\epsilon^2 a^2 + i\epsilon a - 1 = 0$ gives

$$\epsilon = -\frac{i}{2a} \pm \frac{\sqrt{3}}{2a} = -\,i\frac{0.500}{a} \pm \frac{0.866}{a}. \tag{79–15}$$

The wave-length is $4\pi a/\sqrt{3}$ and the damping constant (in time) is $v/2a$. We shall direct our attention rather to the ratio of the half wave-length to the diameter of the sphere and to the logarithmic decrement δ. These dimensionless constants are:

$$\frac{\lambda}{4a} = \frac{\pi}{\sqrt{3}} = 1.81, \quad \delta = \frac{2\pi}{\sqrt{3}} = 3.62. \tag{79–16}$$

The waves are damped out very rapidly, the second oscillation having an amplitude less than 3% that of the first.

The quantity ρF_ϕ is represented by the real part of the complex function (79–11), that is, by

$$A_1 = C_1 \sin^2\theta\, e^{\frac{r-vt}{2a}} \left\{ \left(1 - \frac{a}{2r}\right) \cos\frac{\sqrt{3}}{2a}(r - vt) - \frac{\sqrt{3}a}{2r} \sin\frac{\sqrt{3}}{2a}(r - vt) \right\}. \quad (79\text{–}17)$$

The current through any cross-section of the sphere, given by (78–5), is

$$i = 2\pi c C_1 \sqrt{\epsilon}\, \sin^2\theta\, e^{-\frac{v}{2a}t} \cos\left\{ \frac{\sqrt{3}}{2a} vt - \frac{\sqrt{3}}{2} - \frac{\pi}{3} \right\},$$

or, if we change our origin of time by putting $t = t' + \frac{a}{v}\left(1 + \frac{2\pi}{3\sqrt{3}}\right)$,

$$i = i_{00} \sin^2\theta\, e^{-\frac{v}{2a}t'} \cos\frac{\sqrt{3}}{2a} vt', \quad (79\text{–}18)$$

where i_{00} is the current through the equatorial plane at time $t' = 0$.

The field intensities E_r, E_θ and F_ϕ, obtained from (79–11) by means of (78–4), are

$$\left.\begin{aligned}
E_r &= C_1 \sqrt{\frac{\mu}{\kappa}} \cos\theta\, e^{\frac{r-vt}{2a}} \left\{ -\frac{a}{r^2}\left(1 + \frac{a}{r}\right) \cos\frac{\sqrt{3}}{2a}(r - vt) - \frac{\sqrt{3}a}{r^2}\left(1 - \frac{a}{r}\right) \sin\frac{\sqrt{3}}{2a}(r - vt) \right\}, \\
E_\theta &= C_1 \sqrt{\frac{\mu}{\kappa}} \sin\theta\, e^{\frac{r-vt}{2a}} \left\{ \left(\frac{1}{r} - \frac{a}{2r^2} - \frac{a^2}{2r^3}\right) \cos\frac{\sqrt{3}}{2a}(r - vt) - \frac{\sqrt{3}a}{2r^2}\left(1 - \frac{a}{r}\right) \sin\frac{\sqrt{3}}{2a}(r - vt) \right\}, \\
F_\phi &= C_1 \sin\theta\, e^{\frac{r-vt}{2a}} \left\{ \left(\frac{1}{r} - \frac{a}{2r^2}\right) \cos\frac{\sqrt{3}}{2a}(r - vt) - \frac{\sqrt{3}a}{2r^2} \sin\frac{\sqrt{3}}{2a}(r - vt) \right\}.
\end{aligned}\right\} \quad (79\text{–}19)$$

At a great distance from the sphere the components of **E** and **F** of dominant magnitude are

$$\left.\begin{aligned} E_\theta &= \frac{C_1}{r}\sqrt{\frac{\mu}{\kappa}}\sin\theta\, e^{\frac{r-vt}{2a}}\cos\frac{\sqrt{3}}{2a}(r-vt), \\ F_\phi &= \frac{C_1}{r}\sin\theta\, e^{\frac{r-vt}{2a}}\cos\frac{\sqrt{3}}{2a}(r-vt). \end{aligned}\right\} \qquad (79\text{–}20)$$

Comparison with (72–23) shows that the radiation field is that of a damped linear simple harmonic oscillator.

In the case of the second harmonic, represented by (79–12), the boundary condition at the surface of the sphere is $\epsilon^3a^3 + 3i\epsilon^2a^2 - 6\epsilon a - 6i = 0$. The complex roots of this cubic, which represent oscillations, are

$$\epsilon = -i\frac{0.702}{a} \pm \frac{1.807}{a}, \qquad (79\text{–}21)$$

giving

$$\frac{\lambda}{4a} = 0.87, \quad \delta = 2.44. \qquad (79\text{–}22)$$

This solution we shall call the *normal* harmonic. The wave-length is a little less than half that of the fundamental, and the decrement is about two-thirds as great.

The boundary condition, however, is also satisfied by

$$\epsilon = -i\frac{1.596}{a}, \qquad (79\text{–}23)$$

which represents an aperiodic motion of charge in which the heavily damped current never reverses its sense. We shall call this the *abnormal* harmonic.

The boundary condition for the third harmonic, obtained from (79–13), is $\epsilon^4a^4 + 6i\epsilon^3a^3 - 21\epsilon^2a^2 - 45i\epsilon a + 45 = 0$. This biquadratic has two pairs of complex roots. The pair representing the normal harmonic is

$$\epsilon = -i\frac{0.842}{a} \pm \frac{2.760}{a} \qquad (79\text{–}24)$$

which gives

$$\frac{\lambda}{4a} = 0.57, \quad \delta = 1.92, \qquad (79\text{–}25)$$

the wave-length being somewhat less than one-third that of the fundamental, and the decrement about half as great.

The abnormal harmonic is specified by the remaining pair of roots,

$$\epsilon = -\,i\,\frac{2.157}{a} \pm \frac{0.869}{a}, \qquad (79\text{–}26)$$

which represents a heavily damped oscillation for which

$$\frac{\lambda}{4a} = 1.81, \quad \delta = 15.60. \qquad (79\text{–}27)$$

It should be noted that the wave-length of this oscillation is nearly identical with that of the fundamental.

Evidently an aperiodic solution will appear for each of the even harmonics.

Next we shall investigate the forced oscillations of a perfectly conducting sphere when plane polarized waves of wave-length long compared with the diameter of the sphere pass over it. Under these conditions the electric intensity E_1 and magnetic force F_1 in the impinging waves can be represented to a sufficient degree of approximation by

$$\left.\begin{aligned} E_1 &= E_0 e^{-i\omega t}, \\ F_1 &= E_0 \sqrt{\frac{\kappa}{\mu}}\, e^{-i\omega t}, \end{aligned}\right\} \qquad (79\text{–}28)$$

over the small region occupied by the sphere, in accord with (72–14). Here, of course, ω is real.

If we take the polar axis of the sphere in the direction of the impressed electric field E_1, the boundary condition at the surface is

$$E_1 \sin\theta - E_\theta = 0, \quad r = a, \qquad (79\text{–}29)$$

where E_θ is the component in the direction of increasing θ of the field radiated from the sphere.

Evidently we need the solution (79–11) representing the fundamental oscillation of the sphere. From (78–4)

$$E_\theta = C \sin\theta \left\{ \frac{1}{\epsilon r} + \frac{i}{\epsilon^2 r^2} - \frac{1}{\epsilon^3 r^3} \right\} e^{\,i(\epsilon r - \omega t)}. \qquad (79\text{–}30)$$

Therefore the boundary condition requires that

$$C = -\frac{\epsilon^3 a^3}{\sqrt{1 - \epsilon^2 a^2 + \epsilon^4 a^4}} E_0\, e^{i(-\epsilon a + \psi)}, \quad \tan\psi \equiv \frac{\epsilon a}{1 - \epsilon^2 a^2}. \quad (79\text{–}31)$$

This gives for the field intensities in the wave radiated from the sphere

$$\left.\begin{aligned}
E_r &= \frac{E_0 \cos\theta}{\sqrt{1 - \epsilon^2 a^2 + \epsilon^4 a^4}} \left\{ 2\frac{a^3}{r^3} \cos\{\epsilon(r-a) - \omega t + \psi\} \right. \\
&\qquad \left. + 2\frac{\epsilon a^3}{r^2} \sin\{\epsilon(r-a) - \omega t + \psi\} \right\}, \\
E_\theta &= \frac{E_0 \sin\theta}{\sqrt{1 - \epsilon^2 a^2 + \epsilon^4 a^4}} \left\{ \left(\frac{a^3}{r^3} - \frac{\epsilon^2 a^3}{r} \right) \cos\{\epsilon(r-a) - \omega t + \psi\} \right. \\
&\qquad \left. + \frac{\epsilon a^3}{r^2} \sin\{\epsilon(r-a) - \omega t + \psi\} \right\}, \\
F_\phi &= \frac{\sqrt{\frac{\kappa}{\mu}}\, E_0 \sin\theta}{\sqrt{1 - \epsilon^2 a^2 + \epsilon^4 a^4}} \left\{ -\frac{\epsilon^2 a^3}{r} \cos\{\epsilon(r-a) - \omega t + \psi\} \right. \\
&\qquad \left. + \frac{\epsilon a^3}{r^2} \sin\{\epsilon(r-a) - \omega t + \psi\} \right\}.
\end{aligned}\right\} \quad (79\text{–}32)$$

The last gives for the current in the sphere

$$i = 2\pi a c \sqrt{\frac{\kappa}{\mu}} \frac{\epsilon a (1 + \epsilon^2 a^2)}{\sqrt{1 + \epsilon^6 a^6}} E_0 \sin^2\theta \cos(\omega t + \psi_i), \ \tan\psi_i \equiv \frac{1}{\epsilon^3 a^3} \cdot (79\text{–}33)$$

The current leads the electromotive force by the angle ψ_i.

If we put $w \equiv \epsilon^2 a^2$, the square of the current is proportional to

$$\frac{\epsilon^2 a^2 (1 + \epsilon^2 a^2)^2}{1 + \epsilon^6 a^6} = 1 + \frac{2w - 1}{w^2 - w + 1} \equiv f(w) \qquad (79\text{–}34)$$

and the ratio of the half wave-length to the diameter of the sphere is

$$\frac{\lambda}{4a} = \frac{\pi}{2\sqrt{w}}. \qquad (79\text{–}35)$$

Plotting $f(w)$ against $\lambda/4a$ we get the curve shown in Fig. 81. The function $f(w)$ increases as the wave-length decreases, reaching a

maximum for $\lambda/4a = 1.344$. We are not justified, however, in associating $f(w)$ with the square of the current for values of $\lambda/4a$ much smaller than 4, since the boundary condition assumed is valid only for wave-lengths large compared with the diameter of the sphere.

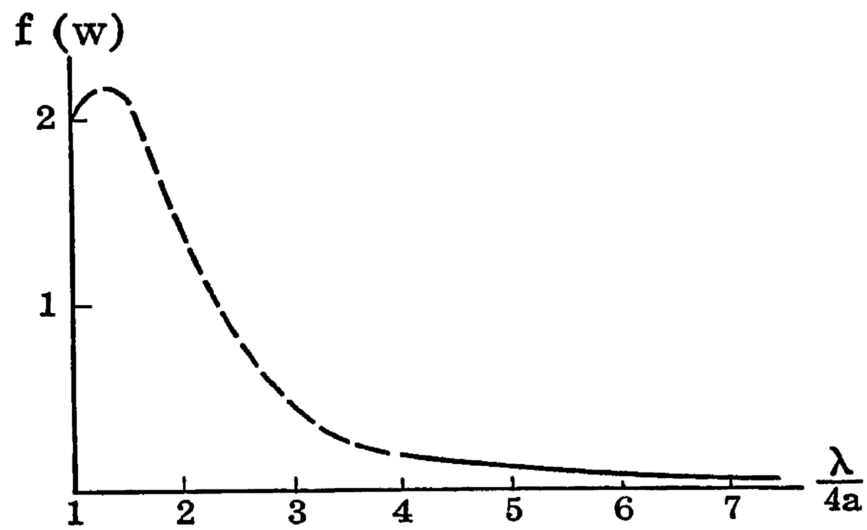

FIG. 81.

At a great distance from the sphere the only important field intensities are

$$\left.\begin{aligned} E_\theta &= -\frac{\epsilon^2 a^3}{\sqrt{1-\epsilon^2 a^2+\epsilon^4 a^4}}\frac{E_0}{r}\sin\theta\cos\left\{\epsilon(r-a)-\omega t+\psi\right\}, \\ F_\phi &= -\frac{\epsilon^2 a^3}{\sqrt{1-\epsilon^2 a^2+\epsilon^4 a^4}}\sqrt{\frac{\kappa}{\mu}}\frac{E_0}{r}\sin\theta\cos\left\{\epsilon(r-a)-\omega t+\psi\right\}. \end{aligned}\right\} \quad (79\text{–}36)$$

Comparing these expressions with (72–23) we obtain the rate of radiation of energy immediately from (72–24). It is

$$\frac{8}{3}\pi c\sqrt{\frac{\kappa}{\mu}}\frac{\epsilon^4 a^6 E_0{}^2}{1-\epsilon^2 a^2+\epsilon^4 a^4}\cos^2\left\{\epsilon(r-a)-\omega t+\psi\right\}. \quad (79\text{–}37)$$

The *radiation resistance* R of the sphere is defined as the mean rate of radiation divided by the mean square current through the equatorial section of the sphere. Hence

$$R = \frac{2}{3\pi c}\sqrt{\frac{\mu}{\kappa}}\frac{\epsilon^2 a^2}{1+\epsilon^2 a^2}. \quad (79\text{–}38)$$

The resistance increases continually with increasing frequency.

80. Free Oscillations of Prolate Spheroidal Conductor. — The prolate spheroidal conductor is of great interest because, when its

eccentricity is nearly unity, it represents to a high degree of approximation the straight wire antenna. In this article we shall solve the differential equations (78–14) for the fundamental and the third harmonic, the latter of which we need for our subsequent analysis of the forced vibrations of an antenna, and shall discuss the free oscillations which the fundamental represents.

(I). *Solution of the Equation in* ξ. We start by making the substitution $X = \sqrt{1 - \xi^2}\, u(\xi)$ in the first of the two equations (78–14). This gives for u the equation

$$\frac{d}{d\xi}\left\{(1 - \xi^2)\frac{du}{d\xi}\right\} - \frac{u}{1 - \xi^2} + \alpha u = \gamma^2\xi^2 u, \qquad (80\text{–}1)$$

the left-hand member of which is identical in form with (79–1). As in the case of the sphere the parameter α is not arbitrary, but is limited to the characteristic values $\alpha_1, \alpha_2, \alpha_3, \cdots$ for which the solution $u(\xi)$ remains finite over the whole range $-1 \leqslant \xi \leqslant 1$. Let us denote the solution corresponding to the characteristic value α_l by $u_l\,(\xi)$. The functions $u_1\,(\xi)$, $u_2\,(\xi)$, $u_3\,(\xi)$, $\cdots$ are called the *characteristic solutions* of the differential equation.

The differential equations for two distinct characteristic solutions $u_k\,(\xi)$ and $u_l\,(\xi)$ are

$$\frac{d}{d\xi}\left\{(1 - \xi^2)\frac{du_k}{d\xi}\right\} - \frac{u_k}{1 - \xi^2} + \alpha_k u_k = \gamma^2\xi^2 u_k\,,$$

$$\frac{d}{d\xi}\left\{(1 - \xi^2)\frac{du_l}{d\xi}\right\} - \frac{u_l}{1 - \xi^2} + \alpha_l u_l = \gamma^2\xi^2 u_l.$$

Multiplying the first by u_l, the second by u_k, and subtracting, we get

$$\frac{d}{d\xi}\left\{(1 - \xi^2)u_l\frac{du_k}{d\xi}\right\} - \frac{d}{d\xi}\left\{(1 - \xi^2)u_k\frac{du_l}{d\xi}\right\} = (\alpha_l - \alpha_k)u_k u_l.$$

Integrating this equation over the range of ξ we have

$$(\alpha_l - \alpha_k)\int_{-1}^{1} u_k u_l d\xi = 0,$$

from which it follows that

$$\int_{-1}^{1} u_k\, u_l d\xi = 0, \quad k \neq l. \qquad (80\text{–}2)$$

A set of functions satisfying this condition are said to be *orthogonal.* Evidently the associated Legendrian functions $P_{lm}(\mu)$, to which the equation (79–2) of the previous article leads, are orthogonal.

To find the characteristic values and the characteristic solutions of (80–1) we express α_l and $u_l(\xi)$ as power series in γ^2, writing

$$\alpha_l = \alpha_{l0} + \alpha_{l1}\gamma^2 + \alpha_{l2}\gamma^4 + \cdots,$$
$$u_l(\xi) = u_{l0}(\xi) + u_{l1}(\xi)\gamma^2 + u_{l2}(\xi)\gamma^4 + \cdots,$$

and substitute in the differential equation. Equating terms in like powers of γ^2 we have then

$$\frac{d}{d\xi}\left\{(1-\xi^2)\frac{du_{l0}}{d\xi}\right\} - \frac{u_{l0}}{1-\xi^2} + \alpha_{l0}u_{l0} = 0, \tag{80–3}$$

$$\frac{d}{d\xi}\left\{(1-\xi^2)\frac{du_{l1}}{d\xi}\right\} - \frac{u_{l1}}{1-\xi^2} + \alpha_{l0}u_{l1} = (\xi^2 - \alpha_{l1})u_{l0}, \tag{80–4}$$

$$\frac{d}{d\xi}\left\{(1-\xi^2)\frac{du_{l2}}{d\xi}\right\} - \frac{u_{l2}}{1-\xi^2} + \alpha_{l0}u_{l2} = (\xi^2 - \alpha_{l1})u_{l1} - \alpha_{l2}u_{l0}, \tag{80–5}$$

and so forth. It is evident from (80–3) that the functions $u_{10}, u_{20}, u_{30}, \cdots$ are orthogonal. In fact, as (80–3) is identical in form with (79–1), we see that $\alpha_{l0} = l(l+1)$ and $u_{l0} = P_{l1}(\xi)$.

To obtain α_{l1} we multiply (80–3) by u_{l1} and (80–4) by u_{l0}, subtract, and integrate over the range of ξ. This gives

$$\int_{-1}^{1}(\xi^2 - \alpha_{l1})u_{l0}^2 d\xi = 0, \tag{80–6}$$

from which it follows that

$$\alpha_{l1} = \frac{\displaystyle\int_{-1}^{1}\xi^2 u_{l0}^2 d\xi}{\displaystyle\int_{-1}^{1} u_{l0}^2 d\xi}. \tag{80–7}$$

Now we express both $(\xi^2 - \alpha_{l1})u_{l0}$ and u_{l1} as series in the functions $u_{10}, u_{20}, u_{30}, \cdots$, writing

$$(\xi^2 - \alpha_{l1})u_{l0} = \sum_k A_k u_{k0}, \tag{80–8}$$

$$u_{l1} = \sum_k B_k u_{k0}. \tag{80–9}$$

Since $(\xi^2 - \alpha_{l1})u_{l0}$ is orthogonal to u_{l0} by (80–6), the first series contains no term in u_{l0}.

Substituting (80–9) in the left-hand member of (80–4) and (80–8) in the right-hand member, remembering that each u_{k0} satisfies (80–3) with $\alpha_{k0} = k(k+1)$, we get

$$B_k = -\frac{A_k}{k(k+1) - l(l+1)}. \tag{80–10}$$

This determines each B_k in terms of A_k. To find A_j multiply (80–8) by u_{j0} and integrate over ξ. On account of the orthogonality of the functions $u_{10}, u_{20}, u_{30}, \cdots$ we get

$$A_j = \frac{\int_{-1}^{1} (\xi^2 - \alpha_{l1})u_{l0}u_{j0}d\xi}{\int_{-1}^{1} u_{j0}^2 d\xi}. \tag{80–11}$$

Continuing this procedure we can find as many terms as desired in the series for α_l and $u_l(\xi)$. Since $u_{l0} = P_{l1}(\xi)$ in the problem under consideration, the formula

$$\int_{-1}^{1} P_{l1}^2(\xi)d\xi = \frac{2}{2l+1}\,\frac{(l+1)!}{(l-1)!} \tag{80–12}$$

is helpful in evaluating some of the integrals.

By this process we find for the fundamental oscillation $(l = 1)$ of the prolate spheroid

$$\alpha_1 = 2 + \frac{1}{5}\gamma^2 - \frac{4}{5^3\cdot 7}\gamma^4 + \frac{8}{3\cdot 5^5\cdot 7}\gamma^6 - \frac{124}{5^6\cdot 7^3\cdot 11}\gamma^8 + \cdots, \tag{80–13}$$

$$\begin{aligned} u_1(\xi) = P_{11}(\xi) &- \frac{1}{3\cdot 5^2}P_{31}(\xi)\gamma^2 \\ &+ \left\{\frac{2}{3^2\cdot 5^4}P_{31}(\xi) + \frac{1}{3^2\cdot 5^2\cdot 7^2}P_{51}(\xi)\right\}\gamma^4 \\ &- \left\{\frac{31}{3\cdot 5^5\cdot 7^2\cdot 11}P_{31}(\xi) + \frac{4}{3\cdot 5^4\cdot 7^2\cdot 13}P_{51}(\xi)\right. \\ &\qquad \left. + \frac{1}{3^4\cdot 5\cdot 7^2\cdot 11\cdot 13}P_{71}(\xi)\right\}\gamma^6 + \cdots, \end{aligned} \tag{80–14}$$

and for the third harmonic ($l = 3$)

$$\alpha_3 = 12 + \frac{7}{3\cdot 5}\gamma^2 + \frac{152}{3^4\cdot 5^3\cdot 11}\gamma^4 + \cdots, \tag{80–15}$$

$$u_3(\xi) = P_{31}(\xi) + \left\{\frac{6}{5^2\cdot 7}P_{11}(\xi) - \frac{2}{3^3\cdot 7}P_{51}(\xi)\right\}\gamma^2 + \cdots. \tag{80–16}$$

(II). *Solution of the Equation in* η *for Large Values of the Variable.* Next we turn our attention to the second of the two equations (78–14). Since we are interested in a diverging wave system the function Y must reduce to $e^{i\gamma\eta}$ for very large values of η. Hence we look for a solution of the form $Y = v(\eta)e^{i\gamma\eta}$ where $v(\eta)$ is a series in positive powers of $1/\eta$. Substituting in the differential equation we find that

$$\frac{d^2v}{d\eta^2} + 2i\gamma\frac{dv}{d\eta} - \frac{\alpha - \gamma^2}{\eta^2 - 1}v = 0, \tag{80–17}$$

or, if we put $z \equiv i/\gamma\eta$,

$$z^2\frac{d^2v}{dz^2} + 2(z + 1)\frac{dv}{dz} - \frac{\alpha - \gamma^2}{1 + \gamma^2 z^2}v = 0. \tag{80–18}$$

Putting for α the characteristic value already found for the harmonic under consideration, we assume a solution of the form

$$v = 1 + g_1(\gamma^2)z + g_2(\gamma^2)z^2 + g_3(\gamma^2)z^3 + \cdots$$

and substitute in (80–18) after expanding the denominator of the last term in the differential equation in a power series by means of the binomial theorem. In this way we find for the fundamental ($l = 1$)

$$\begin{aligned}
Y_1(\eta) = e^{i\gamma\eta}\Big[\, &1 \\
&+ \frac{i}{\gamma}\left\{1 - \frac{2}{5}\gamma^2 - \frac{2}{5^3\cdot 7}\gamma^4 + \frac{4}{3\cdot 5^5\cdot 7}\gamma^6 - \frac{62}{5^6\cdot 7^3\cdot 11}\gamma^8 + \cdots\right\}\frac{1}{\eta} \\
&+ \frac{1}{5}\left\{1 - \frac{69}{5^2\cdot 7}\gamma^2 - \frac{62}{3\cdot 5^4\cdot 7}\gamma^4 + \frac{3943}{3\cdot 5^5\cdot 7^3\cdot 11}\gamma^6 + \cdots\right\}\frac{1}{\eta^2} \\
&+ \frac{1}{5}\frac{i}{\gamma}\left\{1 - \frac{94}{5^2\cdot 7}\gamma^2 + \frac{2014}{3^2\cdot 5^4\cdot 7}\gamma^4 + \frac{34{,}588}{3\cdot 5^5\cdot 7^3\cdot 11}\gamma^6 + \cdots\right\}\frac{1}{\eta^3} \\
&+ \frac{3}{5\cdot 7}\left\{1 - \frac{289}{3^3\cdot 5^2}\gamma^2 + \frac{75{,}914}{3^3\cdot 5^4\cdot 7^2\cdot 11}\gamma^4 + \cdots\right\}\frac{1}{\eta^4}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{3}{5\cdot 7}\frac{i}{\gamma}\left\{1 - \frac{46}{3\cdot 5^2}\gamma^2 + \frac{761{,}914}{3^3\cdot 5^4\cdot 7^2\cdot 11}\gamma^4 + \cdots\right\}\frac{1}{\eta^5} \\
&+ \frac{1}{3\cdot 7}\left\{1 - \frac{371}{3\cdot 5^2\cdot 11}\gamma^2 + \cdots\right\}\frac{1}{\eta^6} \\
&+ \frac{1}{3\cdot 7}\frac{i}{\gamma}\left\{1 - \frac{182}{5^2\cdot 11}\gamma^2 + \cdots\right\}\frac{1}{\eta^7} \\
&+ \frac{1}{3\cdot 11}\left\{1 + \cdots\right\}\frac{1}{\eta^8} \\
&+ \frac{1}{3\cdot 11}\frac{i}{\gamma}\left\{1 + \cdots\right\}\frac{1}{\eta^9} \\
&+ \cdots \Bigg], \qquad (80\text{–}19)
\end{aligned}
$$

and for the third harmonic ($l = 3$)

$$
\begin{aligned}
Y_3(\eta) = e^{i\gamma\eta}\Bigg[&1 \\
&+ 6\frac{i}{\gamma}\left\{1 - \frac{2}{3^2\cdot 5}\gamma^2 + \frac{38}{3^5\cdot 5^3\cdot 11}\gamma^4 + \cdots\right\}\frac{1}{\eta} \\
&- 15\frac{1}{\gamma^2}\left\{1 - \frac{22}{3^2\cdot 5^2}\gamma^2 + \frac{398}{3^5\cdot 5^4}\gamma^4 + \cdots\right\}\frac{1}{\eta^2} \\
&- 15\frac{i}{\gamma^3}\left\{1 - \frac{8}{5^2}\gamma^2 + \frac{3242}{3^3\cdot 5^4\cdot 11}\gamma^4 + \cdots\right\}\frac{1}{\eta^3} \\
&- 10\frac{1}{\gamma^2}\left\{1 - \frac{1667}{2\cdot 3^3\cdot 5^2\cdot 11}\gamma^2 + \cdots\right\}\frac{1}{\eta^4} \\
&- 10\frac{i}{\gamma^3}\left\{1 - \frac{2521}{3^3\cdot 5^2\cdot 11}\gamma^2 + \cdots\right\}\frac{1}{\eta^5} \\
&- \frac{75}{11}\frac{1}{\gamma^2}\left\{1 + \cdots\right\}\frac{1}{\eta^6} \\
&- \frac{75}{11}\frac{i}{\gamma^3}\left\{1 + \cdots\right\}\frac{1}{\eta^7} \\
&+ \cdots \Bigg]. \qquad (80\text{–}20)
\end{aligned}
$$

The calculation of the successive terms in the solutions (80–19) and (80–20) affords us an excellent check on the correctness of the characteristic values α_1 and α_3 already computed, for the terms in the latter are of just the magnitude necessary to cause each alternate coefficient of z (after the first four in (80–20)) to start with a power of γ two greater than the previous coefficient.

We shall discuss only the fundamental free oscillation. The boundary condition at the surface of the spheroid demands that E_ξ and consequently $\frac{dY_1}{d\eta}$ shall vanish when $\eta = \eta_0$, where $1/\eta_0$ is the eccentricity of the surface of the conductor. Equating the derivative of (80–19) to zero and solving by successive approximations for the factor ω/v contained implicitly in $\gamma \equiv (\omega/v)f$, we obtain

$$\frac{\omega}{v} = -\frac{i}{2a}\left\{1 - 0.0606\frac{1}{\eta_0^4} - 0.0573\frac{1}{\eta_0^6} - 0.0466\frac{1}{\eta_0^8} + \cdots\right\}$$
$$\pm\frac{\sqrt{3}}{2a}\left\{1 + 0.2667\frac{1}{\eta_0^2} + 0.1233\frac{1}{\eta_0^4} + 0.0684\frac{1}{\eta_0^6} + 0.0426\frac{1}{\eta_0^8} + \cdots\right\}, \quad (80\text{–}21)$$

where $a = f\eta_0$ is the semi-major axis of the spheroid. From this expression the wave-length λ and the logarithmic decrement δ can be obtained by comparison with (79–14). Unfortunately, however, the series converge too slowly to yield accurate values for eccentricities greater than 0.8. Consequently we must look for another solution of the differential equation which will converge more rapidly in the interesting neighborhood of $\eta_0 = 1$.

(III). *Solution of the Equation in η for Small Values of the Variable.* To obtain solutions of the second of the two equations (78–14), which converge rapidly for values of η in the neighborhood of unity, we make the substitution $t \equiv \eta^2 - 1$. Then the differential equation becomes

$$4t(1+t)\frac{d^2Y}{dt^2} + 2t\frac{dY}{dt} + \left\{\gamma^2(1+t) - \alpha\right\}Y = 0. \quad (80\text{–}22)$$

As this equation is of the second order the complete solution is a linear combination of two independent primitives, which we shall denote by U and V. We look, therefore, for two independent solutions in powers of γ^2, the coefficients of which are functions of t.

Consider the fundamental, for which α has the value α_1 given by (80–13). If we neglect all terms in γ^2 and its powers, (80–22) becomes

$$4t(1+t)\frac{d^2Y_1}{dt^2} + 2t\frac{dY_1}{dt} - 2Y_1 = 0,$$

which is satisfied by both t and $t \log \dfrac{\sqrt{1+t}-1}{\sqrt{1+t}+1} + 2\sqrt{1+t}.$ These, then, are the zero order terms in the primitives U_1 and V_1 respectively. Next we return to the complete equation (80–22) with $\alpha = \alpha_1$ and assume a solution of the form

$$U_1 = t + U_{11}(t)\gamma^2 + U_{12}(t)\gamma^4 + \cdots.$$

After determining the polynomials $U_{11}(t)$, $U_{12}(t)$, etc., by substituting this in the differential equation (80–22), we assume a second solution of the form

$$V_1 = U_1 \log \frac{\sqrt{1+t}-1}{\sqrt{1+t}+1} + \sqrt{1+t}\, w_1.$$

Putting V_1 for Y in (80–22) we find that w_1 satisfies the differential equation

$$4t(1+t)\frac{d^2w_1}{dt^2} + 6t\frac{dw_1}{dt} + \{\gamma^2(1+t) - \alpha_1\}w_1 + 8\frac{dU_1}{dt} - \frac{4}{t}U_1 = 0,$$

from which it can be determined as a series in γ^2 the coefficients of which are polynomials in t.

In this way we obtain for the fundamental ($\alpha = \alpha_1$)

$$\begin{aligned} U_1 = t &- \frac{1}{2\cdot 5}t^2\gamma^2 - \left(\frac{1}{2\cdot 5^3\cdot 7}t^2 - \frac{1}{2^3\cdot 5\cdot 7}t^3\right)\gamma^4 \\ &+ \left(\frac{1}{3\cdot 5^5\cdot 7}t^2 + \frac{1}{2^2\cdot 3^2\cdot 5^3\cdot 7}t^3 - \frac{1}{2^4\cdot 3^3\cdot 5\cdot 7}t^4\right)\gamma^6 + \cdots, \end{aligned} \tag{80–23}$$

$$\begin{aligned} V_1 = U_1 \log \frac{\sqrt{1+t}-1}{\sqrt{1+t}+1} + \sqrt{1+t}\Bigg\{&2 + \left(\frac{4}{5} - \frac{1}{5}t\right)\gamma^2 \\ &+ \left(\frac{284}{5^3\cdot 7} - \frac{151}{5^3\cdot 7}t + \frac{1}{2^2\cdot 5\cdot 7}t^2\right)\gamma^4 \\ &+ \left(\frac{8632}{3\cdot 5^5\cdot 7} - \frac{115{,}646}{3^4\cdot 5^5\cdot 7}t + \frac{667}{3^4\cdot 5^3\cdot 7}t^2 \right.\\ &\left. - \frac{1}{2^3\cdot 3^3\cdot 5\cdot 7}t^3\right)\gamma^6 + \cdots \Bigg\}, \end{aligned} \tag{80–24}$$

and for the third harmonic ($\alpha = \alpha_3$)

$$U_3 = t + \frac{5}{2^2}t^2 - \left(\frac{1}{3\cdot 5}t^2 + \frac{5}{2^3\cdot 3^2}t^3\right)\gamma^2 + \cdots, \tag{80–25}$$

$$V_3 = U_3 \log\frac{\sqrt{1+t}-1}{\sqrt{1+t}+1} + \sqrt{1+t}\left\{\frac{1}{3} + \frac{5}{2}t + \left(\frac{2}{3^3\cdot 5} - \frac{11}{2\cdot 3^3\cdot 5}t - \frac{5}{2^2\cdot 3^2}t^2\right)\gamma^2 + \cdots\right\}. \tag{80–26}$$

For the fundamental, the most general solution of the differential equation in η is

$$Y_1 = A_1U_1 + B_1V_1, \tag{80–27}$$

where the coefficients A_1 and B_1 may be functions of the parameter γ. In order to represent a diverging wave system (that is, to satisfy the boundary condition of our problem at infinity) this solution must be identical with (80–19). Therefore we expand the logarithm in a power series and replace t by $\eta^2 - 1$ in (80–27), and expand the exponential in a power series in (80–19), and then compare coefficients. In this way we find that

$$A_1 = -\frac{1}{3}\gamma^2 a_1, \quad B_1 = \frac{3}{4}\frac{i}{\gamma}b_1, \tag{80–28}$$

where

$$a_1 \equiv 1 - \frac{1}{2\cdot 5^2}\gamma^2 + \frac{187}{2^3\cdot 5^4\cdot 7^2}\gamma^4 - \frac{26{,}021}{2^4\cdot 3^4\cdot 5^6\cdot 7^2}\gamma^6 + \cdots$$
$$= 1 - 0.020{,}000\gamma^2 + 0.000{,}763\gamma^4 - 0.000{,}026\gamma^6 + \cdots, \tag{80–29}$$

$$b_1 \equiv 1 - \frac{19}{2\cdot 5^2}\gamma^2 - \frac{2609}{2^3\cdot 5^4\cdot 7^2}\gamma^4 + \frac{32{,}593}{2^4\cdot 3^4\cdot 5^5\cdot 7^2}\gamma^6 + \cdots$$
$$= 1 - 0.380{,}000\gamma^2 - 0.010{,}649\gamma^4 + 0.000{,}164\gamma^6 + \cdots. \tag{80–30}$$

In carrying through this operation each term in A_1 and B_1 can be calculated by comparing two or more different groups of terms in the two solutions in such a way that every term in each solution is used at least once. Therefore the calculation of A_1 and B_1 can be made to afford a perfect check on the accuracy of every numerical coefficient in (80–19) on the one hand and in (80–23) and (80–24) on the other.

Similarly the solution which satisfies the boundary condition at infinity in the case of the third harmonic is

$$Y_3 = A_3 U_3 + B_3 V_3, \tag{80–31}$$

in which

$$A_3 = \frac{4}{3 \cdot 5^2 \cdot 7} \gamma^4 a_3, \quad B_3 = -\frac{1575}{2^3} \frac{i}{\gamma^3} b_3, \tag{80–32}$$

where

$$a_3 = 1 + \cdots, \tag{80–33}$$

$$b_3 = 1 - \frac{37}{2 \cdot 3^3 \cdot 5^2} \gamma^2 + \cdots = 1 - 0.027\gamma^2 + \cdots. \tag{80–34}$$

From (80–27) we can obtain the wave-length λ and the logarithmic decrement δ of the fundamental oscillation for eccentricities of the conductor near to unity. We shall confine our attention to eccentricities so close to unity that $t_0 = \eta_0{}^2 - 1$ is negligible except in the argument of the logarithm. As before the boundary condition at the surface of the conductor is $\dfrac{dY_1}{d\eta} = 0$ for $\eta = \eta_0$. Equating the derivative of (80–27) to zero and neglecting all terms in positive powers of t we get

$$b_1 = \frac{4}{9} i\gamma^3 l \frac{a_1}{m_1}, \tag{80–35}$$

where a_1 and b_1 are the series (80–29) and (80–30) respectively, and

$$l \equiv \frac{1}{\log \dfrac{\eta_0 + 1}{\eta_0 - 1} - 2}, \tag{80–36}$$

$$\begin{aligned} m_1 &\equiv 1 - \frac{1}{5}\gamma^2 l + \frac{9}{5^3 \cdot 7}\gamma^4 l - \frac{886}{3^4 \cdot 5^5 \cdot 7}\gamma^6 l + \cdots \\ &= 1 - 0.200{,}000\gamma^2 l + 0.010{,}286\gamma^4 l - 0.000{,}500\gamma^6 l + \cdots. \end{aligned} \tag{80–37}$$

Evidently l is very small for eccentricities close to unity.

Consider first the limiting case $l = 0$ of eccentricity unity, the ratio of the semi-minor axis b to the semi-major axis a of the elliptical section of the spheroid being zero. The boundary condition (80–35) reduces to $b_1 = 0$. Consequently γ is entirely real and the decrement is zero. This does not mean that no energy is radiated from the spheroid, but rather that the energy stored in the electromagnetic

field is infinite compared with the energy radiated during a period of oscillation. Solving for γ we find

$$\gamma = 1.5708 \tag{80–38}$$

correct to five significant figures. This is exactly $\pi/2$. Hence, as $f = a$ in this case, the half wave-length is exactly equal to the major axis of the spheroid.

To discuss eccentricities slightly different from unity, we multiply (80–35) by m_1/a_1. Combining the series represented by a_1, b_1 and m_1 this yields the formula

$$1 - (0.360{,}000 + 0.200{,}000l)\gamma^2 - (0.018{,}612 - 0.082{,}286l)\gamma^4 + (0.000{,}093 - 0.000{,}481l)\gamma^6 = 0.444{,}444il\gamma^3. \tag{80–39}$$

We know that γ cannot differ greatly from $\pi/2$ over the range to be considered. Hence we put

$$\gamma = \frac{\pi}{2}(1 - g - ih).$$

A few trials indicate that sufficient accuracy is obtained by retaining terms in g, h, gh, h^2 and neglecting higher powers. Then (80–39) leads to the simultaneous equations

$$(2.2214 - 0.9735l)g + (1.5472 - 2.4039l)h^2 = 5.1677l(h - 2gh),$$
$$(2.2214 - 0.9735l)h - (3.0944 - 4.8078l)gh = 1.7226l(1 - 3g - 3h^2),$$

which we may solve for g and h for any given value of l. In the range of eccentricities under consideration it is unnecessary to distinguish between a and f.

In Table I are listed values of the ratio $\lambda/4a$ of the half wave-length to the major axis of the spheroid, and of the logarithmic decrement δ, for six small values of l. In the second column are

TABLE I

WAVE-LENGTHS λ AND LOGARITHMIC DECREMENTS δ FOR FUNDAMENTAL FREE OSCILLATION OF SPHEROIDS OF ECCENTRICITIES NEAR TO UNITY

l	b/a	$\lambda/4a$	δ
0.000	0.000	1.000	0.000
0.020	$1.022\ (10)^{-11}$	1.001	0.098
0.050	$3.344\ (10)^{-5}$	1.004	0.247
0.080	$1.420\ (10)^{-3}$	1.009	0.396
0.100	$4.958\ (10)^{-3}$	1.015	0.494
0.125	$1.348\ (10)^{-2}$	1.023	0.613

given the corresponding values of the ratio b/a of the semi-minor to the semi-major axis of the elliptical section of the spheroid. Finally in Fig. 82 are plotted (*a*) the ratio $\lambda/4a$ and (*b*) the logarithmic decrement δ against the eccentricity $1/\eta_0$ all the way from eccentricity zero (sphere) to eccentricity unity (straight line), the data for eccentricities less than 0.8 being obtained from (80–21).

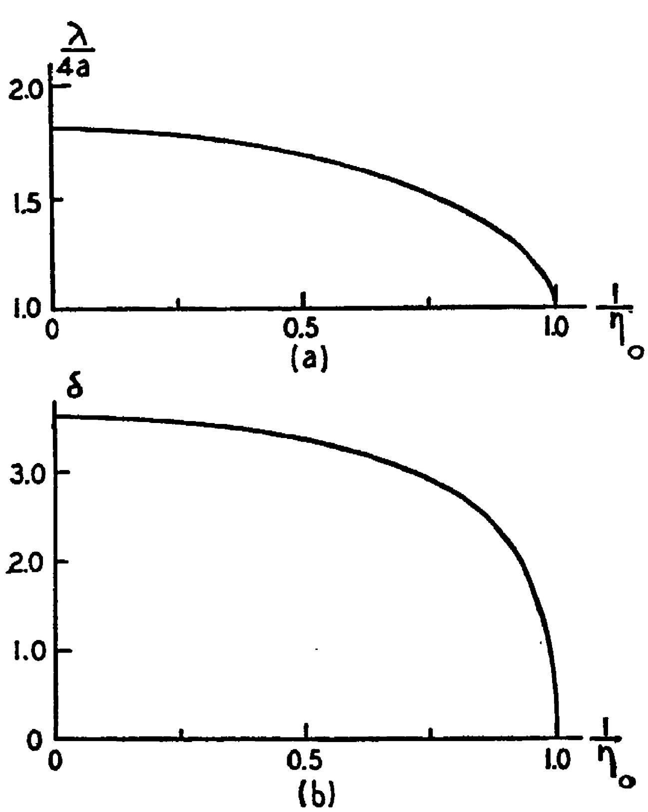

FIG. 82.

As regards the third harmonic, the series a_3 and b_3 have not been carried to a sufficient number of terms to enable us to compute the wave-length and the logarithmic decrement of free oscillation.

81. Forced Oscillations of Prolate Spheroidal Conductor.—In this article we shall discuss the oscillations of a perfectly conducting prolate spheroid forced by an electric field $E_0e^{-i\omega t}$ parallel to its axis, which, as before, we shall take as the X axis of our coordinate system. Such an exciting field can be produced by allowing plane electromagnetic waves to pass over the spheroid. As, however, the assumption of a uniform field over the region occupied by the conductor is a valid approximation only when the wave-length is long compared with the width of the spheroid, we are limited in our discussion of oscillations in the neighborhood of resonance to eccentricities close to unity. Fortunately this is the region of greatest interest.

First we set up the boundary condition at the surface of the spheroid. If χ is the angle which the unit vector $\boldsymbol{\xi}_1$ (Art. 78) makes with the X axis, we find, by differentiating (78–9), that

$$\tan\chi = -\frac{x}{\rho}\frac{\eta^2 - 1}{\eta^2} = -\frac{\xi}{\eta}\sqrt{\frac{\eta^2 - 1}{1 - \xi^2}}$$

with the aid of (78–11). Consequently

$$\cos\chi = \eta\sqrt{\frac{1 - \xi^2}{\eta^2 - \xi^2}}. \tag{81–1}$$

If, then, E_ξ is the tangential component of the electric field produced by the oscillations of the spheroid, the boundary condition at the surface of the conductor is

$$E_\xi + \eta_0 \sqrt{\frac{1 - \xi^2}{\eta_0^2 - \xi^2}}\, E_0 e^{-i\omega t} = 0, \tag{81-2}$$

which must be satisfied for all values of ξ.

Expressing E_ξ in terms of A by means of (78–4) and remembering that $f\eta_0$ is the semi-major axis a of the spheroid, the boundary condition reduces to

$$\begin{aligned}\left(\frac{\partial A}{\partial \eta}\right)_0 &= i\sqrt{\frac{\kappa}{\mu}}\,\gamma a(1 - \xi^2) E_0 e^{-i\omega t} \\ &= i\sqrt{\frac{\kappa}{\mu}}\,\gamma a P_{11}(\xi)\sqrt{1 - \xi^2}\, E_0 e^{-i\omega t},\end{aligned} \tag{81-3}$$

where the subscript 0 appended to the derivative indicates that $\eta = \eta_0$.

Evidently A must be a linear combination of the solutions of (78–13) representing the odd harmonics, that is,

$$\begin{aligned}A = \{&C_1 u_1(\xi) Y_1(\eta) + C_3 u_3(\xi) Y_3(\eta) \\ &+ C_5 u_5(\xi) Y_5(\eta) + \cdots\}\sqrt{1 - \xi^2}\, e^{-i\omega t},\end{aligned} \tag{81-4}$$

where $u_1(\xi)$ and $u_3(\xi)$ are given by (80–14) and (80–16) respectively, and $Y_1(\eta)$ and $Y_3(\eta)$ by (80–27) and (80–31) respectively. The coefficients C_1, C_3, C_5, $\cdots$ fall off so rapidly with increasing index that we can neglect the terms in the fifth and higher harmonics. Determining C_1 and C_3 by substituting (81–4) in (81–3) and equating coefficients of like associated Legendrian functions, we find

$$C_1 = i\sqrt{\frac{\kappa}{\mu}}\,\gamma a E_0 \left\{1 - \frac{2}{5^4 \cdot 7}\gamma^4 + \frac{8}{3 \cdot 5^6 \cdot 7}\gamma^6 + \cdots\right\} \Big/ \left(\frac{dY_1}{d\eta}\right)_0, \tag{81-5}$$

$$C_3 = i\sqrt{\frac{\kappa}{\mu}}\,\gamma a E_0 \left\{\frac{1}{3 \cdot 5^2}\gamma^2 - \frac{2}{3^2 \cdot 5^4}\gamma^4 + \cdots\right\} \Big/ \left(\frac{dY_3}{d\eta}\right)_0. \tag{81-6}$$

To save writing we shall put

$$k \equiv 2\pi c\sqrt{\frac{\kappa}{\mu}}\, a E_0. \tag{81-7}$$

Then it follows from (78–5) and (81–4) that the current i_0 at the center $\xi = 0$ of the antenna is

$$i_0 = ik\gamma \left\{ \left(1 - \frac{2}{5^4 \cdot 7}\gamma^4 + \frac{8}{3 \cdot 5^6 \cdot 7}\gamma^6 + \cdots \right) u_1(0) \frac{Y_1(\eta_0)}{\left(\frac{dY_1}{d\eta}\right)_0} \right.$$

$$\left. + \frac{1}{3 \cdot 5^2}\gamma^2 \left(1 - \frac{2}{3 \cdot 5^2}\gamma^2 + \cdots \right) u_3(0) \frac{Y_3(\eta_0)}{\left(\frac{dY_3}{d\eta}\right)_0} + \cdots \right\} e^{-i\omega t}. \quad (81\text{–}8)$$

From (80–27) we find

$$Y_1(\eta_0) = \frac{3}{2}\frac{i}{\gamma} c_1, \quad (81\text{–}9)$$

$$\left(\frac{dY_1}{d\eta}\right)_0 = -\frac{3}{2}\frac{i}{\gamma}\frac{b_1 m_1}{l} - \frac{2}{3}\gamma^2 a_1, \quad (81\text{–}10)$$

to a sufficient degree of approximation for eccentricities so near to unity that t_0 may be neglected everywhere except in the argument of the logarithm. Here

$$c_1 \equiv 1 + \frac{1}{2 \cdot 5^2}\gamma^2 - \frac{89}{2^3 \cdot 5^4 \cdot 7^2}\gamma^4 + \frac{733}{2^4 \cdot 3^4 \cdot 5^5 \cdot 7^2}\gamma^6 + \cdots = u_1(0)$$

$$= 1 + 0.020{,}000\gamma^2 - 0.000{,}363\gamma^4 + 0.000{,}004\gamma^6 + \cdots. \quad (81\text{–}11)$$

We note that c_1 is the reciprocal of the series a_1 given by (80–29).

Similarly from (80–31) we obtain

$$Y_3(\eta_0) = -\frac{525}{2^3}\frac{i}{\gamma}, \quad (81\text{–}12)$$

$$\left(\frac{dY_3}{d\eta}\right)_0 = \frac{1575}{2^2}\frac{i}{\gamma^3}\frac{b_3}{l}\left(1 - \tfrac{2}{3}l\right) + \frac{8}{3 \cdot 5^2 \cdot 7}\gamma^4, \quad (81\text{–}13)$$

to a sufficient degree of approximation. In fact, we need retain only the first term in the series (80–34) for b_3 and may neglect the real part of (81–13) as compared with the imaginary part.

Put

$$s_1 \equiv 1 - \frac{2}{5^4 \cdot 7}\gamma^4 + \frac{8}{3 \cdot 5^6 \cdot 7}\gamma^6 + \cdots$$

$$= 1 - 0.000{,}457\gamma^4 + 0.000{,}024\gamma^6 + \cdots. \quad (81\text{–}14)$$

Then, substituting (81–9), (81–10), (81–12), (81–13) in (81–8) we find for the current i_0 at the center of the antenna

$$i_0 = k\left[\frac{(l\gamma s_1 c_1{}^2)(\frac{4}{9}l\gamma^3 a_1)}{(\frac{4}{9}l\gamma^3 a_1)^2 + (b_1 m_1)^2} - \mathrm{i}\left\{\frac{(l\gamma s_1 c_1{}^2)(b_1 m_1)}{(\frac{4}{9}l\gamma^3 a_1)^2 + (b_1 m_1)^2} - \frac{l\gamma^3}{300}(1 + \tfrac{2}{3}l)\right\}\right]e^{-\mathrm{i}\omega t} \quad (81\text{–}15)$$

to a sufficient degree of approximation. It is seen that the third harmonic contributes only a very small term to the imaginary part of the current.

The square of the current amplitude i_{00} at the center is

$$i_{00}^2 = k^2\left[\frac{(l\gamma s_1 c_1{}^2)^2}{(\frac{4}{9}l\gamma^3 a_1)^2 + (b_1 m_1)^2}\right]\left[1 - \frac{\gamma^2}{150}b_1(1 + \tfrac{2}{3}l)\right] \quad (81\text{–}16)$$

with neglect of the square of the third harmonic. To find the frequency of resonance we must equate to zero the derivative of this with respect to γ, or, more conveniently, with respect to γ^2. This leads to the formula

$$b_1 = 0.14750 l^2\gamma^6 \frac{1 - 0.00560l}{(1 - 0.43837l)^2}, \quad (81\text{–}17)$$

where, in the very small right-hand member, we have replaced γ by $\pi/2$ in the numerator and denominator of the fractional factor. The justification for this lies in the fact that the values of γ computed from (81–17) are found to differ very little from $\pi/2$ in the range under consideration.

In Table II are given the ratios $\lambda/4a$, $i_0/k\,e^{-\mathrm{i}\omega t}$, i_{00}/k and the angle ψ_i by which the current leads the electromotive force, at resonance, for six values of l. In the second column of the table is given

TABLE II

WAVE-LENGTHS AND CURRENTS AT RESONANCE

l	b/a	$\lambda/4a$	$i_0/ke^{-\mathrm{i}\omega t}$	i_{00}/k	ψ_i
0.000	0.000	1.0000	1.045	1.045	0°.0
0.020	$1.022\,(10)^{-11}$	1.0004	1.045 − 0.028i	1.045	1°.5
0.050	$3.344\,(10)^{-6}$	1.0027	1.045 − 0.071i	1.047	3°.9
0.080	$1.420\,(10)^{-3}$	1.007	1.045 − 0.113i	1.050	6°.2
0.100	$4.958\,(10)^{-3}$	1.011	1.045 − 0.141i	1.055	7°.7
0.125	$1.348\,(10)^{-2}$	1.017	1.046 − 0.175i	1.061	9°.5

the ratio of the semi-minor axis b to the semi-major axis a of the spheroid corresponding to the assumed value of l.

It will be noted that the real part of the current is substantially the same for all values of l considered, and that the current leads the electromotive force at resonance for all values of l greater than zero by an angle ψ_i which increases with increasing l. For $l = 0$ the half wave-length is exactly equal to the major axis of the spheroid as in

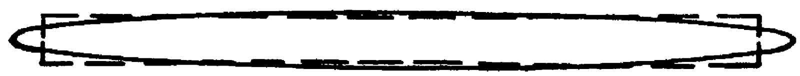

FIG. 83.

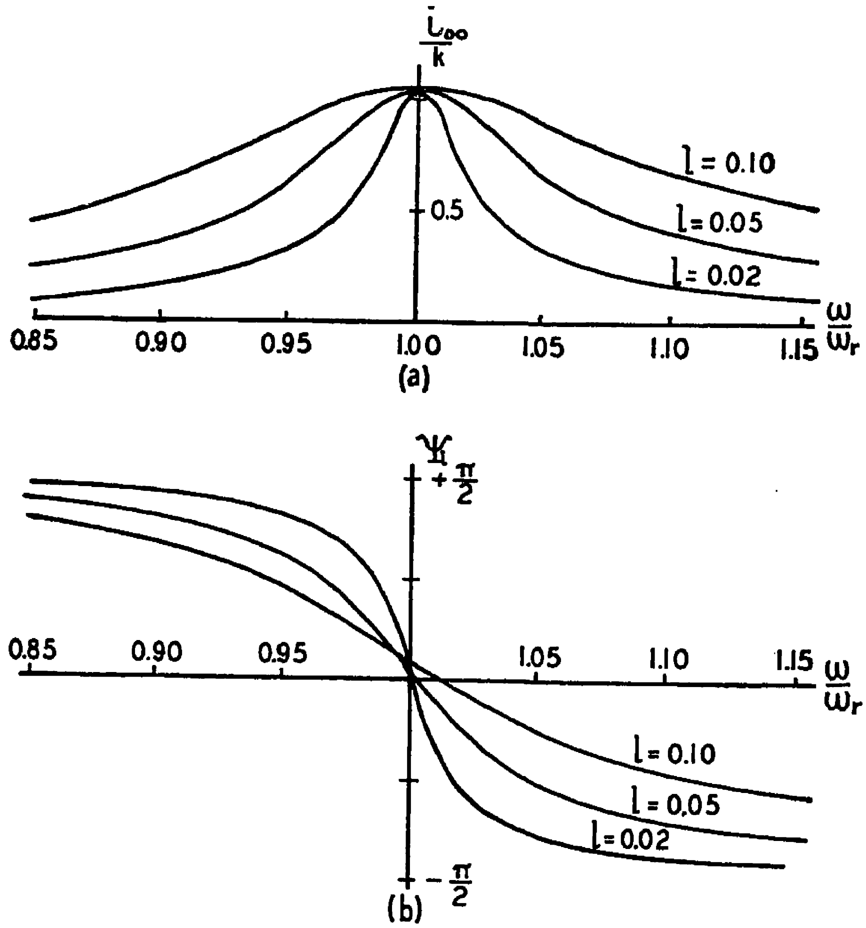

FIG. 84.

the case of free oscillation, but for values of l greater than zero the resonant wave-length is slightly shorter than the wave-length of free oscillation. Experiments on antennas in the form of right circular cylinders indicate that the half wave-length at resonance is from five to six per cent longer than the antenna. This is in satisfactory qualitative agreement with our calculated ratios of $\lambda/2$ to $2a$ for the prolate spheroid, for the diameter of a prolate spheroid falls to half of its maximum value at a distance $x = 0.866a$ from the center whatever

its eccentricity may be, and consequently the prolate spheroid which best fits a right circular cylinder is one which is slightly wider at the center and somewhat longer, as illustrated in Fig. 83. We may call this prolate spheroid the *equivalent spheroid* for the given cylinder.

In Fig. 84 are plotted resonance curves for $l = 0.02$, 0.05 and 0.10. The abscissa in each curve is the ratio $\omega/\omega_r = \gamma/\gamma_r$ of the frequency to the frequency at resonance. In (a) is given the ratio i_{00}/k at the center of the antenna, and in (b) the angle ψ_i by which the current leads the electromotive force.

Next we shall investigate the distribution of current along the antenna. In this calculation we shall neglect the small contribution to the current made by the third and higher harmonics. Indeed, when $l = 0$, it is clear from (81–15) that the current is due entirely to the fundamental.

Let i_ξ be the current at a distance from the center of the antenna specified by the coordinate ξ. As $f\eta_0 = a$, it follows from (78–11) that $\xi = x/a$. Then we have from (78–5) and (81–4) that

$$\begin{aligned}\frac{i_\xi}{i_0} &= \frac{\sqrt{1-\xi^2}\,u_1(\xi)}{u_1(0)} \\ &= (1-\xi^2)\left\{1 - \frac{\gamma^2}{2.5}\left(1 - \frac{4}{5^2\cdot 7}\gamma^2 + \frac{8}{3\cdot 5^4\cdot 7}\gamma^4\right)\xi^2 \right. \\ &\qquad \left. + \frac{\gamma^4}{2^3\cdot 5\cdot 7}\left(1 - \frac{8}{3^2\cdot 5^2}\gamma^2\right)\xi^4 - \frac{\gamma^6}{2^4\cdot 3^3\cdot 5\cdot 7}\xi^6 + \cdots\right\}. \qquad (81\text{–}18)\end{aligned}$$

If we put p for the ratio of the frequency for which it is desired to evaluate (81–18) to the frequency for resonance with $l = 0$, then $p = \gamma/1.5708$ and (81–18) becomes

$$\begin{aligned}\frac{i_\xi}{i_0} &= (1-\xi^2)\{1 - 0.24674p^2(1 - 0.05640p^2 + 0.00371p^4)\xi^2 \\ &\qquad + 0.02174p^4(1 - 0.08773p^2)\xi^4 - 0.00099p^6\xi^6 + \cdots\}. \qquad (81\text{–}19)\end{aligned}$$

When $p = 1$ this expression reduces to

$$\frac{i_\xi}{i_0} = (1-\xi^2)\{1 - 0.2337\xi^2 + 0.0198\xi^4 - 0.0010\xi^6 + \cdots\}, \qquad (81\text{–}20)$$

which is exactly $\cos\frac{\pi}{2}\xi$ to the fourth decimal place. The latter function is found to represent the measured distribution of current along thin antennas at resonance within the experimental error. At fre-

quencies off resonance our formula shows that there is an appreciable deviation from this simple cosine law.

When the antenna is oscillating steadily, the mean rate of absorption of energy must equal the mean rate of radiation of energy. Therefore we can calculate the radiation resistance from either of these equal quantities. As a check on our analysis we shall calculate it from both.

To compute the rate of absorption of energy we need the electromotive force $d\mathcal{E}$ along each element of distance $d\lambda_\xi$ at the surface of the spheroid in the direction of increasing ξ. From (78–12)

$$d\lambda_\xi = f\sqrt{\frac{\eta_0{}^2 - \xi^2}{1 - \xi^2}}\, d\xi, \tag{81–21}$$

and consequently, with the aid of (81–1) and the relation $f\eta_0 = a$,

$$d\mathcal{E} = E_0\, e^{-i\omega t} \cos \chi d\lambda_\xi = E_0\, e^{-i\omega t}\, a d\xi. \tag{81–22}$$

This must be multiplied by the part of the current in phase with the exciting field, integrated over ξ and then averaged with respect to the time. Since the part of the current in phase with the electromotive force involves the fundamental only, the ratio of this part of the current at any ξ to the same part of the current at the center of the antenna is given exactly by (81–18). Hence from (81–15) and (81–22) we obtain for the mean rate of absorption of energy

$$\overline{\mathcal{P}} = \frac{1}{2} k E_0 \frac{(l\gamma s_1 c_1{}^2)(\frac{4}{9} l\gamma^3 a_1)}{(\frac{4}{9} l\gamma^3 a_1)^2 + (b_1 m_1)^2} \frac{a}{c_1} \int_{-1}^{1} \sqrt{1 - \xi^2}\, u_1(\xi) d\xi. \tag{81–23}$$

But

$$\int_{-1}^{1} \sqrt{1 - \xi^2}\, u_1(\xi) d\xi = \int_{-1}^{1} P_{11}(\xi) u_1(\xi) d\xi = \frac{4}{3}$$

from (80–14). Consequently, recalling the relation $a_1 c_1 = 1$ and expressing k in terms of E_0 by means of (81–7),

$$\overline{\mathcal{P}} = \frac{64}{27} \pi^3 \sqrt{\frac{\kappa}{\mu}}\, c E_0{}^2 \frac{a^2 f^2}{\lambda^2} \frac{a_1{}^4}{s_1} \frac{(l\gamma s_1 c_1{}^2)^2}{(\frac{4}{9} l\gamma^3 a_1)^2 + (b_1 m_1)^2} \tag{81–24}$$

in terms of the amplitude E_0 of the exciting field, or

$$\overline{\mathcal{P}} = \frac{16}{27} \pi \sqrt{\frac{\mu}{\kappa}} \frac{i_{00}^2}{c} \frac{f^2}{\lambda^2} \frac{a_1{}^4}{s_1} \left\{ 1 + \frac{\gamma^2}{150} b_1 (1 + \tfrac{2}{3} l) \right\} \tag{81–25}$$

in terms of the amplitude i_{00} of the current at the center of the antenna.

Dividing the mean rate of absorption of energy by the mean square current $\frac{1}{2}i_{00}^2$ at the center of the antenna, we find for the radiation resistance R

$$R = \frac{32}{27}\pi\sqrt{\frac{\mu}{\kappa}}\frac{f^2}{c\lambda^2}\frac{a_1^4}{s_1}\left\{1 + \frac{\gamma^2}{150}b_1(1 + \tfrac{2}{3}l)\right\}. \tag{81-26}$$

We may replace f by a for eccentricities nearly unity, obtaining for the radiation resistance R_p in ohms

$$R_p = 87.670\sqrt{\frac{\mu}{\kappa}}\left(\frac{4a}{\lambda}\right)^2\frac{a_1^4}{s_1}\left\{1 + \frac{\gamma^2}{150}b_1\,(1 + \tfrac{2}{3}l)\right\}. \tag{81-27}$$

In Table III are given the radiation resistances at resonance for the six values of l considered before. In a typical antenna the ohmic resistance is of the order of magnitude of one per cent or less of the radiation resistance. Consequently we have made only a negligible error in treating the oscillating spheroid as a perfect conductor. Measured values of the resistance at resonance agree well with those in the table.

TABLE III

RADIATION RESISTANCE AT RESONANCE

l	$\sqrt{\frac{k}{\mu}}\,R_p$ (ohms)
0.000	73.1
0.020	73.0
0.050	72.8
0.080	72.3
0.100	71.8
0.125	71.1

To illustrate how the radiation resistance varies with frequency, the quantity $\sqrt{\kappa/\mu}\,R_p$ is plotted against the ratio ω/ω_r of frequency to frequency at resonance in Fig. 85 over a range extending from a frequency 15 per cent below resonance to one 15 per cent above resonance. The upper curve is for $l = 0.050$ and the lower for $l = 0.100$. Both curves are very nearly straight lines over the range of frequencies considered.

Finally we turn our attention to the electromagnetic field of the oscillating antenna. The three non-vanishing field components E_ξ, E_η, F_ϕ are given in terms of A and its derivatives by (78-4). Using

the expressions for a_u and a_v specified by (78–12) for prolate spheroidal coordinates, we have

$$\left.\begin{aligned} E_\xi &= i\sqrt{\frac{\mu}{\kappa}}\frac{1}{\gamma f}\frac{1}{\sqrt{(1-\xi^2)(\eta^2-\xi^2)}}\frac{\partial A}{\partial \eta}, \\ E_\eta &= -i\sqrt{\frac{\mu}{\kappa}}\frac{1}{\gamma f}\frac{1}{\sqrt{(\eta^2-1)(\eta^2-\xi^2)}}\frac{\partial A}{\partial \xi}, \\ F_\phi &= \frac{1}{f}\frac{1}{\sqrt{(1-\xi^2)(\eta^2-1)}}A. \end{aligned}\right\} \qquad (81\text{–}28)$$

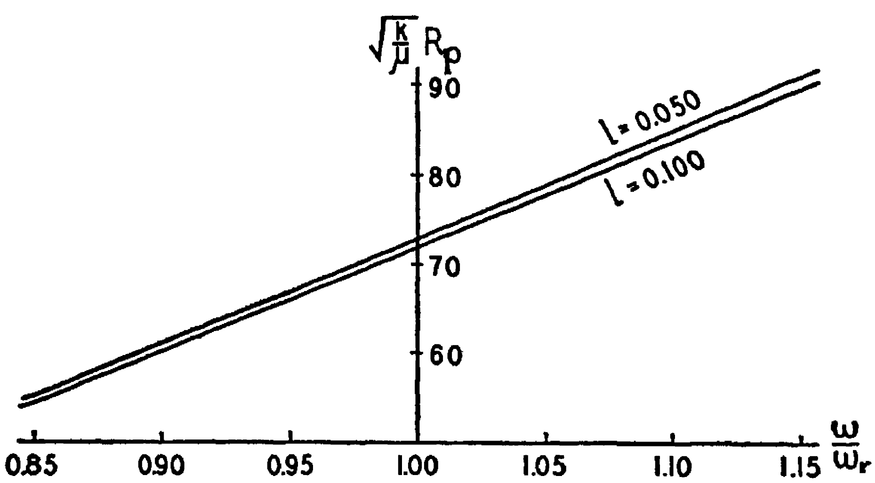

FIG. 85.

Since the coefficient C_3 in (81–4) can be expressed in terms of C_1 by the relation

$$C_3 = -\frac{2b_1}{3^2\cdot 5^4\cdot 7}\gamma^4 C_1$$

to a quite sufficient degree of accuracy for frequencies near to resonance through the use of (81–5), (81–6), (81–10) and (81–13), we have for A

$$A = -\frac{ik}{3\pi c}l\gamma^2 s_1\frac{\frac{4}{9}l\gamma^3 a_1 - ib_1 m_1}{(\frac{4}{9}l\gamma^3 a_1)^2 + (b_1 m_1)^2}\left\{u_1(\xi)Y_1(\eta) - \frac{2b_1}{3^2\cdot 5^4\cdot 7}\gamma^4 u_3(\xi)Y_3(\eta)\right\}\sqrt{1-\xi^2}\,e^{-i\omega t}. \quad (81\text{–}29)$$

At a distance from the nearest point on the antenna not less than the semi-interfocal distance f the field components are specified to better than one per cent by using the functions (80–19) and (80–20) for $Y_1(\eta)$ and $Y_3(\eta)$ respectively, although very close to the antenna it is necessary to use (80–27) and (80–31).

Evaluating the three expressions (81–28) from (81–29) with the use of (80–19) and (80–20), and ignoring all terms in powers of $1/\eta$ higher than the first, we find for the components of the radiation field

$$\left.\begin{aligned} E_\xi &= \frac{k}{3\pi c}\sqrt{\frac{\mu}{\kappa}}\,\frac{l\gamma^2 s_1}{f}\,\frac{1}{\sqrt{\eta^2-\xi^2}}\,\frac{1}{\sqrt{(\frac{4}{9}l\gamma^3 a_1)^2+(b_1 m_1)^2}} \\ &\qquad\left\{u_1(\xi)-\frac{2b_1\gamma^4}{3^2\cdot 5^4\cdot 7}\,u_3(\xi)\right\}e^{i(\gamma\eta-\omega t+\pi/2-\psi)}, \\ E_\eta &= 0, \\ F_\phi &= -\frac{k}{3\pi c}\,\frac{l\gamma^2 s_1}{f}\,\frac{1}{\sqrt{\eta^2-1}}\,\frac{1}{\sqrt{(\frac{4}{9}l\gamma^3 a_1)^2+(b_1 m_1)^2}} \\ &\qquad\left\{u_1(\xi)-\frac{2b_1\gamma^4}{3^2\cdot 5^4\cdot 7}\,u_3(\xi)\right\}e^{i(\gamma\eta-\omega t+\pi/2-\psi)}, \end{aligned}\right\} \quad (81\text{–}30)$$

where

$$\tan\psi \equiv \frac{b_1 m_1}{\frac{4}{9}l\gamma^3 a_1}. \qquad (81\text{–}31)$$

For small l, then, ψ is nearly zero at resonance, nearly $\pi/2$ below resonance, and nearly $-\pi/2$ above resonance.

As the Poynting flux in the direction of increasing η is $-cE_\xi F_\phi$ and the element of area of a spheroidal surface is $2\pi f^2\sqrt{(\eta^2-\xi^2)(\eta^2-1)}\,d\xi$, the mean rate of radiation of energy is

$$\overline{\mathscr{R}} = \frac{16}{9}\pi^3\sqrt{\frac{\kappa}{\mu}}\,cE_0^2\,\frac{a^2 f^2}{\lambda^2}\,a_1^4\,\frac{(l\gamma s_1 c_1^2)^2}{(\frac{4}{9}l\gamma^3 a_1)^2+(b_1 m_1)^2}\int_{-1}^{1}\{u_1(\xi)\}^2 d\xi, \quad (81\text{–}32)$$

in which we have made use of the relation $a_1 c_1 = 1$ and have neglected, as before, the square of the third harmonic. From (80–14) and (80–12) we find

$$\int_{-1}^{1}\{u_1(\xi)\}^2 d\xi = \frac{4}{3}\left\{1+\frac{2}{5^4\cdot 7}\gamma^4-\frac{8}{3\cdot 5^6\cdot 7}\gamma^6+\cdots\right\} = \frac{4}{3s_1}, \quad (81\text{–}33)$$

where s_1 is the series defined by (81–14). Therefore

$$\overline{\mathscr{R}} = \frac{64}{27}\pi^3\sqrt{\frac{\kappa}{\mu}}\,cE_0^2\,\frac{a^2 f^2}{\lambda^2}\,\frac{a_1^4}{s_1}\,\frac{(l\gamma s_1 c_1^2)^2}{(\frac{4}{9}l\gamma^3 a_1)^2+(b_1 m_1)^2}. \qquad (81\text{–}34)$$

This agrees with (81–24), confirming our previous calculation of the radiation resistance.

CHAPTER 8

ELECTROMAGNETIC THEORY OF LIGHT

82. Field Equations. — Light waves differ from the electromagnetic waves discussed in the last two chapters only in their very short wave-length. This minuteness of wave-length, however, introduces two important simplifications into the electromagnetic theory of light. First, we need consider only plane waves, for the nearest point to a source of light at which we can make observations is so many wave-lengths from the radiating atom or molecule that any small portion of the wave front has the essential properties of a plane wave. Second, the inertia of the Ampèrian currents in a magnetic medium is so great that they are unable to follow to any appreciable extent the rapid oscillations of a light wave. Consequently all media act as if they had effectively unit permeability, and, except in the case of an optically active medium, we may put $\mathbf{B} = \mathbf{F} = \mathbf{H}$. Therefore we need consider, in general, only dielectric properties and conductivity in discussing the propagation of light in a homogeneous material medium.

In an anisotropic medium

$$\mathbf{D} = \mathbf{K}\cdot\mathbf{E}, \quad \rho\mathbf{V} = \mathbf{\Sigma}\cdot\mathbf{E}, \tag{82–1}$$

where $\mathbf{K}$ is the permittivity dyadic and $\mathbf{\Sigma}$ the conductivity dyadic; whereas in an isotropic medium the simpler relations

$$\mathbf{D} = \kappa\mathbf{E}, \quad \rho\mathbf{V} = \sigma\mathbf{E}, \tag{82–2}$$

suffice, where κ is the permittivity and σ the conductivity. The isotropic medium can, however, be considered as a special case of the anisotropic medium in which $\mathbf{K} = \kappa\mathbf{I}$ and $\mathbf{\Sigma} = \sigma\mathbf{I}$, where $\mathbf{I}$ is the unit dyadic.

Since we intend to confine our discussion to a steady state of simple harmonic radiation, the field vectors $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{H}$ are of the form

$$\mathbf{D}, \mathbf{E}, \mathbf{H} = \mathrm{A}_0 e^{i\omega(\mathbf{S}\cdot\mathbf{r}-t)}, \tag{82–3}$$

where $\mathbf{A}_0$ is a constant vector amplitude, $\mathbf{S}$ the vector wave-slowness, and $\mathbf{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$ the position vector of the field-point under consideration. As in our previous discussion of simple harmonic waves,

$$\frac{\partial}{\partial t} = -\,\mathrm{i}\omega. \tag{82–4}$$

Therefore, the equation of continuity (62–1) gives for the free charge density

$$\rho = -\frac{\mathrm{i}}{\omega}\nabla\cdot\overline{\overline{\boldsymbol{\Sigma}}}\cdot\mathbf{E}. \tag{82–5}$$

If, then, we put

$$\boldsymbol{\Psi} \equiv \mathbf{K} + \frac{\mathrm{i}}{\omega}\boldsymbol{\Sigma} \tag{82–6}$$

and write

$$\mathbf{D}_e \equiv \boldsymbol{\Psi}\cdot\mathbf{E}, \tag{82–7}$$

the field equations (62–12) become

$$\left.\begin{array}{llll} \nabla\cdot\mathbf{D}_e = 0, & (a) & \nabla\cdot\mathbf{H} = 0, & (b) \\ \nabla\times\mathbf{E} = -\dfrac{1}{c}\dot{\mathbf{H}}, & (c) & \nabla\times\mathbf{H} = \dfrac{1}{c}\dot{\mathbf{D}}_e. & (d) \end{array}\right\} \tag{82–8}$$

As these equations are identical with the field equations for a non-conducting dielectric with electric displacement $\mathbf{D}_e$, we shall call $\mathbf{D}_e$ the *effective electric displacement*. If the medium is non-conducting, $\mathbf{D}_e$ becomes the true electric displacement $\mathbf{D}$.

Taking the curl of (82–8*c*) and the time derivative of (82–8*d*) we eliminate $\mathbf{H}$ in the usual manner, obtaining the wave equation

$$\nabla\cdot\nabla\mathbf{E} - \nabla\nabla\cdot\mathbf{E} = \frac{1}{c^2}\ddot{\mathbf{D}}_e. \tag{82–9}$$

While $\nabla\cdot\mathbf{D}_e = 0$ in accord with (82–8*a*) in all cases, this equation leads to the vanishing of $\nabla\cdot\mathbf{E}$ only in the case of an isotropic medium. Therefore we cannot omit the second term on the left of (82–9) in general.

Since, in this chapter, we shall be concerned only with waves of constant vector amplitude $\mathbf{A}_0$, the coordinates appear only in the exponent of (82–3). So, as $\mathbf{S}\cdot\mathbf{r} = S_x x + S_y y + S_z z$, it follows that the operator ∇ is given by

$$\nabla = \mathrm{i}\omega\mathbf{S}. \tag{82–10}$$

Therefore, if we make use of (82–4) and (82–10) and write S_0 for the wave slowness $1/c$ *in vacuo*, the field equations (82–8) and the wave equation (82–9) become

$$\left.\begin{array}{llll} \mathbf{S}\cdot\mathbf{D}_e = 0, & (a) & \mathbf{S}\cdot\mathbf{H} = 0, & (b) \\ \mathbf{S}\times\mathbf{E} = S_0\mathbf{H}, & (c) & \mathbf{S}\times\mathbf{H} = -S_0\mathbf{D}_e, & (d) \end{array}\right\} \quad (82\text{–}11)$$

and

$$\mathbf{S}\cdot\mathbf{SE} - \mathbf{SS}\cdot\mathbf{E} = S_0^2\mathbf{D}_e. \quad (82\text{–}12)$$

We see from (82–11*a*) and (82–11*b*) that $\mathbf{D}_e$ and $\mathbf{H}$ are always perpendicular to $\mathbf{S}$ and therefore lie in the wave front, and, from (82–12), that $\mathbf{E}$, although it may not lie in the wave front in an anisotropic medium, always lies in the plane of $\mathbf{D}_e$ and $\mathbf{S}$. Furthermore, by taking the scalar product of (82–11*d*) by $\mathbf{H}$ we find that $\mathbf{H}$ is at right angles to $\mathbf{D}_e$ as well as to $\mathbf{S}$, and by taking the vector product of (82–11*d*) by $\mathbf{H}$ we see that $\mathbf{D}_e \times \mathbf{H}$ has the direction of $\mathbf{S}$. In fact, by taking the vector product of (82–11*d*) by $\mathbf{S}$, we obtain the expression

$$\mathbf{H} = \frac{S_0}{S^2}\mathbf{S}\times\mathbf{D}_e \quad (82\text{–}13)$$

for the magnetic intensity. Finally, from (82–12) we find that, if $E_\perp$ is the component of $\mathbf{E}$ perpendicular to $\mathbf{S}$,

$$E_\perp = \frac{S_0^2}{S^2}D_e. \quad (82\text{–}14)$$

As the phenomena of light propagation are so different in isotropic and anisotropic media we shall not carry our general analysis farther but shall study four special cases: (I) a conducting medium which is isotropic in both permittivity and conductivity, (II) a non-conducting anisotropic crystal, (III) a non-conducting isotropic dielectric in a uniform external magnetic field, (IV) a non-conducting isotropic dielectric which is optically active.

83. Homogeneous Isotropic Medium. — In the case of a static or slowly varying field the permittivity κ of an isotropic medium is a constant independent of frequency. In fact, we were able to make a rough calculation of the value of this important constant in article 63. In a rapidly oscillating field, however, the situation is quite different, and κ is a complex function of frequency which we must determine from the equation of motion of the bound electrons. The only effect

of an oscillating electric field on the permanent electric dipoles in a dielectric is to cause angular oscillation, and the moments of inertia of the permanent dipoles are presumably too large to permit them to follow to any appreciable extent the rapid changes of electric intensity in a light wave. Hence we attribute dielectric properties at high frequencies solely to the induced dipoles in the medium, that is, to the oscillations of electrons bound to the atom by an elastic force proportional to the displacement of the particle from its position of equilibrium. In addition we assume that these electrons are subject to a damping force proportional to the velocity.

(I) *Unlimited Wave Trains.* First we shall consider the steady state existing after waves have been traversing the medium for a time long compared with the period of oscillation. If $\mathbf{r}$ is the vector displacement of a bound electron from its equilibrium position, the equation of motion of the electron in an impressed electric field $\mathbf{E}_1$ and magnetic field $\mathbf{H}$ is

$$\frac{d^2\mathbf{r}}{dt^2} + 2l\frac{d\mathbf{r}}{dt} + k_0{}^2\mathbf{r} = \frac{e}{m}\left\{\mathbf{E}_1 + \frac{1}{c}\frac{d\mathbf{r}}{dt} \times \mathbf{H}\right\}, \qquad (83\text{–}1)$$

provided we retain only the first term in the kinetic reaction (57–13), in which the variation of mass with velocity is neglected. In this equation $-2ml\dfrac{d\mathbf{r}}{dt}$ is the damping force and $-mk_0{}^2\mathbf{r}$ the force of restitution. While $\mathbf{H}$ represents the mean magnetic intensity in the medium due to the light wave, $\mathbf{E}_1$ is the sum of the mean electric intensity $\mathbf{E}$ and $\frac{1}{3}\mathbf{P}$, in accord with (63–3).

Since $\mathbf{E}_1$ and $\mathbf{H}$ are of the same order of magnitude, and $\left|\dfrac{d\mathbf{r}}{dt}\right|$ is very small compared with c, the second term on the right-hand side of (83–1) is negligible compared with the first, and will therefore be omitted. Then, making use of (82–4),

$$\left\{(k_0{}^2 - \omega^2) - 2i\omega l\right\}\mathbf{r} = \frac{e}{m}\mathbf{E}_1$$

and, if there are N such bound electrons per unit volume, the polarization of the dielectric is

$$\mathbf{P} = Ne\mathbf{r} = \frac{1}{(k_0{}^2 - \omega^2) - 2i\omega l}\frac{Ne^2}{m}\mathbf{E}_1. \qquad (83\text{–}2)$$

The coefficient of $\mathbf{E}_1$ in this equation is the constant α appearing in (63–6). Now, in accord with (63–10),

$$\kappa = \frac{1 + \frac{2}{3}\alpha}{1 - \frac{1}{3}\alpha} = 1 + \frac{\alpha}{1 - \frac{1}{3}\alpha}.$$

So, if we put $k^2 \equiv k_0{}^2 - Ne^2/3m$, we have

$$\kappa = 1 + \frac{\dfrac{Ne^2}{m}}{(k^2 - \omega^2) - 2i\omega l}. \tag{83–3}$$

Finally, writing

$$\kappa = \kappa' + i\kappa'', \tag{83–4}$$

where κ' and κ'' are respectively the real and the imaginary parts of the permittivity, we find that

$$\kappa' = 1 + \frac{(k^2 - \omega^2)\dfrac{Ne^2}{m}}{(k^2 - \omega^2)^2 + 4\omega^2 l^2}, \tag{83–5}$$

$$\kappa'' = \frac{2\omega l \dfrac{Ne^2}{m}}{(k^2 - \omega^2)^2 + 4\omega^2 l^2}. \tag{83–6}$$

We have assumed a single type of bound electron. If, more generally, there are present several types distinguished by different values of the coefficients k and l,

$$\kappa' = 1 + \frac{e^2}{m}\sum_i \frac{N_i(k_i{}^2 - \omega^2)}{(k_i{}^2 - \omega^2)^2 + 4\omega^2 l_i{}^2}, \tag{83–7}$$

$$\kappa'' = \frac{e^2}{m}\sum_i \frac{2N_i\omega l_i}{(k_i{}^2 - \omega^2)^2 + 4\omega^2 l_i{}^2}, \tag{83–8}$$

where N_i is the number of electrons of type i per unit volume.

For the propagation of plane simple harmonic light waves in an isotropic medium, then, the constitutive relation (82–7) becomes

$$\mathbf{D}_e = \left\{\kappa' + i\left(\kappa'' + \frac{\sigma}{\omega}\right)\right\}\mathbf{E}. \tag{83–9}$$

Except for the manner in which they vary with frequency, it is impossible to distinguish between the physical effects of the conductivity

σ of the medium and of the imaginary part κ'' of the dielectric constant.

As $\mathbf{E}$ has the direction of $\mathbf{D}_e$ in an isotropic medium, $\mathbf{S}\cdot\mathbf{E} = 0$. Therefore the wave equation (82–12) reduces to

$$S^2 = S_0{}^2 \left\{ \kappa' + \mathrm{i}\left(\kappa'' + \frac{\sigma}{\omega} \right) \right\} \tag{83–10}$$

if we express $\mathbf{D}_e$ in terms of $\mathbf{E}$ by means of (83–9). Since this equation requires the wave-slowness to be complex, we put $\mathbf{S} = \mathbf{S}' + \mathrm{i}\mathbf{S}''$. As was shown in article 74, S' is the reciprocal of the actual velocity of propagation, and $\omega S''$ measures the attenuation of the wave as it progresses into the medium. The index of refraction of the medium is

$$n \equiv \frac{S}{S_0} = \sqrt{\kappa' + \mathrm{i}\left(\kappa'' + \frac{\sigma}{\omega} \right)} \tag{83–11}$$

As n is in general complex, it is customary to write

$$n = \nu(1 + \mathrm{i}\chi). \tag{83–12}$$

Then ν represents the ratio of the velocity of light *in vacuo* to the actual velocity in the medium, and χ is a measure of the absorption of the medium. In fact $\omega S'' = \omega S_0 \nu\chi = \omega S'\chi$. But $\omega S' = 2\pi/\lambda$, where λ is the wave-length in the medium. Hence the field vectors in a wave advancing through the medium in the X direction contain the factor $e^{-2\pi\chi x/\lambda}$, indicating that the amplitude falls to $1/e$ times its original value in a distance $\lambda/2\pi\chi$. The quantity χ is sometimes called the *index of absorption*.

Comparing (83–12) with (83–11) we find that

$$\nu^2(1 - \chi^2) = \kappa', \quad 2\nu^2\chi = \kappa'' + \frac{\sigma}{\omega}. \tag{83–13}$$

The term κ'' in the second expression is a measure of the electromagnetic energy dissipated in maintaining the oscillations of the bound electrons and the term in σ of the energy converted into heat in producing a current of free electrons. Eliminating χ we obtain

$$\nu^4 - \kappa'\nu^2 - \tfrac{1}{4}\left(\kappa'' + \frac{\sigma}{\omega} \right)^2 = 0$$

or

$$\nu^2 = \frac{\kappa'}{2}\left\{\sqrt{1 + \left(\frac{\kappa'' + \frac{\sigma}{\omega}}{\kappa'}\right)^2} + 1\right\}, \tag{83–14}$$

and then

$$\chi = \frac{\kappa'}{\left(\kappa'' + \frac{\sigma}{\omega}\right)}\left\{\sqrt{1 + \left(\frac{\kappa'' + \frac{\sigma}{\omega}}{\kappa'}\right)^2} - 1\right\}. \tag{83–15}$$

In a non-conducting dielectric in the region of transparency $\sigma = 0$ and $\kappa'' \ll \kappa'$. Hence $\nu^2 = \kappa'$ and $\chi = 0$. On the other hand, in a metal of high conductivity, such as silver, κ' and κ'' are very small compared with σ/ω and, to a first approximation, $\nu^2 = \sigma/2\omega$ and $\chi = 1$.

Next we shall write down the magnitudes of the three field vectors, E, D_e and H. If

$$E = A_0 e^{-\omega \mathbf{S}''\cdot\mathbf{r}} e^{i\omega(\mathbf{S}'\cdot\mathbf{r}-t)}, \tag{83–16}$$

then, from (82–14),

$$D_e = n^2 E = n^2 A_0 e^{-\omega \mathbf{S}''\cdot\mathbf{r}} e^{i\omega(\mathbf{S}'\cdot\mathbf{r}-t)} \tag{83–17}$$

since $\mathbf{E}$ is perpendicular to $\mathbf{S}$, and, from (82–13),

$$H = nE = nA_0 e^{-\omega \mathbf{S}''\cdot\mathbf{r}} e^{i\omega(\mathbf{S}'\cdot\mathbf{r}-t)}. \tag{83–18}$$

When the index of refraction n is complex, no two of the three field vectors are in phase. The vectors $\mathbf{E}$ and $\mathbf{H}$ are at right angles to the direction of propagation of the wave and at right angles to each other in such a sense that $\mathbf{E} \times \mathbf{H}$ has the direction of propagation.

The dispersion of a non-conducting dielectric is specified by (83–14) with $\sigma = 0$, the quantities κ' and κ'' being given by (83–5) and (83–6) if only one absorption band lies in the neighborhood of the frequency range under consideration, or by (83–7) and (83–8) if the effect of two or more absorption bands has to be taken into account. We shall discuss the first case in some detail, assuming the damping constant l to be small compared with the natural angular frequency k of the bound electrons.

First, for frequencies remote from resonance (ω not nearly equal

to k) we can neglect $2\omega l$ compared with $k^2 - \omega^2$. In this range of frequencies (83–5) and (83–6) give

$$\kappa' = 1 + \frac{a}{k^2 - \omega^2}, \quad \frac{\kappa''}{\kappa'} \text{ negligible,} \tag{83–19}$$

where we have put $a \equiv Ne^2/m$. Consequently (83–14) and (83–15) reduce to

$$\nu^2 = \kappa' = 1 + \frac{a}{k^2 - \omega^2}, \tag{83–20}$$

$$\chi \text{ negligible.} \tag{83–21}$$

As χ is negligibly small this is a region of transparency. The index of refraction is entirely real, ν^2 increasing with increasing frequency. For $\omega < k$, $\nu > 1$, whereas for $\omega > k$, $\nu < 1$. For common transparent dielectrics the nearest important absorption band is in the ultraviolet and consequently $\nu > 1$ in the visible region.

When $\omega = k$,

$$\kappa' = 1, \quad \kappa'' = \frac{a}{2\omega l}, \tag{83–22}$$

and

$$\nu^2 = \frac{1}{2}\left\{\sqrt{1 + \frac{a^2}{4k^2l^2}} + 1\right\}, \tag{83–23}$$

$$\chi = \frac{2kl}{a}\left\{\sqrt{1 + \frac{a^2}{4k^2l^2}} - 1\right\}. \tag{83–24}$$

If $2kl \ll a$, as is generally the case in solid or liquid dielectrics, χ is of the order of magnitude of unity and the radiation is strongly absorbed. In this region the dielectric is opaque.

To cover the entire range of frequency, we put $k^2 - \omega^2 \equiv \rho a \cos\theta$ and $2\omega l \equiv \rho a \sin\theta$. Then $\rho^2 a^2 = (k^2 - \omega^2)^2 + 4\omega^2 l^2$ and $\tan\theta = 2\omega l/(k^2 - \omega^2)$. Consequently $\kappa' = 1 + \frac{1}{\rho}\cos\theta$, $\kappa'' = \frac{1}{\rho}\sin\theta$ and

$$\nu^2 = \frac{1}{2\rho}\left\{\sqrt{\rho^2 + 2\rho\cos\theta + 1} + (\rho + \cos\theta)\right\}, \tag{83–25}$$

$$\chi = \frac{1}{\sin\theta}\left\{\sqrt{\rho^2 + 2\rho\cos\theta + 1} - (\rho + \cos\theta)\right\}, \tag{83–26}$$

showing that ν and χ are functions of only two parameters, ρ and θ.

If, now, we put $x \equiv \rho \cos\theta = (k^2 - \omega^2)/a$, $y \equiv \rho \sin\theta = 2\omega l/a$, we obtain, on eliminating ω, the parabola $a^2y^2 = 4l^2(k^2 - ax)$, whose intercepts are k^2/a on the X axis and $\pm\ 2lk/a$ on the Y axis. The upper half of this parabola is plotted in Fig. 86, where $\overline{AO} = 1$. If

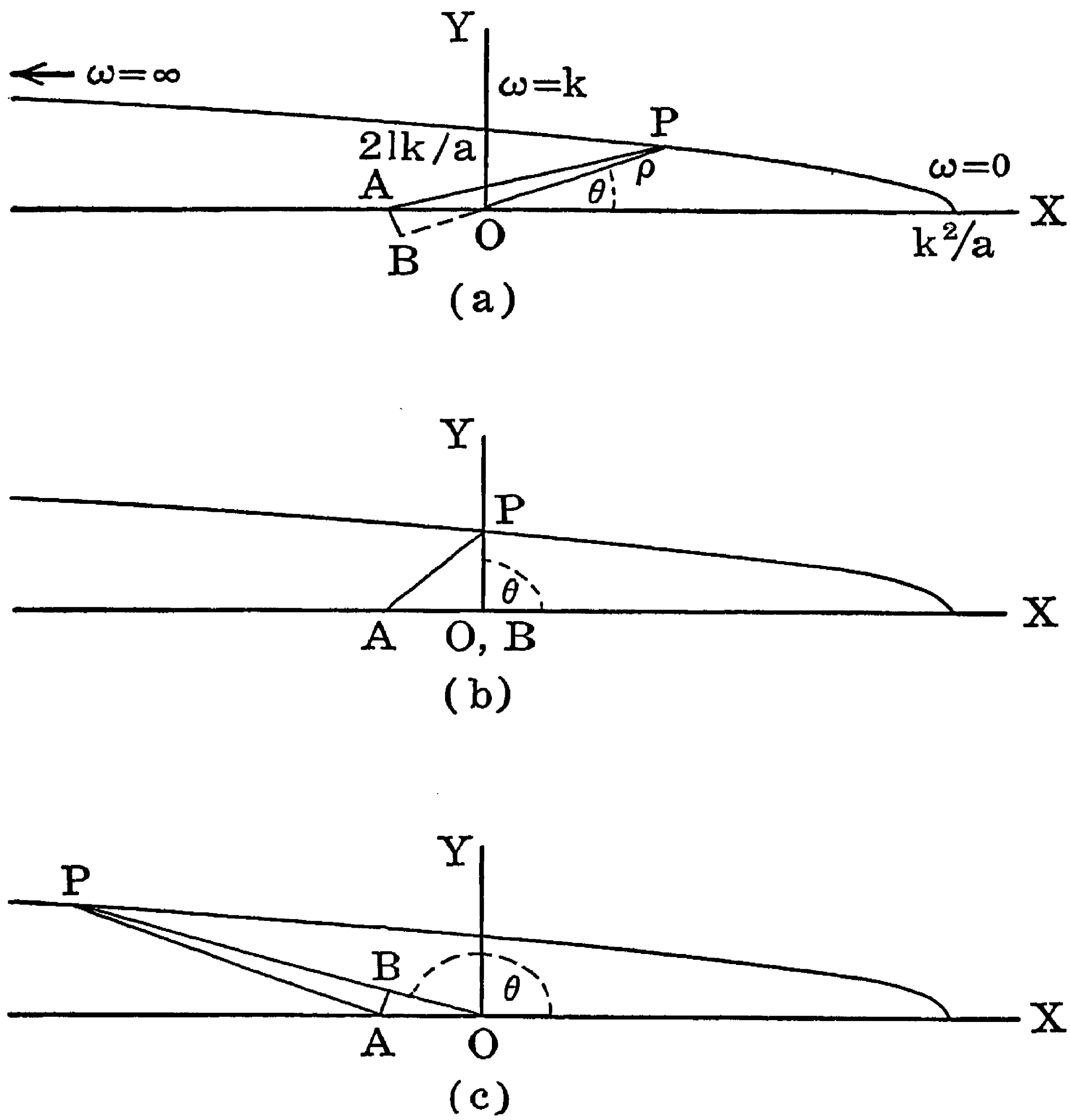

FIG. 86.

P is a point on the curve, $\overline{AP} = \sqrt{\rho^2 + 2\rho\cos\theta + 1}$, $\overline{BP} = \rho + \cos\theta$, and $\overline{AB} = \sin\theta$. Consequently

$$\nu^2 = \frac{\overline{AP} + \overline{BP}}{2\overline{OP}}, \tag{83–27}$$

$$\chi = \frac{\overline{AP} - \overline{BP}}{\overline{AB}}. \tag{83–28}$$

As ω increases from 0 to ∞, θ increases from 0 to π, being equal to

$\pi/2$ for the resonance frequency $\omega = k$. The three cases $\theta < \pi/2$, $\theta = \pi/2$, $\theta > \pi/2$ are illustrated in the three parts (*a*), (*b*), (*c*) of the figure, which is drawn for the values $k^2/a = 5$, $l^2/a = 0.05$. These make $l/k = 0.1$.

The dispersion curves for these values of the parameters are drawn to scale in Fig. 87. The solid curve represents $\nu - 1$ and the broken curve χ, both being plotted against the ratio ω/k of the frequency of the radiation to the natural frequency of the electronic oscillators.

The dispersion is said to be *normal* in the region *mn* of transparency where the real part of the index of refraction increases with increasing frequency. In the absorption band *no* the quantity ν

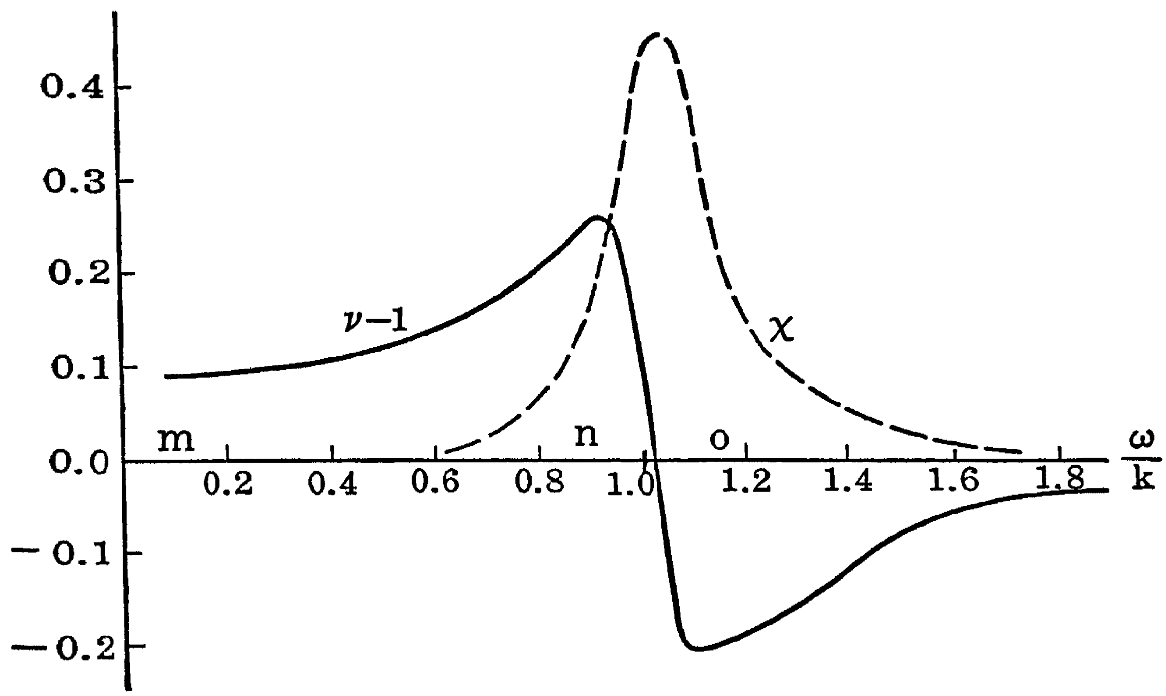

FIG. 87.

decreases very rapidly to a value less than unity as the frequency becomes greater. This phenomenon is known as *anomalous dispersion*. Finally, as the absorption band is passed, ν rises rapidly, approaching unity asymptotically as the frequency increases. If a second absorption band lies a little further to the right, it will soon make its effect felt by increasing ν to a value greater than unity.

A special case of the foregoing theory, which is of much interest, is that of a homogeneous free electron or ion gas. As was shown in article 63 the electric field acting on free particles is represented by the mean electric intensity **E**. Hence $\kappa = 1 + \alpha$, where α is the ratio of P to E, and, as the particles are subject to no force of restitution and, presumably, to no appreciable damping, $k_0 = l = 0$. There-

fore, replacing $\mathbf{E}_1$ by $\mathbf{E}$ in (83–2), we find that κ is entirely real and is given by

$$\kappa = 1 - \frac{Ne^2}{m\omega^2} = 1 - \frac{Ne^2}{4\pi^2 mc^2}\lambda^2 \tag{83–29}$$

for a wave-length λ. In actual examples of interest, the number of electrons per unit volume is so small that the magnitude of the second term is very much less than that of the first. Then

$$n = \sqrt{\kappa} = 1 - \frac{Ne^2}{8\pi^2 mc^2}\lambda^2. \tag{83–30}$$

As in the region of anomalous dispersion, we have here an index of refraction less than unity. Since $n - 1$ is equal to a very small constant multiplied by λ^2, the effect is inappreciable in the optical region, but becomes of importance in the region of long waves, such as those used in radio-communication. As the phase velocity increases with increasing density of free ions, the bending of radio waves so as to follow the curved surface of the earth can be explained by the increased ionization at higher altitudes.

It is important to note that although the phase velocity specified by (83–30) is greater than c, the group velocity is less. For, if we denote the first by V and the second by U, we have

$$V = c + \frac{Ne^2}{8\pi^2 mc}\lambda^2 \tag{83–31}$$

from (83–30). Hence the group velocity[1] U is

$$U = V - \lambda\frac{dV}{d\lambda} = c - \frac{Ne^2}{8\pi^2 mc}\lambda^2. \tag{83–32}$$

The energy which the bound electrons in a dielectric gas absorb from radiation passing through the medium is generally re-radiated as fast as it is absorbed in the form of waves which are related both in phase and in polarization to the incident waves. This phenomenon, known as *coherent scattering*, we shall now investigate. As the absorption of a gas is small, we can take κ to be real in the following calculation.

[1] L. Page, *Theoretical Physics*, 2nd Edit. p. 251.

Since $\mathbf{P} = (\kappa - 1)\mathbf{E}$, the electric moment $\mathbf{p}_E$ of a single molecule in a homogeneous gas containing N' molecules per unit volume is

$$\mathbf{p}_E = \frac{\kappa - 1}{N'}\mathbf{E} = \frac{\kappa - 1}{N'}\mathbf{E}_0 e^{-i\omega t},$$

where $\mathbf{E}_0$ is the vector amplitude and ω the angular frequency of the incident radiation. Take the X axis (Fig. 88) in the direction of propagation and the Y axis in the direction of $\mathbf{E}$. Then the Z axis has the direction of $\mathbf{H}$. Introducing spherical coordinates r, θ, ϕ

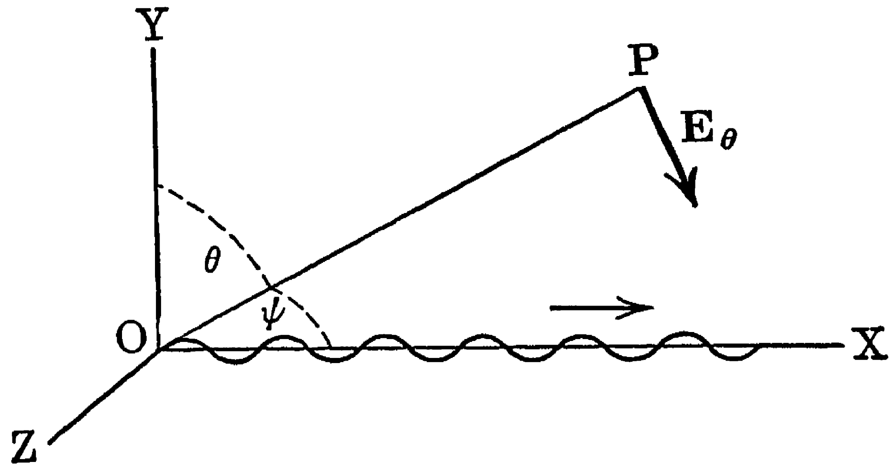

FIG. 88.

with $\overline{OY}$ as polar axis, we have from (72–23) for the radiation field at a distant point P of an induced dipole located at the origin O the two non-vanishing components

$$E_\theta = -\frac{(\kappa - 1)\epsilon^2}{4\pi\gamma N' r} E_0 \sin\theta\, e^{i(\epsilon r - \omega t)}$$

$$H_\phi = -\frac{\sqrt{\kappa}(\kappa - 1)\epsilon^2}{4\pi\gamma N' r} E_0 \sin\theta\, e^{i(\epsilon r - \omega t)},$$

where $\epsilon \equiv \frac{\omega}{v} = \omega S$ and, if the molecules are treated as spheres, $\gamma = (2\kappa + 1)/3$. Since, however, we are interested in a gas, for which κ is very nearly equal to unity, we can put $\gamma = 1$, $\sqrt{\kappa} = 1$ and $\kappa - 1 = n^2 - 1 = 2(n - 1)$, where n is the index of refraction. Hence we write

$$E_\theta = H_\phi = -\frac{2\pi(n - 1)}{N'\lambda^2 r} E_0 \sin\theta\, e^{i(\epsilon r - \omega t)} \qquad (83\text{–}33)$$

where λ is the wave-length. This gives for the mean Poynting flux at P

$$\bar{s} = c\overline{E_\theta H_\phi} = \frac{2\pi^2 c(n - 1)^2}{N'^2\lambda^4 r^2} E_0^2 \sin^2\theta, \qquad (83\text{–}34)$$

and for the mean rate of radiation of energy

$$\bar{\mathscr{R}} = \int_0^\pi \bar{s}\, 2\pi r^2 \sin\theta\, d\theta = \frac{16\pi^3 c(n-1)^2}{3N'^2\lambda^4} E_0^2. \qquad (83\text{–}35)$$

As the molecules of a gas are not regularly spaced they may be considered to scatter independently. Hence, as there are N' molecules per unit volume, the energy scattered by the molecules located in a rectangular parallelopiped of length dx and unit cross-section is

$$\frac{16\pi^3 c(n-1)^2}{3N'\lambda^4} E_0^2\, dx. \qquad (83\text{–}36)$$

The mean flow of energy into this parallelopiped in the incident beam is $I = \frac{1}{2}cE_0^2$. Hence, as (83–36) represents energy lost by the incident radiation, we are led to the equation

$$dI = -\frac{32\pi^3}{3N'\lambda^4}(n-1)^2 I\, dx \qquad (83\text{–}37)$$

for the increment dI in the intensity of the beam in progressing a distance dx through the medium.

As the scattering coefficient β is defined by the relation $dI = -\beta I\, dx$, this gives

$$\beta = \frac{32\pi^3}{3N'\lambda^4}(n-1)^2 \qquad (83\text{–}38)$$

By measuring β Fowle [2] has determined N' from this formula, obtaining therefrom the value $6.02(10)^{23}$ molecules per gram molecule for Loschmidt's constant, which agrees within the experimental error with the generally accepted value of $6.06(10)^{23}$.

Since the scattering coefficient (83–38) is inversely proportional to the fourth power of the wave-length, blue light is scattered much more than red. Therefore the sky is blue whereas the sun, particularly when near the horizon, has a more reddish tinge than would be the case if the earth had no atmosphere.

We have treated the incident radiation as if it were plane polarized with the electric vector parallel to the Y axis (Fig. 88). If it is plane polarized with the electric vector parallel to the Z axis, the angle θ in (83–33) and (83–34) must be taken as the angle between $\overline{OP}$

[2] F. E. Fowle, Astro. Jour. 40, p. 435 (1914).

and the Z axis. Now, if ψ is the angle which $\overline{OP}$ makes with the X axis, and γ the angle which the plane of $\overline{OX}$ and $\overline{OP}$ makes with the XY plane, $\sin^2 \theta = 1 - \sin^2 \psi \cos^2 \gamma$ in the first case and $\sin^2 \theta = 1 - \sin^2 \psi \sin^2 \gamma$ in the second. So, if the incident beam is unpolarized, $\sin^2 \theta$ in the expression for the Poynting flux must be replaced by the mean of $1 - \sin^2 \psi \cos^2 \gamma$ and $1 - \sin^2 \psi \sin^2 \gamma$, that is, by $\frac{1}{2}(1 + \cos^2 \psi)$. This gives

$$\bar{s} = \frac{\pi^2 c(n-1)^2}{N'^2 \lambda^4 r^2} E_0{}^2 (1 + \cos^2 \psi), \tag{83–39}$$

which, obviously, must lead to the same expressions (83–35) for the mean rate of radiation and (83–38) for the coefficient of scattering.

Evidently, then, the radiation scattered from an unpolarized incident beam is polarized. When $\sin \psi = \pm 1$, the polarization is complete. If, for instance, $\gamma = 0$, so that the point of observation P lies on the Y axis, the scattered light is due entirely to the component along the Z axis of the oscillations of the scattering electrons and is plane polarized with the electric vector parallel to the Z axis. These theoretical conclusions are in good agreement with observation.

(II) *Limited Wave Trains.* We have seen that the phase velocity of electromagnetic waves in a dielectric on the high frequency side of an absorption band, as well as in a free electron gas, is greater than the velocity of light *in vacuo*. In the early years of the twentieth century many physicists interpreted this phenomenon to mean that a signal can be transmitted with a speed greater than c. Since we now deduce electromagnetic theory from the relativity principle, we can confidently assert that such a conclusion is erroneous; nevertheless a detailed analysis of the problem is illuminating. In order to transmit a signal by means of a simple harmonic wave train it is necessary to interrupt or in some way to modulate the wave train. Sommerfeld and Brillouin [8] have investigated the essential features of the problem by determining the velocity in a dielectric of the forerunners of a wave train which is limited in length.

Consider an electric field which, at a point in the medium selected as origin, is equal to $\sin \frac{2\pi t}{P}$ for $0 < t < T$, where $T = NP$ is a whole number N of periods P, and is equal to zero for $t < 0$ and for

[8] Sommerfeld and Brillouin, Ann. d. Physik 44, p. 177 (1914).

$t > T$, as indicated in Fig. 89. We can express $E(t)$ from time $-\tau/2$ to time $\tau/2$ by the Fourier series

$$E(t) = a_0 + a_1 \cos\left(\frac{2\pi}{\tau}\right)t + a_2 \cos 2\left(\frac{2\pi}{\tau}\right)t + \cdots$$
$$+ b_1 \sin\left(\frac{2\pi}{\tau}\right)t + b_2 \sin 2\left(\frac{2\pi}{\tau}\right)t + \cdots.$$

As

$$\int_{-\tau/2}^{\tau/2} E(\xi) \cos n\left(\frac{2\pi}{\tau}\right)\xi\, d\xi = \tfrac{1}{2}a_n\tau,$$

$$\int_{-\tau/2}^{\tau/2} E(\xi) \sin n\left(\frac{2\pi}{\tau}\right)\xi\, d\xi = \tfrac{1}{2}b_n\tau,$$

for $n = 1, 2, 3, \cdots$, we have

$$E(t) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} E(\xi)d\xi + \frac{2}{\tau}\sum_{n=1}^{\infty}\int_{-\tau/2}^{\tau/2} E(\xi)\cos\frac{2\pi n}{\tau}(t-\xi)d\xi. \quad (83\text{–}40)$$

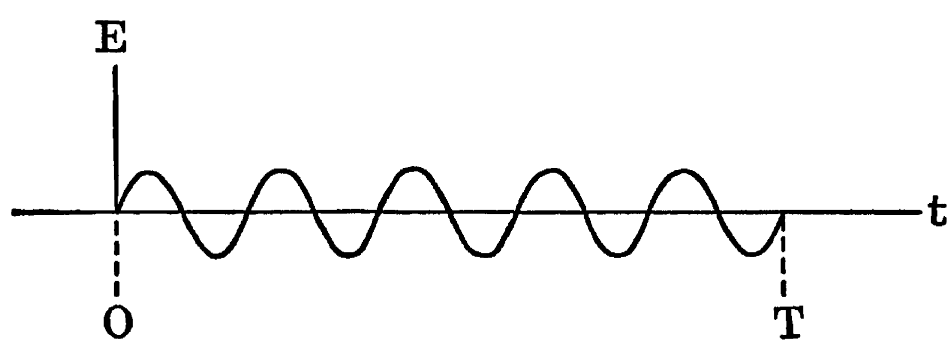

FIG. 89.

Now make τ very large and write $\omega \equiv 2\pi n/\tau$. Then, as $\Delta\omega = 2\pi/\tau$, the sum expressed by (83–40) becomes the Fourier integral

$$E(t) = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\tau/2}^{\tau/2} E(\xi)\cos\omega(t-\xi)d\xi. \quad (83\text{–}41)$$

Since $E(\xi)$ is equal to $\sin\dfrac{2\pi\xi}{P}$ in the interval $0 < \xi < T$ and vanishes outside this interval in the problem we are considering,

$$E(t) = \frac{1}{\pi}\int_0^{\infty} d\omega \int_0^{T} \sin\frac{2\pi\xi}{P}\cos\omega(t-\xi)d\xi. \quad (83\text{–}42)$$

Evaluating the integral with respect to ξ, remembering that T is an integral number of periods, we find

$$E(t) = \frac{2}{P}\int_0^\infty \frac{d\omega}{\omega^2 - \left(\frac{2\pi}{P}\right)^2}\left\{\cos\omega(t - T) - \cos\omega t\right\}$$

$$= \frac{4}{P}\int_0^\infty \frac{\sin\frac{\omega T}{2}}{\omega^2 - \left(\frac{2\pi}{P}\right)^2}\sin\omega(t - \tfrac{1}{2}T)d\omega. \qquad (83\text{–}43)$$

In the last expression we have represented the original oscillation of finite duration as the sum of an infinite number of sinusoidal oscillations of infinite duration having frequencies ranging all the way from 0 to ∞. The intensity of each partial oscillation of angular frequency ω is proportional to

$$J \equiv \left\{\frac{4}{P}\frac{\sin\frac{\omega T}{2}}{\omega^2 - \left(\frac{2\pi}{P}\right)^2}\right\}^2.$$

This function, which has its principal maximum at $\omega = 2\pi/P$, is represented by the curve in Fig. 90. In the double width of its

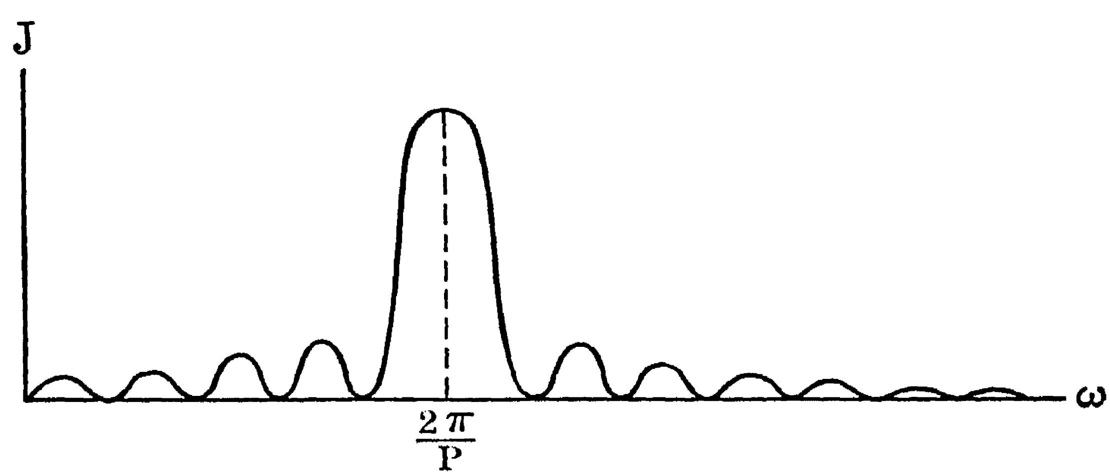

Fig. 90.

principal maximum, the curve is similar to that for the intensity in the single slit diffraction pattern.

Understanding, as usual, that the physical quantity is represented

by the real part of a function of a complex variable, we can write (83–43) in the form

$$E(t) = \frac{1}{2\pi}\left[\int_0^\infty \frac{d\omega}{\omega - \frac{2\pi}{P}}\{\cos\omega(t-T) - \cos\omega t\}\right.$$

$$\left. - \int_0^\infty \frac{d\omega}{\omega + \frac{2\pi}{P}}\{\cos\omega(t-T) - \cos\omega t\}\right]$$

$$= \frac{1}{2\pi}\left[\int_0^\infty \frac{d\omega}{\omega - \frac{2\pi}{P}}\{e^{-i\omega(t-T)} - e^{-i\omega t}\}\right.$$

$$\left. + \int_{-\infty}^0 \frac{d\omega}{\omega - \frac{2\pi}{P}}\{e^{-i\omega(t-T)} - e^{-i\omega t}\}\right]$$

$$= \frac{1}{2\pi}\int_\infty^{-\infty} \frac{d\omega}{\omega - \frac{2\pi}{P}} e^{-i\omega t} - \frac{1}{2\pi}\int_\infty^{-\infty} \frac{d\omega}{\omega - \frac{2\pi}{P}} e^{-i\omega(t-T)} \qquad (83–44)$$

To obtain the second integral in the intermediate step, we have replaced ω by $-\omega$, making the appropriate change in sign in the limits of the integral, and then have interchanged the limits so as to bring the plus sign in front of the integral.

Consider the first integral

$$E_1(t) = \frac{1}{2\pi}\int_\infty^{-\infty} \frac{d\omega}{\omega - \frac{2\pi}{P}} e^{-i\omega t} \qquad (83–45)$$

in (83–44). Writing $\omega = x' + iy'$, we shall take for the path of integration in the complex plane (Fig. 91) the line ab lying along the real axis just above the pole $\omega = 2\pi/P$ of the integrand. Now we can distort this path as much as we choose, provided we do not pass over a singularity of the integrand, without affecting the value of the integral. Since

$$e^{-i\omega t} = e^{y't}e^{-ix't}$$

we see that, by distorting the path to cd, along which y' is sufficiently large, we can make the factor $e^{y't}$ as small as we choose for any

$t < 0$. Hence $E_1(t) = 0$ for all $t < 0$. For $t > 0$, we distort the path to $efghij$. For the sections ef and ij the value of y' is an infinitely large negative number. Therefore the integral along these two sections vanishes. Furthermore the integral along hi annuls that along fg as they are taken in opposite senses. So we are left with the

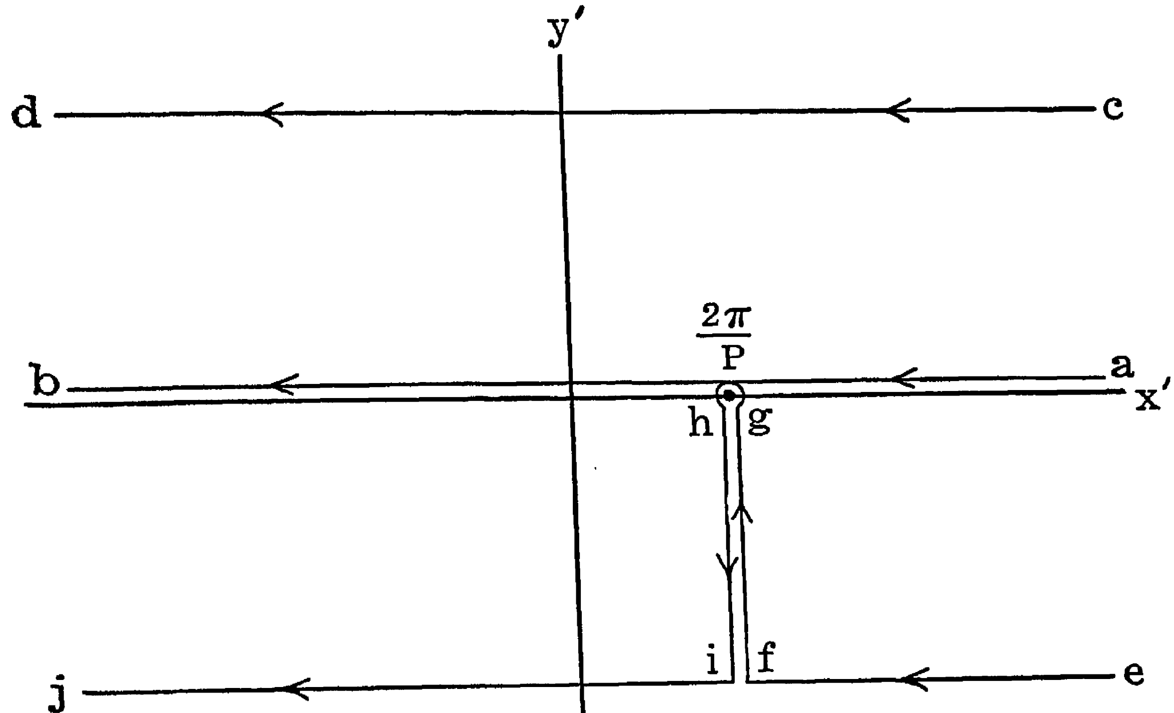

FIG. 91.

integral around the pole $\omega = 2\pi/P$. By Cauchy's integral theorem, then, $E_1(t) = \text{i}e^{-\text{i}\frac{2\pi t}{P}} = \sin\frac{2\pi t}{P}$ for all $t > 0$.

In a similar manner it is seen that the second integral

$$E_2(t) = -\frac{1}{2\pi}\int_{\infty}^{-\infty}\frac{d\omega}{\omega - \frac{2\pi}{P}}e^{-\text{i}\omega(t-T)} \tag{83–46}$$

in (83–44) vanishes for all $t < T$ and is equal to $-\sin\frac{2\pi t}{P}$ for all $t > T$. While the first integral initiates the oscillation at $t = 0$, the second terminates it at $t = T$. Although the combination of the two was needed to make the Fourier integral (83–42) converge, we can now dispense with the second and represent the field by $E_1(t)$ alone, since we are interested only in the speed of the head of the wave train.

Suppose, now, that the partial oscillations of which $E_1(t)$ is the sum are due to plane waves progressing through the medium in

the X direction. As each of these partial oscillations is of infinite duration, we can apply the theory we have developed in the earlier part of this article and write

$$E_1(t, x) = \frac{1}{2\pi}\int_{\infty}^{-\infty} \frac{d\omega}{\omega - \frac{2\pi}{P}} e^{-i\omega(t - Sx)}, \qquad (83\text{–}47)$$

the wave-slowness S being given by

$$\frac{S}{S_0} = \sqrt{1 + \frac{a}{(k^2 - \omega^2) - 2i\omega l}} = \sqrt{\frac{k^2 + a - 2i\omega l - \omega^2}{k^2 - 2i\omega l - \omega^2}} \qquad (83\text{–}48)$$

in accord with (83–3), where $a \equiv Ne^2/m$ as before.

In this case the singularities of the integrand in the complex $\omega = x' + iy'$ plane consist, in addition to the pole $\omega = 2\pi/P$, of a pair of branch cuts the extremities of which are determined by the values of ω which cause the numerator or the denominator of (83–48) to vanish. Equating the denominator to zero, we get the points

$$\omega = -il \pm \sqrt{k^2 - l^2}$$

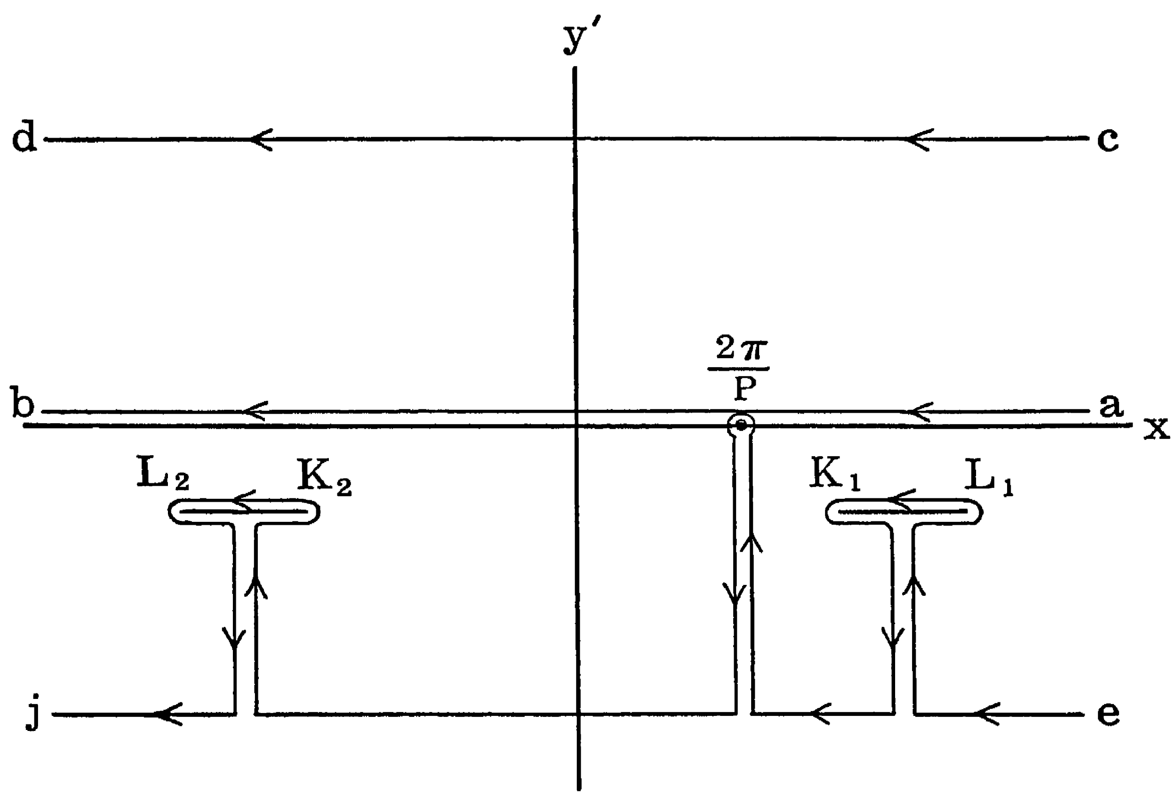

FIG. 92.

indicated on Fig. 92 by K_1 and K_2, and equating the numerator to zero, we find the points

$$\omega = -il \pm \sqrt{k^2 + a - l^2}$$

shown at L_1 and L_2.

Now we put $t' = t - S_0 x$ and investigate the electric field at any point x at all times for which $t' < 0$. Evidently these are times earlier than that at which the front of the wave train would reach the point x if it travelled with the velocity of light *in vacuo*. As the branch cuts lie in the lower half of the complex plane, we can change the path of integration from ab to cd without affecting the value of the integral. But, by making the value of y' for cd large enough, we can make the modulus of ω as great as we choose, with the result that $S = S_0$ from (83–48). Hence $t - Sx$ becomes $t - S_0 x = t'$ for this path, and, as in the case illustrated in Fig. 91, the function E_1 vanishes for all $t' < 0$. So we have proved that, even in the frequency range where the phase velocity of an infinite train of waves is greater than the velocity of light *in vacuo*, the head of a limited wave train never travels through an isotropic medium with a velocity greater than c.

By distorting the path of integration to $e \cdots j$ (Fig. 92) we can investigate the disturbance at times for which $t' > 0$. Then the integral reduces to the sum of the integral around the pole at $2\pi/P$ and the integrals around the branch cuts K_1L_1 and K_2L_2. Since $S = S' + iS''$ is complex, the first of these integrals is $e^{-2\pi S''x/P} \sin \dfrac{2\pi}{P}(t - S'x)$. If we designate the integrals around the branch cuts by b_1 and b_2, respectively,

$$E_1(t, x) = e^{-\frac{2\pi}{P} S''x} \sin \frac{2\pi}{P}(t - S'x) + b_1 + b_2$$

for $t' > 0$. The first term represents the steady oscillation existing in the case of an infinite wave train. We shall not evaluate b_1 and b_2, but shall content ourselves with pointing out that, since the imaginary part of ω is negative along both branch cuts, these terms contain a damping factor in the time, which causes them to approach zero asymptotically for large values of t'. A detailed calculation of these integrals shows that E_1 does not remain zero for $t' > 0$, but that a disturbance of very small amplitude and very high frequency starts at $t' = 0$, the amplitude increasing to $e^{-2\pi S''x/P}$ and the frequency decreasing to $1/P$ as time goes on. Therefore the initial disturbance travels with exactly the velocity of light *in vacuo*, quite irrespective of the dispersive properties of the medium.

84. Homogeneous Anisotropic Dielectric. — In article 63 energy considerations were invoked to show that the permittivity dyadic $\mathbf{K}$

of a perfect anisotropic dielectric must be symmetric. If, then, we orient the coordinate axes so as to put **K** in its normal form,

$$\mathbf{K} = \kappa_x \boldsymbol{ii} + \kappa_y \boldsymbol{jj} + \kappa_z \boldsymbol{kk}.$$

The directions defined by the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are known as the *electrical axes* of the crystal. As we shall consider only non-conducting transparent crystals, the vector $\mathbf{D}_e$ used in article 82 becomes the electric displacement **D**, which is connected to **E** by the constitutive relation

$$\mathbf{D} = \mathbf{K}\cdot\mathbf{E}, \tag{84–1}$$

where the elements of **K** are real.

The only modification required in the field equations (82–11) and the wave equation (82–12) is to replace $\mathbf{D}_e$ by **D**. As noted in article 82, the vectors **D**, **E** and **S** lie in a plane perpendicular to **H** with **D** at right angles to **S** in such a sense that $\mathbf{D} \times \mathbf{H}$ has the direction of propagation of the wave. The Poynting flux $\mathbf{s} = c\mathbf{E} \times \mathbf{H}$ is at right angles to both **E** and **H** and consequently lies in the plane of **D**, **E** and **S**, making the same angle α with **S** that **E** makes with **D**. The direction of the flow of energy, therefore, does not in general coincide with the wave-normal. Consequently a train of limited wave fronts side-steps as it advances into the medium. This is indicated in Fig. 93, where the relative orientation of the five vectors **D**, **E**, **S**, **s** and **H** are shown, the first four lying in the plane of the paper and the last being directed along the normal to this plane toward the reader.

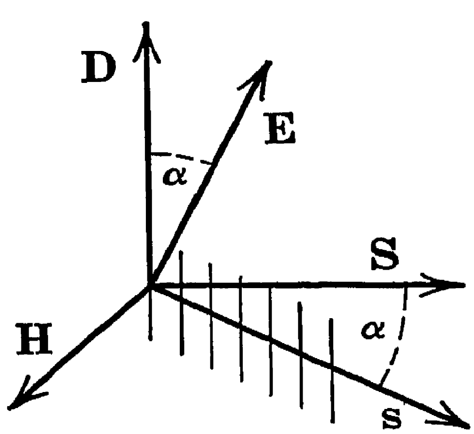

Fig. 93.

We can eliminate either **D** or **E** from the wave equation (82–12) by means of the constitutive relation (84–1), obtaining

$$S_0^2\mathbf{K}\cdot\mathbf{E} + \mathbf{SS}\cdot\mathbf{E} - \mathbf{S}\cdot\mathbf{SE} = 0 \tag{84–2}$$

in the first case, and

$$S_0^2\mathbf{D} + \mathbf{SS}\cdot\mathbf{K}^{-1}\cdot\mathbf{D} - \mathbf{S}\cdot\mathbf{SK}^{-1}\cdot\mathbf{D} = 0 \tag{84–3}$$

in the second. As (84–3) is the equation to which we shall devote most of our attention, we shall simplify our notation by putting

$$\boldsymbol{\Phi} \equiv \frac{1}{S_0^2}\mathbf{K}^{-1} = a^2\boldsymbol{ii} + b^2\boldsymbol{jj} + c^2\boldsymbol{kk}, \tag{84–4}$$

where

$$a \equiv \frac{1}{S_0\sqrt{\kappa_x}}, \quad b \equiv \frac{1}{S_0\sqrt{\kappa_y}}, \quad c \equiv \frac{1}{S_0\sqrt{\kappa_z}}. \tag{84–5}$$

Evidently a, b and c are the three phase velocities corresponding, respectively, to the three principal permittivities κ_x, κ_y and κ_z. Then (84–3) becomes

$$\mathbf{D} + \mathbf{SS}\cdot\mathbf{\Phi}\cdot\mathbf{D} - \mathbf{S}\cdot\mathbf{S}\mathbf{\Phi}\cdot\mathbf{D} = 0. \tag{84–6}$$

Now let $\boldsymbol{\gamma}$ be a vector having the direction of $\mathbf{D}$ and the magnitude of S. As (84–6) is linear and homogeneous in $\mathbf{D}$, the same equation is satisfied by $\boldsymbol{\gamma}$. Hence

$$\boldsymbol{\gamma} + \mathbf{SS}\cdot\mathbf{\Phi}\cdot\boldsymbol{\gamma} - \mathbf{S}\cdot\mathbf{S}\mathbf{\Phi}\cdot\boldsymbol{\gamma} = 0. \tag{84–7}$$

Taking the scalar product of $\boldsymbol{\gamma}$ by this equation, remembering that $\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = \mathbf{S}\cdot\mathbf{S}$ and that $\boldsymbol{\gamma}\cdot\mathbf{S} = 0$, we get

$$\boldsymbol{\gamma}\cdot\mathbf{\Phi}\cdot\boldsymbol{\gamma} = 1. \tag{84–8}$$

Therefore, if the origin of the vector $\boldsymbol{\gamma}$ is fixed and its direction varied, its terminus describes an ellipsoid, an elliptical section of which is shown in Fig. 94. Differentiating (84–8),

$$d\boldsymbol{\gamma}\cdot\mathbf{\Phi}\cdot\boldsymbol{\gamma} = 0, \tag{84–9}$$

since $\mathbf{\Phi}$ is symmetric. The vector $\boldsymbol{\beta} \equiv \mathbf{\Phi}\cdot\boldsymbol{\gamma}$ is therefore normal to the tangent plane through the terminus P of $\boldsymbol{\gamma}$. But, since $\mathbf{\Phi}$ is proportional to the reciprocal of $\mathbf{K}$, $\boldsymbol{\beta}$ has the direction of $\mathbf{E}$. Furthermore, as (84–8) may be written $\boldsymbol{\gamma}\cdot\boldsymbol{\beta} = 1$, the magnitude of $\boldsymbol{\beta}$ is the reciprocal of the distance $\overline{OQ}$ on the figure. Now we have seen that $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{S}$ lie in the same plane, with $\mathbf{S}$ perpendicular to $\mathbf{D}$. So $\mathbf{S}$ lies in the plane of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ at right angles to $\boldsymbol{\gamma}$, and is equal in magnitude to γ. For an assigned direction of the electric displacement, then, there is in general one and only one line along which the wave can advance. This is very different from the situation existing in an isotropic medium, where the wave may progress in any direction perpendicular to $\mathbf{D}$.

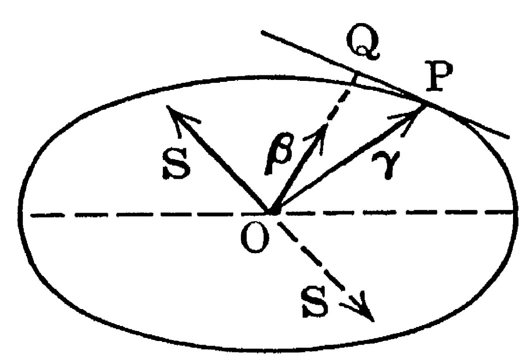

Fig. 94.

If the direction of propagation is given, $\boldsymbol{\gamma}$ or $\mathbf{D}$ is similarly restricted in direction. Since $\boldsymbol{\gamma}$ must be perpendicular to $\mathbf{S}$, we take a section

(Fig. 95) of the ellipsoid (84–8) at right angles to **S**. Then, as **S**, $\boldsymbol{\gamma}$ and the normal to the tangent plane are coplanar, the electric displacement is limited to the directions $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ parallel to the axes of the elliptical section. A wave polarized with **D** parallel to $\boldsymbol{\gamma}_1$ advances with a wave-slowness equal to γ_1, one with **D** parallel to $\boldsymbol{\gamma}_2$ with a wave-slowness equal to γ_2. A wave propagated in the specified direction in which **D** has neither of these two possible directions splits into two perpendicularly polarized waves advancing with different velocities. For this reason a crystal gives rise to the phenomenon of double refraction.

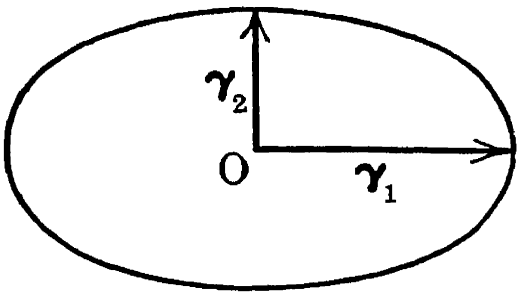

FIG. 95.

We shall designate the wave velocity by **V**. This vector has the direction of **S** and a magnitude equal to the reciprocal of S. Also we shall introduce a vector **v**, known as the *ray velocity*, which has the direction of the Poynting flux **s** and a magnitude equal to the speed of flow of energy. From Fig. 93 we see that $v = V \sec \alpha$ and that $\mathbf{v}\cdot\mathbf{S} = 1$. We shall find the equation of the surface described by the terminus of the vector **v** when its origin is held fixed at the origin O of coordinates. This surface, we shall prove, is also the envelope of all plane wave-fronts which passed through the origin one second earlier. Known as the *Fresnel wave-surface*, it completely describes the propagation of plane waves in an anisotropic dielectric.

Writing $\boldsymbol{\beta}$ in place of $\boldsymbol{\Phi}\cdot\boldsymbol{\gamma}$ in (84–7) this equation becomes

$$\boldsymbol{\gamma} = \mathbf{S}\cdot\mathbf{S}\boldsymbol{\beta} - \mathbf{S}\mathbf{S}\cdot\boldsymbol{\beta} = \mathbf{S}\times(\boldsymbol{\beta}\times\mathbf{S}), \qquad (84\text{–}10)$$

and

$$\boldsymbol{\gamma}\times\mathbf{v} = \{\mathbf{S}\times(\boldsymbol{\beta}\times\mathbf{S})\}\times\mathbf{v} = \mathbf{v}\cdot\mathbf{S}\boldsymbol{\beta}\times\mathbf{S} - \mathbf{v}\cdot\boldsymbol{\beta}\times\mathbf{S}\mathbf{S} = \boldsymbol{\beta}\times\mathbf{S}$$

since $\mathbf{v}\cdot\boldsymbol{\beta}\times\mathbf{S} = 0$ and $\mathbf{v}\cdot\mathbf{S} = 1$. Therefore

$$\mathbf{v}\times(\boldsymbol{\gamma}\times\mathbf{v}) = \mathbf{v}\times(\boldsymbol{\beta}\times\mathbf{S}) = \mathbf{v}\cdot\mathbf{S}\boldsymbol{\beta} - \mathbf{v}\cdot\boldsymbol{\beta}\mathbf{S} = \boldsymbol{\beta}$$

as **v** and $\boldsymbol{\beta}$ are perpendicular (Fig. 93). Since $\boldsymbol{\beta} = \boldsymbol{\Phi}\cdot\boldsymbol{\gamma}$ we may write this equation in the form

$$(\boldsymbol{\Phi} + \mathbf{v}\mathbf{v} - v^2\mathbf{I})\cdot\boldsymbol{\gamma} = 0, \qquad (84\text{–}11)$$

where **I** is the unit dyadic. As (84–11) indicates that the dyadic $\mathbf{X} \equiv \boldsymbol{\Phi} + \mathbf{v}\mathbf{v} - v^2\mathbf{I}$ is planar, either its antecedents or its consequents must be coplanar. If we put $\mathbf{v} \equiv ix + jy + kz$, $r^2 \equiv x^2 + y^2 + z^2$,

then x, y, z are the coordinates of a point on the Fresnel wave-surface. The dyadic $\mathbf{X}$ may then be written

$$\begin{aligned}\mathbf{X} = \boldsymbol{i}\{&\boldsymbol{i}(a^2 - r^2 + x^2) + \boldsymbol{j}xy + \boldsymbol{k}xz\} \\ + \boldsymbol{j}\{&\boldsymbol{i}yx + \boldsymbol{j}(b^2 - r^2 + y^2) + \boldsymbol{k}yz\} \\ + \boldsymbol{k}\{&\boldsymbol{i}zx + \boldsymbol{j}zy + \boldsymbol{k}(c^2 - r^2 + z^2)\}.\end{aligned} \qquad (84\text{–}12)$$

As the antecedents of $\mathbf{X}$ are not coplanar, the consequents must be. Therefore the triple scalar product of the consequents vanishes, and the equation of the Fresnel wave-surface is

$$\begin{vmatrix} a^2 - r^2 + x^2 & xy & xz \\ yx & b^2 - r^2 + y^2 & yz \\ zx & zy & c^2 - r^2 + z^2 \end{vmatrix} = 0, \quad (84\text{–}13)$$

which reduces to

$$(r^2 - b^2)(r^2 - c^2)x^2 + (r^2 - c^2)(r^2 - a^2)y^2 + (r^2 - a^2)(r^2 - b^2)z^2 = (r^2 - a^2)(r^2 - b^2)(r^2 - c^2). \quad (84\text{–}14)$$

Before discussing this surface we must prove that it is the envelope of all plane wave-fronts which passed through the origin one second earlier. According to (84–9), $d\boldsymbol{\gamma}\cdot\boldsymbol{\beta} = 0$. Hence, as $\boldsymbol{\gamma}\cdot\boldsymbol{\beta} = 1$, it follows that $\boldsymbol{\gamma}\cdot d\boldsymbol{\beta} = 0$. Now the scalar product of the differential of (84–10) with $\boldsymbol{\beta}$ is

$$d\boldsymbol{\gamma}\cdot\boldsymbol{\beta} = \mathbf{S}\cdot\mathbf{S}\,\boldsymbol{\beta}\cdot d\boldsymbol{\beta} - \mathbf{S}\cdot\boldsymbol{\beta}\,\mathbf{S}\cdot d\boldsymbol{\beta} + 2\boldsymbol{\beta}\cdot\boldsymbol{\beta}\,\mathbf{S}\cdot d\mathbf{S} - 2\mathbf{S}\cdot\boldsymbol{\beta}\,\boldsymbol{\beta}\cdot d\mathbf{S} = 0.$$

But the scalar product of (84–10) with $d\boldsymbol{\beta}$ is

$$\boldsymbol{\gamma}\cdot d\boldsymbol{\beta} = \mathbf{S}\cdot\mathbf{S}\,\boldsymbol{\beta}\cdot d\boldsymbol{\beta} - \mathbf{S}\cdot\boldsymbol{\beta}\,\mathbf{S}\cdot d\boldsymbol{\beta} = 0.$$

Subtracting, and dividing by 2, we have

$$\boldsymbol{\beta}\cdot\boldsymbol{\beta}\,\mathbf{S}\cdot d\mathbf{S} - \mathbf{S}\cdot\boldsymbol{\beta}\,\boldsymbol{\beta}\cdot d\mathbf{S} = \{\boldsymbol{\beta}\times(\mathbf{S}\times\boldsymbol{\beta})\}\cdot d\mathbf{S} = 0.$$

Now $\mathbf{v}$ has the direction of $\boldsymbol{\beta}\times(\mathbf{S}\times\boldsymbol{\beta})$, since it lies in the plane of $\mathbf{S}$ and $\boldsymbol{\beta}$ at right angles to $\boldsymbol{\beta}$. Therefore $\mathbf{v}\cdot d\mathbf{S} = 0$. But we have seen that $\mathbf{v}\cdot\mathbf{S} = 1$. Consequently $d\mathbf{v}\cdot\mathbf{S} = 0$. The last relation proves that the plane tangent to the Fresnel wave-surface at the terminus of $\mathbf{v}$ is perpendicular to $\mathbf{S}$. Therefore the tangent plane is the wave-front which left the origin one second earlier, and the surface described by the terminus of $\mathbf{v}$ is the envelope of all such wave-fronts. The

wave velocity **V** is given in magnitude as well as in direction by the perpendicular dropped from the origin to the tangent plane, and the electric displacement, since it lies in the plane of **V** and **v** (Fig. 93), has the direction of the line of intersection of the plane of these two vectors with the tangent plane.

In our discussion of the Fresnel wave-surface (84–14) we can assume that $a < b < c$ without loss of generality. The trace of the surface on the YZ plane, obtained by putting $x = 0$, is

$$(r^2 - a^2)\left(\frac{y^2}{c^2} + \frac{z^2}{b^2} - 1\right) = 0,$$

which represents a circle of radius a inside an ellipse of semi-axes c and b. On the ZX plane we find

$$(r^2 - b^2)\left(\frac{z^2}{a^2} + \frac{x^2}{c^2} - 1\right) = 0,$$

a circle of radius b which cuts an ellipse of semi-axes a and c; and on the XY plane

$$(r^2 - c^2)\left(\frac{x^2}{b^2} + \frac{y^2}{a^2} - 1\right) = 0,$$

a circle of radius c outside an ellipse of semi-axes b and a.

As the surface is symmetrical with respect to the coordinate planes, we need construct only the one octant shown in Fig. 96*a*. The surface consists of two sheets which intersect at Q. The state of polarization, determined by the fact that **V**, **v** and **D** are coplanar, is indicated by double-headed arrows where **D** is tangent to a trace, and by dots where **D** is normal.

The *primary optic axes* are those lines along which the wave velocity V is the same for both states of polarization. They are given, therefore, by the normals $\overline{OP}$ to the common tangents $\overline{AB}$ to the two traces in the XZ plane. Evidently there are two primary optic axes in the general crystal with three unequal permittivities which we are here discussing, the angle which the optic axis not shown makes with the Z axis being the negative of the angle θ which $\overline{OP}$ makes with $\overline{OZ}$. For this reason the crystal is called *bi-axial*. To find the angle θ we must make the straight line

$$z \cos\theta + x \sin\theta = b,$$

which is tangent to the circular trace for any value of θ, also tangent to the elliptical trace

$$\frac{z^2}{a^2} + \frac{x^2}{c^2} = 1.$$

Eliminating x between these two equations and then equating to zero the discriminant of the resulting quadratic in z, so as to make the two points of intersection of the straight line with the ellipse coincident, we find

$$\sin\theta = \pm\sqrt{\frac{b^2 - a^2}{c^2 - a^2}}, \quad \cos\theta = \pm\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}. \qquad (84\text{–}15)$$

The *secondary optic axes* are those lines along which the ray velocity v is the same for both states of polarization. They are given,

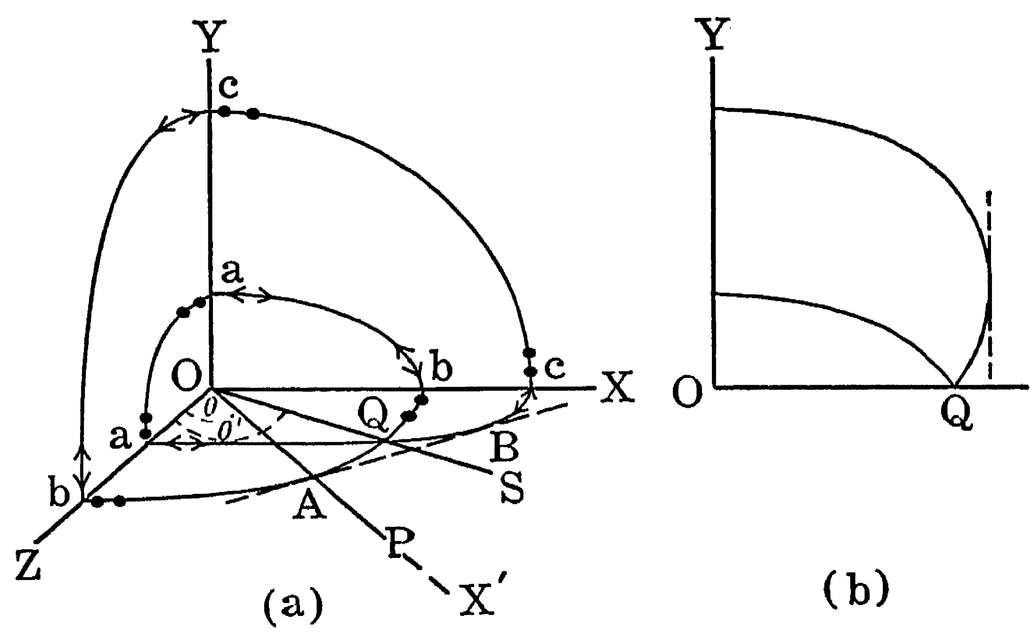

Fig. 96.

therefore, by the lines $\overline{OS}$ through the points of intersection of the two traces on the ZX plane. As with the primary axes, there are two secondary optic axes in the general crystal, the angle which the one makes with the Z axis being the negative of that made by the other. We easily find for the angle θ' between $\overline{OS}$ and $\overline{OZ}$,

$$\sin\theta' = \pm\frac{c}{b}\sqrt{\frac{b^2 - a^2}{c^2 - a^2}}, \quad \cos\theta' = \pm\frac{a}{b}\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}. \qquad (84\text{–}16)$$

Since the coefficients a, b, c are functions of κ_x, κ_y, κ_z, respectively, and these in turn are functions of frequency, the directions of the optic axes vary with the frequency of the radiation passing through

a bi-axial crystal. In the visible region, however, this variation is small.

The outer sheet of the Fresnel wave-surface has a dimple at the point Q, as illustrated in Fig. 96*b*. Now we shall prove the remarkable fact that the plane through $\overline{AB}$ perpendicular to the ZX plane, whose trace on the section shown in Fig. 96*b* is indicated by the broken line, makes contact with the surface without cutting it, along the circumference of a circle. This property of the surface gives rise to the phenomenon of *conical refraction*, which we shall discuss shortly.

First we introduce a new set of axes $X'YZ'$ with the X' axis along the primary optic axis $\overline{OP}$. The transformations to the new coordinates are

$$z = x' \cos\theta + z' \sin\theta,$$

$$x = x' \sin\theta - z' \cos\theta,$$

where $\sin\theta$ and $\cos\theta$ are defined by (84–15) with positive signs before the radical. In these coordinates the equation of the plane through $\overline{AB}$ is $x' = b$. Eliminating x' between this equation and the equation (84–14) of the Fresnel wave-surface expressed in terms of x', y, z', we get

$$b^2(y^2 + z'^2)^2 + 2b(y^2 + z'^2)z' \sqrt{(c^2 - b^2)(b^2 - a^2)} + z'^2(c^2 - b^2)(b^2 - a^2) = 0$$

for the equation of the curves in which the plane intersects the wave-surface. But, as the expression on the left of this equation is a perfect square, the two intersections are the same, that is, the contact is tangential. Taking the square root we find for the curve of contact the circle

$$y^2 + \left\{ z' + \frac{1}{2b} \sqrt{(c^2 - b^2)(b^2 - a^2)} \right\}^2 = \frac{1}{4b^2}(c^2 - b^2)(b^2 - a^2). \quad (84\text{–}17)$$

Consider, now, a slab of crystal (Fig. 97) cut with its parallel faces perpendicular to a primary optic axis, the points O, A, B corresponding to the similarly lettered points in Fig. 96*a*. If a plane wave *uw*, incident normally on the lower face of the crystal, is diaphragmed by the screen MN with a small circular aperture at O, the wave spreads through the crystal in the form of a hollow cone, which emerges above as a hollow cylinder. The polarization, indicated in part in the elevation, is shown in more detail in the plan above. This phenomenon is known as *internal conical refraction*.

Again, if the crystal is cut with its parallel faces perpendicular to a secondary optic axis (Fig. 98), the incident light consisting of rays converging on the aperture in the screen *MN*, the rays emerging from

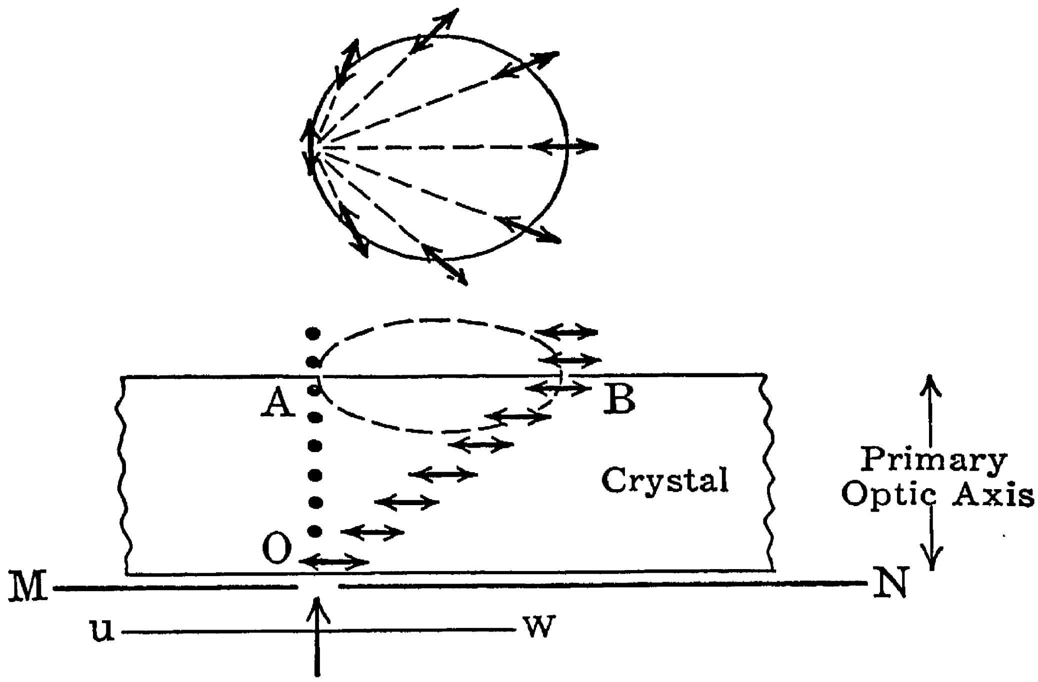

FIG. 97.

the opposite aperture in the screen *M′N′* form a hollow cone above the crystal. In this case we have *external conical refraction*.

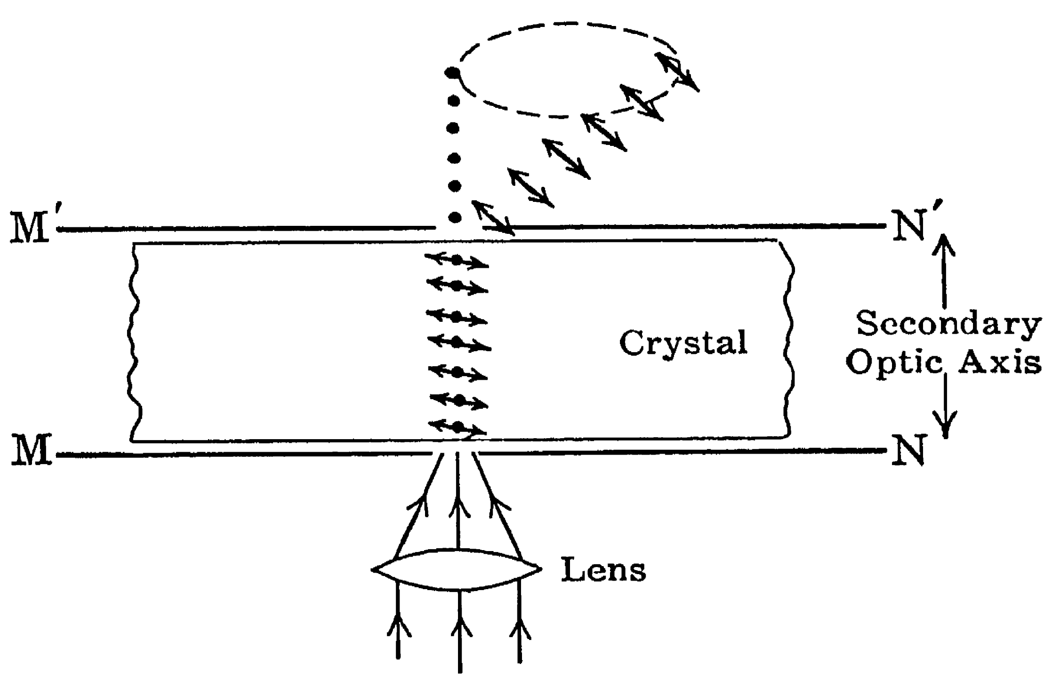

FIG. 98.

The phenomena of conical refraction, easily observed with aragonite, were not discovered until Hamilton had proved their existence theoretically.

Up to this point we have devoted our attention to bi-axial crystals in which the three principal permittivities are all different. Now we shall consider special cases in which two of the three elements of the dyadic Φ are the same.

If $c = b$ the Fresnel wave-surface becomes a prolate spheroid of semi-axes a and b inside a sphere of radius b, as indicated in Fig. 99*a*. All the optic axes become coincident with the X axis, so that the crystal is uni-axial and there is no longer any distinction between primary and secondary axes. Furthermore, as the optic axis coincides with one of the electrical axes, its direction is independent of the frequency of the radiation traversing the medium. Waves in the

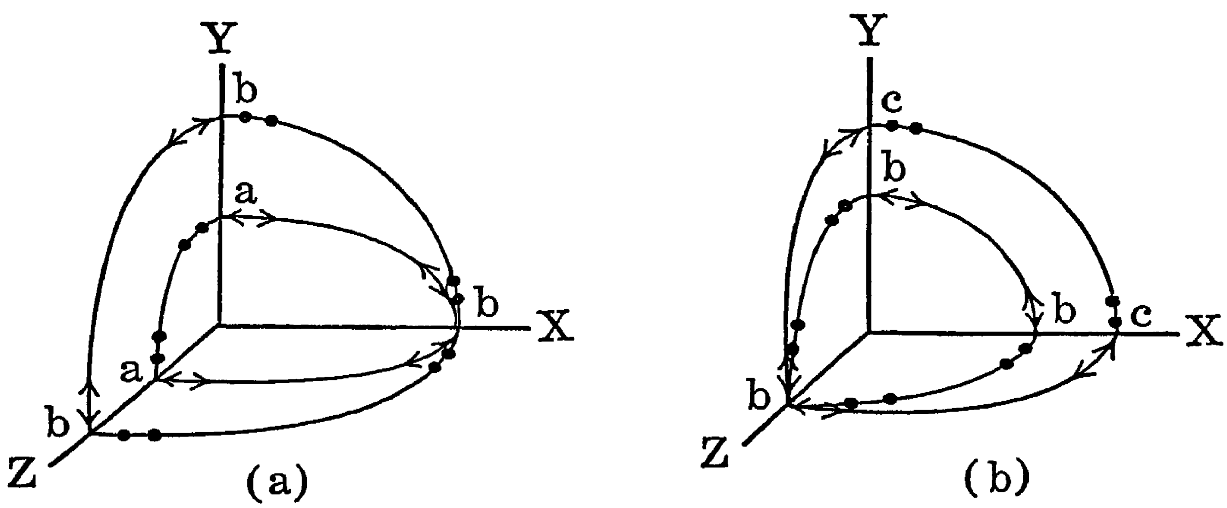

FIG. 99.

state of polarization characteristic of the spherical sheet are called *ordinary* since they have a velocity independent of the direction of propagation, whereas waves in the state of polarization characteristic of the spheroidal sheet are known as *extraordinary* since the velocity depends upon the direction of propagation. Huygens' construction for finding the angle of refraction of the ordinary ray o and the extraordinary ray e is shown in Fig. 100*a*, the incident ray being denoted by i. Except for the case where both rays move along the optic axis, the index of refraction n_e of the extraordinary ray is greater than the index of refraction n_o of the ordinary ray. A crystal of the type under consideration is known as *prolate*. Quartz is an example.

If $a = b$ the Fresnel wave-surface becomes the sphere inside the oblate spheroid shown in Fig. 99*b*, the single optic axis of the uni-axial crystal coinciding with the Z axis. The ordinary waves, as in the previous case, are those in the state of polarization character-

istic of the spherical sheet, and the extraordinary waves those in the state of polarization characteristic of the spheroidal sheet. Huygens' construction for finding the angle of refraction is shown in Fig. 100*b*. Evidently n_e is less than n_o except for propagation along the optic axis, when the two are equal. A crystal of this type, of which Iceland spar is an example, is called *oblate*.

The extreme index of refraction n_E of the extraordinary ray, as well as the index of refraction n_o of the ordinary ray, can be measured for any uni-axial crystal by cutting a prism with the optic axis parallel to the refracting edge. Then unpolarized plane waves, incident

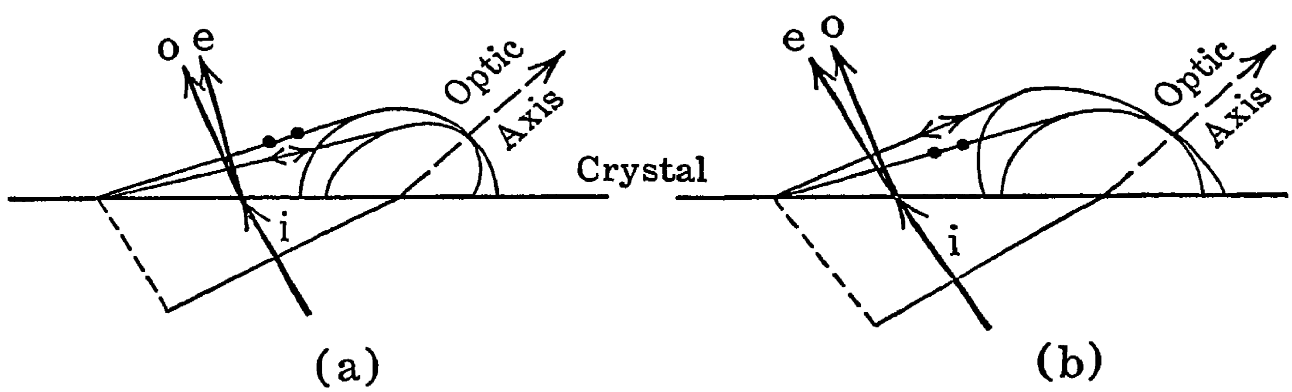

FIG. 100.

on the prism, split into two beams polarized at right angles which travel through the prism with different velocities, and therefore suffer different deviations. If δ is the deviation at minimum deviation and α the angle of the prism, the index of refraction is given by the elementary formula

$$n = \frac{\sin\dfrac{\alpha+\delta}{2}}{\sin\dfrac{\alpha}{2}}.$$

Consider a slab cut from a uni-axial crystal with its two parallel faces parallel to the optic axis. Plane waves, incident normally on one of the surfaces, will in general split into two perpendicularly polarized components which travel with different velocities through the crystal. If the thickness of the slab is such as to cause one component to lose a quarter wave-length on the other at emergence, circularly polarized incident light will emerge as plane polarized light in which the electric vector makes an angle of 45° with the optic axis, and *vice versa*. Such a device is known as a *quarter-wave plate*. Since

the thickness of a quarter-wave plate depends upon the wave-length used, a plate of adjustable thickness, composed of two wedges having triangular sections in a plane at right angles to the optic axis, is often used. By sliding one wedge over the other the thickness of the combination is varied. This device is the essential feature of the *Babinet compensator*. Obviously it can be used to produce relative retardations of the two components other than a quarter wave-length, if desired.

The Nicol prism (Fig. 101) consists of a rhomb of Iceland spar, an oblate crystal, with its optic axis perpendicular to the plane of the figure. The rhomb is split along the plane AB into two halves, which are cemented together with Canada balsam. Light incident on the left-hand face splits into two components on entering the prism, of which the ordinary ray, on account of its lower velocity, is

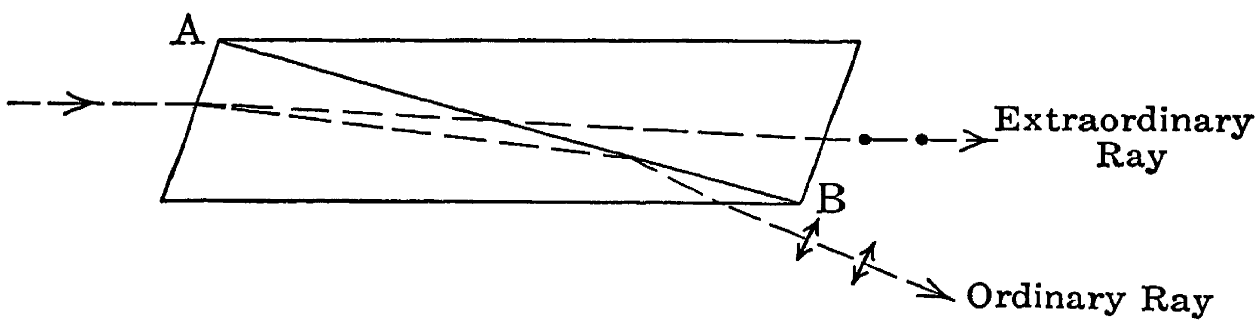

FIG. 101.

refracted the more. This ray meets the surface AB at an angle greater than the critical angle, and is totally reflected. Hence the emergent light consists solely of the extraordinary ray, which is plane polarized with the electric vector parallel to the optic axis.

85. Homogeneous Isotropic Dielectric in Uniform Magnetic Field. — If light passes through an isotropic dielectric placed in a strong external magnetostatic field $\mathbf{H}_0$, we cannot neglect the term involving $\mathbf{H}_0$ on the right-hand side of the equation of motion (83–1) of a bound electron. So (82–4) gives us

$$\left\{(k_0{}^2 - \omega^2) - 2i\omega l\right\}\mathbf{r} = \frac{e}{m}\mathbf{E}_1 - i\frac{e\omega}{mc}\mathbf{r}\times\mathbf{H}_0. \qquad (85\text{–}1)$$

In accord with (29–2) we can write $\mathbf{r}\times\mathbf{H}_0 = \boldsymbol{\Theta}\cdot\mathbf{r}$, where $\boldsymbol{\Theta}$ is the skew-symmetric dyadic

$$\begin{aligned}\boldsymbol{\Theta} \equiv\ & 0\boldsymbol{ii} + H_{0z}\boldsymbol{ij} - H_{0y}\boldsymbol{ik}\\ & - H_{0z}\boldsymbol{ji} + 0\boldsymbol{jj} + H_{0x}\boldsymbol{jk}\\ & + H_{0y}\boldsymbol{ki} - H_{0x}\boldsymbol{kj} + 0\boldsymbol{kk}.\end{aligned} \qquad (85\text{–}2)$$

Therefore, if **I** is the unit dyadic,

$$\left[\{(k_0^2 - \omega^2) - 2i\omega l\}\mathbf{I} + i\frac{e\omega}{mc}\Theta\right]\cdot\mathbf{r} = \frac{e}{m}\mathbf{E}_1.$$

Designating the number of bound electrons per unit volume by N, the polarization is $\mathbf{P} = Ne\mathbf{r}$, and, as $\mathbf{E}_1 = \mathbf{E} + \frac{1}{3}\mathbf{P}$ by (63–3), we have

$$\left[\{(k_0^2 - \omega^2) - 2i\omega l\}\mathbf{I} + i\frac{e\omega}{mc}\Theta\right]\cdot\mathbf{P} = \frac{Ne^2}{m}(\mathbf{E} + \tfrac{1}{3}\mathbf{P}),$$

or, if we put $k^2 \equiv k_0^2 - Ne^2/3m$ as in article 83,

$$\left[\{(k^2 - \omega^2) - 2i\omega l\}\mathbf{I} + i\frac{e\omega}{mc}\Theta\right]\cdot\mathbf{P} = \frac{Ne^2}{m}\mathbf{E}.$$

Put

$$\mathbf{\Phi} \equiv \frac{m}{Ne^2}\left[\{(k^2 - \omega^2) - 2i\omega l\}\mathbf{I} + i\frac{e\omega}{mc}\Theta\right]\cdot \tag{85–3}$$

Then, as $\mathbf{D} = \mathbf{E} + \mathbf{P}$,

$$\mathbf{\Phi}\cdot\mathbf{D} = (\mathbf{\Phi} + \mathbf{I})\cdot\mathbf{E},$$

and, multiplying by $\mathbf{\Phi}^{-1}$,

$$\mathbf{D} = (\mathbf{I} + \mathbf{\Phi}^{-1})\cdot\mathbf{E} = \mathbf{\Psi}\cdot\mathbf{E}, \tag{85–4}$$

where $\mathbf{\Psi} \equiv \mathbf{I} + \mathbf{\Phi}^{-1}$. This constitutive relation between **D** and **E** differs from (84–1) for an anisotropic medium in that $\mathbf{\Psi}$ is not a symmetric dyadic.

In calculating $\mathbf{\Phi}^{-1}$ from (85–3) by means of (32–8) we can neglect terms in the squares or products of two components of $\mathbf{H}_0$ as compared with linear terms, since the effect of the magnetic field is small. Thus we find, with the aid of (83–3),

$$\mathbf{\Phi}^{-1} = (\kappa - 1)\mathbf{I} - \frac{i\omega}{Nec}(\kappa - 1)^2\Theta, \tag{85–5}$$

and

$$\mathbf{\Psi} = \kappa\mathbf{I} - \frac{i\omega}{Nec}(\kappa - 1)^2\Theta. \tag{85–6}$$

Consequently, as $\Theta\cdot\mathbf{E} = \mathbf{E}\times\mathbf{H}_0 = -\mathbf{H}_0\times\mathbf{E}$, the constitutive relation between **D** and **E** may be written

$$\mathbf{D} = \mathbf{\Psi}\cdot\mathbf{E} = \kappa\mathbf{E} + \frac{i\omega}{Nec}(\kappa - 1)^2\mathbf{H}_0\times\mathbf{E}. \tag{85–7}$$

However, as $\mathbf{D}$ always lies in the wave-front, it is more important to express $\mathbf{E}$ in terms of $\mathbf{D}$. This requires the calculation of $\mathbf{\Psi}^{-1}$. Neglecting, as before, terms in squares or products of the components of $\mathbf{H}_0$, we find

$$\mathbf{\Psi}^{-1} = \frac{1}{\kappa}\mathrm{I} + \frac{i\omega}{Nec}\left(\frac{\kappa - 1}{\kappa}\right)^2 \Theta, \tag{85-8}$$

which gives

$$\mathbf{E} = \mathbf{\Psi}^{-1}\cdot\mathbf{D} = \frac{1}{\kappa}\mathbf{D} - \frac{i\omega}{Nec}\left(\frac{\kappa - 1}{\kappa}\right)^2 \mathbf{H}_0 \times \mathbf{D}. \tag{85-9}$$

To save writing, we will express this is the form

$$\mathbf{E} = \frac{1}{\kappa}\mathbf{D} - i\gamma\mathbf{H}_0 \times \mathbf{D}, \quad \gamma \equiv \frac{\omega}{Nec}\left(\frac{\kappa - 1}{\kappa}\right)^2.$$

Putting this for $\mathbf{E}$ in the wave equation (82–12), remembering that $\mathbf{S}\cdot\mathbf{D} = 0$, we have

$$\left(\frac{S^2}{\kappa} - S_0{}^2\right)\mathbf{D} - i\gamma(S^2\mathbf{H}_0 \times \mathbf{D} - \mathbf{S}\cdot\mathbf{H}_0 \times \mathbf{D}\,\mathbf{S}) = 0. \tag{85-10}$$

As we have not assumed any particular orientation for $\mathbf{H}_0$ relative to the coordinate axes, we can limit our discussion to plane waves advancing in the X direction without loss of generality. Then, as $\mathbf{D}$ is perpendicular to $\mathbf{S}$ by (82–11a), we can put $\mathbf{D} = \boldsymbol{j}D_y + \boldsymbol{k}D_z$. Now

$$S^2\mathbf{H}_0 \times \mathbf{D} - \mathbf{S}\cdot\mathbf{H}_0 \times \mathbf{D}\,\mathbf{S} = S^2 H_{0x}(-\boldsymbol{j}D_z + \boldsymbol{k}D_y)$$

and the two non-vanishing components of the vector equation (85–10) are

$$\left.\begin{aligned} \left(\frac{S^2}{\kappa} - S_0{}^2\right) D_y &= -\, i\gamma S^2 H_{0x} D_z, \\ \left(\frac{S^2}{\kappa} - S_0{}^2\right) D_z &= i\gamma S^2 H_{0x} D_y. \end{aligned}\right\} \tag{85-12}$$

Multiplying these two equations together we find

$$\frac{S^2}{\kappa} - S_0{}^2 = \pm\, \gamma S^2 H_{0x}, \tag{85-13}$$

which gives for the index of refraction

$$\frac{S}{S_0} = \sqrt{\kappa}\,\left\{1 \pm \tfrac{1}{2}\kappa\gamma H_{0x}\right\}\cdot \tag{85-14}$$

We shall confine our discussion to the case where κ and therefore S is real. Using the lower sign on the right-hand side of (85–13) and substituting in (85–12) we find $D_z = -\,\mathrm{i}D_y$ and hence the resultant electric displacement $\mathbf{D}_r$ is of the form

$$\begin{aligned}\mathbf{D}_r &= (\boldsymbol{j} - \mathrm{i}\boldsymbol{k})A_0 e^{\mathrm{i}\omega(S_r x - t)}\\ &= A_0[\boldsymbol{j}\cos\omega(S_r x - t) + \boldsymbol{k}\sin\omega(S_r x - t)],\end{aligned} \qquad (85\text{–}15)$$

where S_r is the value of S corresponding to the lower sign in (85–14). This represents a right-circularly polarized plane wave since, at any instant, the terminus of the vector $\mathbf{D}_r$ lies on a right-handed helix with axis parallel to $\mathbf{S}$. Similarly, the upper sign in (85–13) gives $D_z = \mathrm{i}D_y$ and the resultant electric displacement $\mathbf{D}_l$ is of the form

$$\begin{aligned}\mathbf{D}_l &= (\boldsymbol{j} + \mathrm{i}\boldsymbol{k})A_0 e^{\mathrm{i}\omega(S_l x - t)}\\ &= A_0[\boldsymbol{j}\cos\omega(S_l x - t) - \boldsymbol{k}\sin(S_l x - t)],\end{aligned} \qquad (85\text{–}16)$$

where S_l is the value of S obtained by using the upper sign in (85–14). This is the equation of a left-circularly polarized plane wave since, at any instant, the terminus of the vector $\mathbf{D}_l$ lies on a left-handed helix with axis parallel to $\mathbf{S}$. So the medium will not, in general, transmit a plane polarized wave of constant vector amplitude, but does transmit two circularly polarized waves of contrary polarizations with different velocities. If $\mathbf{H}_0$ is at right angles to $\mathbf{S}$, H_{0x} is zero and the two velocities are the same. For this direction of $\mathbf{H}_0$ a plane polarized wave of constant vector amplitude can be transmitted. As the difference of the velocities of the two circularly polarized waves is proportional to H_{0x}, it is greatest when $\mathbf{H}_0$ is in the same line as $\mathbf{S}$. Since γ is negative, the velocity of the left-circularly polarized wave is the greater when $\mathbf{H}_0$ is parallel to $\mathbf{S}$ and *vice versa* when $\mathbf{H}_0$ is opposite to $\mathbf{S}$.

If both states of circular polarization are transmitted simultaneously, the amplitudes of the two being the same, the resultant electric displacement $\mathbf{D}_p$ is the vector sum of (85–15) and (85–16). Thus, if we put $\bar{S} \equiv \frac{1}{2}(S_r + S_l) = \sqrt{\kappa}\, S_0$, we get

$$\begin{aligned}\mathbf{D}_p &= \mathbf{D}_r + \mathbf{D}_l\\ &= A_0[\boldsymbol{j}\{\cos\omega(S_r x - t) + \cos\omega(S_l x - t)\}\\ &\qquad + \boldsymbol{k}\{\sin\omega(S_r x - t) - \sin\omega(S_l x - t)\}]\\ &= 2A_0[\boldsymbol{j}\cos(\tfrac{1}{2}\omega\kappa\gamma H_{0x}\bar{S}x)\\ &\qquad - \boldsymbol{k}\sin(\tfrac{1}{2}\omega\kappa\gamma H_{0x}\bar{S}x)]\cos\omega(\bar{S}x - t).\end{aligned} \qquad (85\text{–}17)$$

At any point in the medium this represents a linear oscillation, but the direction of the oscillation changes with x. We have, therefore, a plane polarized wave the plane of which rotates about the X axis as it advances into the medium. If α is the angle which the plane of polarization makes with some fixed plane through the X axis, the angle through which the plane of polarization rotates per unit distance of advance is

$$\frac{d\alpha}{dx} = -\tfrac{1}{2}\omega\kappa\gamma\overline{S}H_{0x} = -\frac{2\pi^2}{Ne\lambda^2}\frac{(n^2-1)^2}{n^3}H_{0x}, \qquad (85\text{–}18)$$

where $n = \overline{S}/S_0 = \sqrt{\kappa}$ is the index of refraction of the medium in the absence of an impressed magnetic field and λ is the wave-length. Evidently the magnitude of the rotation is greatest when the magnetic field lies in the same line as that of propagation. Since e is negative, the rotation is positive when $\mathbf{H}_0$ is parallel to $\mathbf{S}$, and negative when $\mathbf{H}_0$ is opposite to $\mathbf{S}$. We note that when plane polarized light is passed through a transparent dielectric in the direction of the magnetic field, and then reflected and caused to traverse the same path in the opposite sense, the rotation is not annulled, but doubled.

The rotation of the plane of polarization of light passing through a dielectric situated in a magnetic field was discovered by Faraday in 1845. It is known as the *Faraday effect*.

86. Optically Active Homogeneous Isotropic Medium. — Some media, both isotropic and anisotropic, have the property of rotating the plane of polarization of plane polarized light propagated through them, even in the absence of an external magnetic field. Such media are said to be *optically active*. An aqueous solution of dextro- or levo-sugar is an example of an optically active isotropic medium; quartz of an optically active anisotropic medium. In this article we shall discuss the electromagnetic theory of the propagation of light in a non-conducting optically active homogeneous isotropic medium, such as a sugar solution.

In order to account for the observed phenomena it is found necessary to assume, in addition to the ordinary polarization $\mathbf{P}_1 = (\kappa - 1)\mathbf{E}$ of an isotropic dielectric, a polarization $\mathbf{P}_2 = -g\dot{\mathbf{F}}$ proportional to the time rate of decrease of the magnetic force, and an intensity of magnetization $\mathbf{I}_2 = g\dot{\mathbf{E}}$ proportional to the time rate of increase of electric intensity, the constant g being the same in the two cases. Models of the molecule containing coupled oscillators vibrating at right angles have been devised to explain the production

of the polarization $\mathbf{P}_2$ and the intensity of magnetization $\mathbf{I}_2$, but their artificiality leads us to omit any detailed description of them here. Instead we shall content ourselves with tracing the electromagnetic consequences of the assumed $\mathbf{P}_2$ and $\mathbf{I}_2$.

As $\mathbf{B} \neq \mathbf{F}$ in this case we need the general form (62–12) of the field equations with $\rho = 0$. Making use of (82–4) and (82–10) they become

$$\left.\begin{aligned} \mathbf{S}\cdot\mathbf{D} &= 0, \quad (a) \qquad & \mathbf{S}\cdot\mathbf{B} &= 0, \quad (b) \\ \mathbf{S}\times\mathbf{E} &= S_0\mathbf{B}, \quad (c) \qquad & \mathbf{S}\times\mathbf{F} &= -S_0\mathbf{D}, \quad (d) \end{aligned}\right\} \tag{86–1}$$

which, with the constitutive relations

$$\mathbf{D} = \kappa\mathbf{E} + i\omega g\mathbf{F}, \tag{86–2}$$

$$\mathbf{B} = \mathbf{F} - i\omega g\mathbf{E}, \tag{86–3}$$

describe completely the propagation of plane electromagnetic waves of angular frequency ω and constant vector amplitude in the medium. We see from (86–1a) and (86–1b) that $\mathbf{D}$ and $\mathbf{B}$ are perpendicular to $\mathbf{S}$ and therefore lie in the wave-front. Therefore these are the fundamental vectors to be considered, and we must solve (86–2) and (86–3) for $\mathbf{E}$ and $\mathbf{F}$, obtaining, since ω^2g^2 is so small compared with κ in any actual medium as to be negligible,

$$\mathbf{E} = \frac{1}{\kappa}\mathbf{D} - \frac{i\omega g}{\kappa}\mathbf{B}, \tag{86–4}$$

$$\mathbf{F} = \mathbf{B} + \frac{i\omega g}{\kappa}\mathbf{D}. \tag{86–5}$$

Eliminating $\mathbf{E}$ from (86–1c) by means of (86–4), and $\mathbf{F}$ from (86–1d) by means of (86–5), we get

$$\frac{1}{\kappa}\mathbf{S}\times\mathbf{D} - \frac{i\omega g}{\kappa}\mathbf{S}\times\mathbf{B} = S_0\mathbf{B},$$

$$\mathbf{S}\times\mathbf{B} + \frac{i\omega g}{\kappa}\mathbf{S}\times\mathbf{D} = -S_0\mathbf{D}.$$

From these two equations we eliminate first $\mathbf{S}\times\mathbf{B}$ and then $\mathbf{S}\times\mathbf{D}$, again neglecting ω^2g^2 as compared with κ. This gives the simpler relations

$$\mathbf{S}\times\mathbf{D} = S_0\{\kappa\mathbf{B} - i\omega g\mathbf{D}\}, \tag{86–6}$$

$$\mathbf{S}\times\mathbf{B} = -S_0\{\mathbf{D} + i\omega g\mathbf{B}\}. \tag{86–7}$$

If we take the vector product of **S** by the first of these, expand the triple vector product on the left, using the relation $\mathbf{S}\cdot\mathbf{D} = 0$, and express $\mathbf{S} \times \mathbf{B}$ and $\mathbf{S} \times \mathbf{D}$ on the right in terms of **D** and **B** with the aid of the original equations, we get the linear relation

$$(S^2 - \kappa S_0{}^2)\mathbf{D} = 2i\omega g\kappa S_0{}^2\mathbf{B}, \tag{86–8}$$

and, if we treat the second equation in the same way, we obtain

$$(S^2 - \kappa S_0{}^2)\,\mathbf{B} = -\,2i\omega g S_0{}^2\mathbf{D}. \tag{86–9}$$

Eliminating either **B** or **D** from (86–8) and (86–9), we find

$$S^2 - \kappa S_0{}^2 = \pm\, 2\omega g\sqrt{\kappa}S_0{}^2, \tag{86–10}$$

which gives, as in the previous article, two indices of refraction,

$$\frac{S}{S_0} = \sqrt{\kappa} \pm \omega g. \tag{86–11}$$

Taking the lower sign in (86–10) and (86–11), we find from (86–8) or (86–9) that $\mathbf{B} = i\mathbf{D}/\sqrt{\kappa}$. Therefore (86–6) becomes

$$\begin{aligned}\mathbf{S} \times \mathbf{D} &= iS_0(\sqrt{\kappa} - \omega g)\mathbf{D}\\ &= iS\mathbf{D}.\end{aligned} \tag{86–12}$$

Let us take the X axis in the direction of propagation. Then, as **D** and **B** are both perpendicular to **S**, $\mathbf{D} = \mathbf{j}D_y + \mathbf{k}D_z$ and $\mathbf{B} = \mathbf{j}B_y + \mathbf{k}B_z$. Consequently we find from (86–12) that $D_z = -\,iD_y$. This is exactly the condition which led to the right-circularly polarized wave (85–15) in article 85. So the medium transmits a right-circularly polarized wave with wave-slowness $S_r = S_0(\sqrt{\kappa} - \omega g)$.

Similarly the upper sign in (86–10) and (86–11) gives $\mathbf{B} = -\,i\mathbf{D}/\sqrt{\kappa}$ and (86–6) becomes

$$\begin{aligned}\mathbf{S} \times \mathbf{D} &= -\,iS_0(\sqrt{\kappa} + \omega g)\mathbf{D}\\ &= -\,iS\mathbf{D},\end{aligned} \tag{86–13}$$

which leads to the relation $D_z = iD_y$ characteristic of a left-circularly polarized wave which travels with the wave-slowness $S_l = S_0(\sqrt{\kappa} + \omega g)$. If $\sqrt{\kappa}$ and g are real neither wave is absorbed, and, if g is positive as well, $S_l > S_r$ and the right-circularly polarized wave has a greater speed than the left-circularly polarized wave.

We shall, however, consider the more general case where g and

possibly $\sqrt{\kappa}$ are complex. Then, if we write $g = g' + ig''$ and put $\gamma' \equiv S_0\omega g'$ and $\gamma'' \equiv S_0\omega g''$, we have

$$S_r = \overline{S}' - \gamma' + i(\overline{S}'' - \gamma''), \tag{86–14}$$

$$S_l = \overline{S}' + \gamma' + i(\overline{S}'' + \gamma''), \tag{86–15}$$

where $\overline{S}'$ is the real part and $\overline{S}''$ the imaginary part of $\sqrt{\kappa}\, S_0$. In the case of the right-circularly polarized wave we have then

$$\begin{aligned}\mathbf{D}_r &= (\boldsymbol{j} - i\boldsymbol{k})A_0 e^{-\omega(\overline{S}''-\gamma'')x} e^{i\omega\{(\overline{S}'-\gamma')x-t\}}\\ &= A_0 e^{-\omega(\overline{S}''-\gamma'')x}[\boldsymbol{j}\cos\omega\{(\overline{S}'-\gamma')x-t\}\\ &\qquad + \boldsymbol{k}\sin\omega\{(\overline{S}'-\gamma')x-t\}],\end{aligned} \tag{86–16}$$

and in the case of the left-circularly polarized wave

$$\begin{aligned}\mathbf{D}_l &= (\boldsymbol{j} + i\boldsymbol{k})A_0 e^{-\omega(\overline{S}''+\gamma'')x} e^{i\omega\{(\overline{S}'+\gamma')x-t\}}\\ &= A_0 e^{-\omega(\overline{S}''+\gamma'')x}[\boldsymbol{j}\cos\omega\{(\overline{S}'+\gamma')x-t\}\\ &\qquad - \boldsymbol{k}\sin\omega\{(\overline{S}'+\gamma')x-t\}].\end{aligned} \tag{86–17}$$

If, now, we combine these two circularly polarized waves to produce a linear oscillation at $x = 0$, the different absorption coefficients give rise to a new phenomenon which was not apparent in the case discussed in the last article. For

$$\begin{aligned}\mathbf{D}_p &= \mathbf{D}_r + \mathbf{D}_l\\ &= A_0 e^{-\omega\overline{S}''x}\{\boldsymbol{j}[e^{\omega\gamma''x}\cos\omega\{(\overline{S}'-\gamma')x-t\}\\ &\qquad + e^{-\omega\gamma''x}\cos\omega\{(\overline{S}'+\gamma')x-t\}]\\ &\quad + \boldsymbol{k}[e^{\omega\gamma''x}\sin\omega\{(\overline{S}'-\gamma')x-t\}\\ &\qquad - e^{-\omega\gamma''x}\sin\omega\{(\overline{S}'+\gamma')x-t\}]\}\\ &= 2A_0 e^{-\omega\overline{S}''x}\{[\boldsymbol{j}\cos\omega\gamma'x - \boldsymbol{k}\sin\omega\gamma'x]\cosh\omega\gamma''x\cos\omega(\overline{S}'x-t)\\ &\quad + [\boldsymbol{j}\sin\omega\gamma'x + \boldsymbol{k}\cos\omega\gamma'x]\sinh\omega\gamma''x\sin\omega(\overline{S}'x-t)\}.\end{aligned} \tag{86–18}$$

The components represented by the two terms inside the braces are at right angles since

$$[\boldsymbol{j}\cos\omega\gamma'x - \boldsymbol{k}\sin\omega\gamma'x]\cdot[\boldsymbol{j}\sin\omega\gamma'x + \boldsymbol{k}\cos\omega\gamma'x] = 0.$$

The first component has amplitude $2A_0$ at $x = 0$, and the second amplitude 0. As x increases the amplitude of the second component

grows, although remaining everywhere less than the amplitude of the first component. Therefore the plane polarized vibration at $x = 0$ becomes elliptically polarized at $x > 0$. This effect, known as *circular dichroism*, was discovered by Cotton in 1896.

The angle through which either component rotates per unit distance of advance through the medium is

$$\frac{d\alpha}{dx} = -\,\omega\gamma' = -\,\omega^2 g' S_0. \tag{86–19}$$

Unlike the situation existing in the Faraday effect, we see here that if plane polarized light is passed through a transparent optically active medium, and then reflected and caused to traverse the same path in the opposite sense, the rotation is annulled.

87. Reflection and Transmission at Surface Separating Two Isotropic Media. — In the earlier articles of this chapter we have been concerned with the propagation of light in a homogeneous medium. Now we shall investigate the reflection and transmission of light at the surface separating two homogeneous isotropic media which have different permittivities and conductivities.

The boundary conditions at the surface, obtained from the field equations (82–8) by the method developed in article 52, are that the normal components of $\mathbf{D}_e$ and $\mathbf{H}$ and the tangential components of $\mathbf{E}$ and $\mathbf{H}$ shall be the same on both sides of the surface.

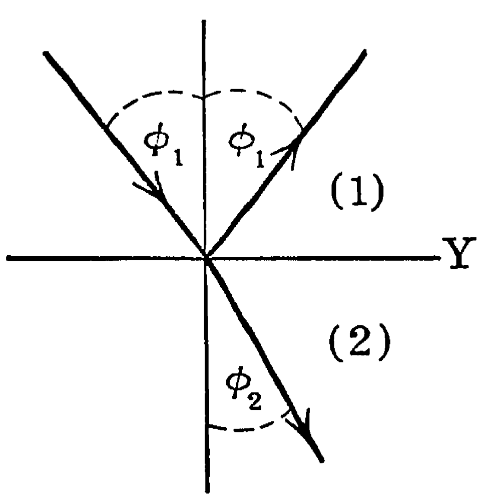

FIG. 102.

Let the YZ plane (Fig. 102) be the surface of separation, quantities pertaining to the medium above being designated by the subscript 1 and those pertaining to the medium below by the subscript 2. We shall suppose the radiation to be incident in the upper medium at an angle ϕ_1 with the normal. In addition to the incident beam we have to consider the radiation reflected at an angle ϕ_1 with the normal and that transmitted at an angle ϕ_2. Whatever the state of polarization of the radiation, we can resolve it into two plane polarized wave trains, one with the electric vector perpendicular to the plane of incidence and the other with the electric vector parallel to this plane. We shall, therefore, treat these two states of plane polarization separately.

Considering first the case where the electric vector is perpendicular to the plane of incidence, we have for the electric intensity in the incident, reflected and transmitted waves, respectively,

$$\left.\begin{aligned}\mathbf{E}_1 &= \mathbf{k}A_1 e^{i\omega\{S_1(x\cos\phi_1 + y\sin\phi_1) - t\}},\\ \mathbf{E}_1' &= \mathbf{k}A_1' e^{i\omega\{S_1(-x\cos\phi_1 + y\sin\phi_1) - t\}},\\ \mathbf{E}_2 &= \mathbf{k}A_2 e^{i\omega\{S_2(x\cos\phi_2 + y\sin\phi_2) - t\}},\end{aligned}\right\} \tag{87–1}$$

and for the magnetic intensity

$$\left.\begin{aligned}\mathbf{H}_1 &= (\mathbf{i}\sin\phi_1 - \mathbf{j}\cos\phi_1)n_1E_1,\\ \mathbf{H}_1' &= (\mathbf{i}\sin\phi_1 + \mathbf{j}\cos\phi_1)n_1E_1',\\ \mathbf{H}_2 &= (\mathbf{i}\sin\phi_2 - \mathbf{j}\cos\phi_2)n_2E_2,\end{aligned}\right\} \tag{87–2}$$

from (83–18), where $n_1 = S_1/S_0$ is the index of refraction of the upper medium, and $n_2 = S_2/S_0$ that of the lower medium. The boundary conditions at the surface $x = 0$ for the tangential component of **E** and the normal and tangential components of **H**, expressed in terms of **E** by (87–2), are respectively

$$E_1 + E_1' = E_2, \tag{87–3}$$

$$(E_1 + E_1')n_1\sin\phi_1 = E_2n_2\sin\phi_2, \tag{87–4}$$

$$(E_1 - E_1')n_1\cos\phi_1 = E_2n_2\cos\phi_2. \tag{87–5}$$

These are all the boundary conditions as there is no normal component of $\mathbf{D}_e$ in the state of polarization under consideration.

Dividing (87–4) by (87–3) we get Snell's law

$$n_1\sin\phi_1 = n_2\sin\phi_2, \tag{87–6}$$

which can also be written

$$S_1\sin\phi_1 = S_2\sin\phi_2. \tag{87–7}$$

On account of (87–7) we see that the arguments of the exponential factors in (87–1) are all the same at $x = 0$. Hence we can replace E_1, E_1', E_2 in (87–3), (87–4), (87–5) by the amplitudes A_1, A_1', A_2, respectively. The *coefficient of reflection* R is defined as the ratio of the amplitude A_1' of the reflected wave to the amplitude A_1 of the incident wave, and the *coefficient of transmission* T as the ratio of the

amplitude A_2 of the transmitted wave to the amplitude A_1 of the incident wave. Dividing (87–5) by (87–4) we get

$$\frac{1 - R_\perp}{1 + R_\perp} = \frac{\sin\phi_1 \cos\phi_2}{\sin\phi_2 \cos\phi_1},$$

where $R_\perp$ is the coefficient of reflection for the case under consideration, in which the electric vector is perpendicular to the plane of incidence. This gives

$$R_\perp = \frac{\sin\phi_2\cos\phi_1 - \sin\phi_1\cos\phi_2}{\sin\phi_2\cos\phi_1 + \sin\phi_1\cos\phi_2} = -\frac{\sin(\phi_1 - \phi_2)}{\sin(\phi_1 + \phi_2)}. \qquad (87\text{–}8)$$

Eliminating E_1' from (87–4) and (87–5) we find for the coefficient of transmission $T_\perp$

$$T_\perp = \frac{2\sin\phi_2\cos\phi_1}{\sin\phi_2\cos\phi_1 + \sin\phi_1\cos\phi_2} = \frac{2\sin\phi_2\cos\phi_1}{\sin(\phi_1 + \phi_2)} \qquad (87\text{–}9)$$

with the aid of (87–6).

When the electric vector is parallel to the plane of incidence the electric intensities in the three waves are

$$\left.\begin{aligned}
\mathbf{E}_1 &= (-\boldsymbol{i}\sin\phi_1 + \boldsymbol{j}\cos\phi_1)A_1 e^{i\omega\{S_1(x\cos\phi_1 + y\sin\phi_1) - t\}},\\
\mathbf{E}_1' &= (-\boldsymbol{i}\sin\phi_1 - \boldsymbol{j}\cos\phi_1)A_1' e^{i\omega\{S_1(-x\cos\phi_1 + y\sin\phi_1) - t\}},\\
\mathbf{E}_2 &= (-\boldsymbol{i}\sin\phi_2 + \boldsymbol{j}\cos\phi_2)A_2 e^{i\omega\{S_2(x\cos\phi_2 + y\sin\phi_2) - t\}},
\end{aligned}\right\} \qquad (87\text{–}10)$$

and the magnetic intensities are

$$\left.\begin{aligned}
\mathbf{H}_1 &= \boldsymbol{k}n_1 E_1,\\
\mathbf{H}_1' &= \boldsymbol{k}n_1 E_1',\\
\mathbf{H}_2 &= \boldsymbol{k}n_2 E_2.
\end{aligned}\right\} \qquad (87\text{–}11)$$

In this case there is no normal component of $\mathbf{H}$. The boundary conditions for the tangential component of $\mathbf{H}$, the normal component of $\mathbf{D}_e$ and the tangential component of $\mathbf{E}$ are, with the aid of (87–11) and (83–17), respectively,

$$(E_1 + E_1')n_1 = E_2 n_2, \qquad (87\text{–}12)$$

$$(E_1 + E_1')n_1^2 \sin\phi_1 = E_2 n_2^2 \sin\phi_2, \qquad (87\text{–}13)$$

$$(E_1 - E_1')\cos\phi_1 = E_2\cos\phi_2. \qquad (87\text{–}14)$$

Dividing (87–13) by (87–12) we get Snell's law (87–6) and (87–7) again. As before (87–7) makes the arguments of the exponentials in (87–10) all the same at $x = 0$ so that we can replace the electric intensities in the boundary conditions by the corresponding amplitudes. Dividing (87–14) by (87–13) we get for the coefficient of reflection $R_{\parallel}$ for the case in which the electric vector is parallel to the plane of incidence

$$\frac{1 - R_{\parallel}}{1 + R_{\parallel}} = \frac{n_1^2 \sin \phi_1 \cos \phi_2}{n_2^2 \sin \phi_2 \cos \phi_1} = \frac{\sin \phi_2 \cos \phi_2}{\sin \phi_1 \cos \phi_1}$$

with the aid of (87–6). This gives

$$R_{\parallel} = \frac{\sin\phi_1 \cos\phi_1 - \sin\phi_2 \cos\phi_2}{\sin\phi_1 \cos\phi_1 + \sin\phi_2 \cos\phi_2} = \frac{\tan(\phi_1 - \phi_2)}{\tan(\phi_1 + \phi_2)}. \quad (87\text{–}15)$$

If we eliminate E_1' from (87–12) and (87–14) and use (87–6) again, we find for the coefficient of transmission $T_{\parallel}$

$$T_{\parallel} = \frac{2 \sin \phi_2 \cos \phi_1}{\sin\phi_1 \cos\phi_1 + \sin\phi_2 \cos\phi_2} = \frac{2 \sin \phi_2 \cos \phi_1}{\sin(\phi_1 + \phi_2) \cos(\phi_1 - \phi_2)}. \quad (87\text{–}16)$$

First we shall consider two dielectrics in the region of transparency. For instance, (1) may be air and (2) glass. We notice that none of the coefficients of reflection or transmission may vanish, except $R_{\parallel}$, which is zero when $\phi_1 + \phi_2 = \pi/2$, a condition existing when the reflected ray is at right angle to the refracted ray. The angle of incidence ϕ_{1B} satisfying this condition, known as *Brewster's angle* or as the *polarizing angle*, is $\tan^{-1}(n_2/n_1)$ from (87–6). For glass in air with $n_2/n_1 = 1.5$ this angle is 56.°3. If unpolarized radiation is incident at this angle, the reflected radiation should be completely plane polarized with the electric vector perpendicular to the plane of incidence. Actually the polarization is never complete on account of surface imperfections. For this angle of incidence the coefficient of reflection of the reflected component is $R_{\perp} = \cos 2\phi_{1B}$. Hence the *reflecting power* for this component, defined as the ratio of the intensity of the reflected to the incident radiation and therefore equal to the square of $R_{\perp}$, is $\rho_{\perp} = \cos^2 2\phi_{1B}$. For $n_2/n_1 = 1.5$ we compute $\rho_{\perp} = 0.148$.

At grazing incidence ($\phi_1 = \pi/2$) we find from (87–8) and (87–15) that $R_{\perp} = R_{\parallel} = -1$, no matter whether n_2 is greater or less than n_1. Therefore the reflecting power is unity for both states of polarization,

the reflected radiation differing in phase by π from the incident radiation.

At normal incidence

$$R_{\perp} = \frac{n_1 - n_2}{n_1 + n_2}, \quad R_{\parallel} = -\frac{n_1 - n_2}{n_1 + n_2}. \tag{87–17}$$

As the two states of polarization are indistinguishable at normal incidence, the discrepancy in sign must be due to the assumed positive senses of $\mathbf{E}_1$ and $\mathbf{E}_1'$. Referring to (87–10) we notice that $\mathbf{E}_1$ and $\mathbf{E}_1'$ for the parallel case have opposite signs when $\phi_1 = 0$. Hence the correct change in phase is given by $R_{\perp}$ for both components. If $n_2 > n_1$ the phase changes on reflection by π, whereas if $n_2 < n_1$ there is no change in phase. The reflecting power for normal incidence is

$$\rho_n = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2. \tag{87–18}$$

For radiation in air incident on glass ($n_2/n_1 = 1.5$) we find that $\rho_n = 0.040$ at normal incidence. Therefore the reflection is small compared with that at grazing incidence.

The phenomenon of total internal reflection is of special interest. For this to occur, it is necessary that $n_1 > n_2$. Consequently it is convenient to put $n \equiv n_1/n_2 > 1$. We may then think of medium (1) as glass and (2) as air. The critical angle of incidence Φ_1 is that for which $\phi_2 = \pi/2$. Therefore $\Phi_1 = \sin^{-1}(1/n)$, which is $41.°8$ for glass of index $n = 1.5$. If $\phi_1 > \Phi_1$, it follows from Snell's law, $n \sin \phi_1 = \sin \phi_2$, that $\sin \phi_2 > 1$ and consequently that $\cos \phi_2 = \sqrt{1 - \sin^2 \phi_2}$ is a pure imaginary. Therefore both coefficients of reflection (87–8) and (87–15) are of the form

$$R = \frac{a - ib}{a + ib} = e^{-i\delta}, \quad \tan\frac{\delta}{2} \equiv \frac{b}{a}.$$

Consequently $A_1' = A_1 e^{-i\delta}$ and the electric intensity in the reflected beam is

$$\begin{aligned} E_1' &= A_1 e^{i[\omega\{S_1(-x\cos\phi_1 + y\sin\phi_1) - t\} - \delta]} \\ &= A_1 \cos[\omega\{S_1(-x\cos\phi_1 + y\sin\phi_1) - t\} - \delta], \end{aligned}$$

from which we conclude that reflection occurs without loss of intensity but with change in phase equal to $-\delta$.

In general the angle δ is not the same for the two components. So, if the incident light contains the two states of polarization with equal amplitudes and in the same phase, which makes it equivalent to plane polarized radiation with the plane of polarization inclined at an angle of 45° with the plane of incidence, the totally reflected light is elliptically polarized. We shall examine this case in more detail, putting $\Delta \equiv \delta_{\parallel} - \delta_{\perp}$ for the difference in phase of the two components after reflection.

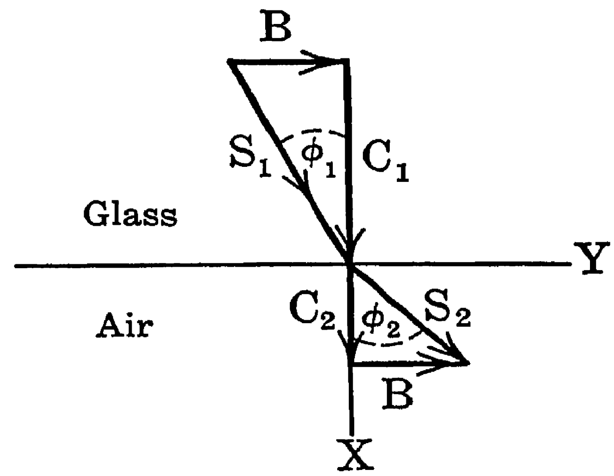

FIG. 103.

Put $B \equiv S_1 \sin \phi_1$. Then from Snell's law (87–7) $B = S_2 \sin \phi_2$. Also put $C_1 \equiv S_1 \cos \phi_1$ and $C_2 \equiv S_2 \cos \phi_2$, so that $S_1{}^2 = B^2 + C_1{}^2$ and $S_2{}^2 = B^2 + C_2{}^2$. We can regard B and C_1 as the components of the vector $\mathbf{S}_1$ and B and C_2 as the components of the vector $\mathbf{S}_2$, as illustrated in Fig. 103. When the angle of incidence is less than the critical angle, B, C_1, C_2 are all real, but after the critical angle is passed C_2 becomes imaginary.

Putting $V = 1/S$ for the wave velocity, the coefficients of reflection (87–8) and (87–15) may be written

$$R_{\perp} = \frac{C_1 - C_2}{C_1 + C_2}, \tag{87–19}$$

$$R_{\parallel} = \frac{V_1{}^2 C_1 - V_2{}^2 C_2}{V_1{}^2 C_1 + V_2{}^2 C_2}. \tag{87–20}$$

Therefore, remembering that $S_1 V_1 = S_2 V_2 = 1$,

$$e^{-i\Delta} = \frac{R_{\parallel}}{R_{\perp}} = \frac{B^2 - C_1 C_2}{B^2 + C_1 C_2}. \tag{87–21}$$

Now we lay off the quantities involved in this formula in the complex plane, plotting real quantities horizontally and imaginary quantities vertically. Thus, in Fig. 104, $\overrightarrow{OM} = S_1{}^2$, $\overrightarrow{ON} = S_2{}^2$, $\overrightarrow{OQ} = B^2$, $\overrightarrow{QM} = S_1{}^2 - B^2 = C_1{}^2$, $\overrightarrow{QN} = S_2{}^2 - B^2 = C_2{}^2$, $\overrightarrow{QP} = C_1 C_2$, $\overrightarrow{QP'} = -C_1 C_2$. As the angle of incidence increases from 0 to $\pi/2$,

Q moves from O through N to M, the point N corresponding to the critical angle Φ_1. Evidently $\overrightarrow{OP'} = B^2 - C_1C_2$ and $\overrightarrow{OP} = B^2 + C_1C_2$, and the phase difference Δ of the two components after reflection is the angle $P'OP$.

As $\overline{QP}$ is the mean proportional of $\overline{QM}$ and $\overline{QN}$, the angle NPM is a right angle. Consequently P and P' lie on the circumference of a circle of diameter $\overline{NM}$. Considering the whole range of Q, then, Q starts from O at normal incidence with P at a distance S_1S_2 to the right and P' at an equal distance to the left. The three points meet at the critical angle N. Here P branches off on the upper half-

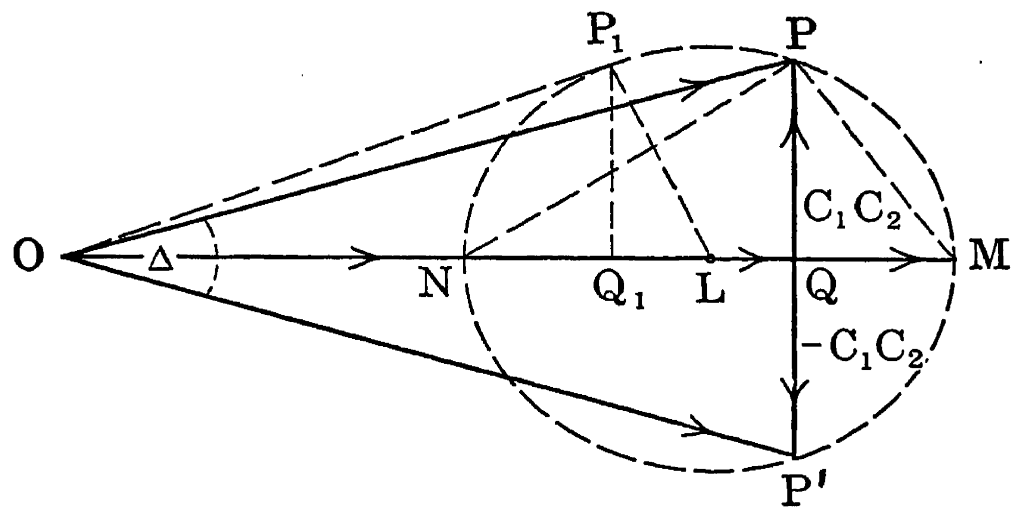

FIG. 104.

circumference of the circle and P' on the lower, the three points again meeting at grazing incidence M. The phase difference Δ, which is zero from O to N, increases to a maximum value Δ_m between N and M, and then falls to zero again at M.

Evidently the maximum phase difference occurs when P is at the point P_1, which makes $\overline{OP_1}$ tangent to the circle, the corresponding position of Q being Q_1. As the radius $\overline{LP_1}$ of the circle is $\frac{1}{2}(S_1{}^2 - S_2{}^2)$,

$$\sin \tfrac{1}{2}\Delta_m = \frac{S_1{}^2 - S_2{}^2}{S_1{}^2 + S_2{}^2} = \frac{n^2 - 1}{n^2 + 1}. \qquad (87\text{–}22)$$

This maximum phase difference takes place for the angle of incidence given by

$$\cos^2 \phi_{1m} = \frac{C_1{}^2}{S_1{}^2} = \frac{\frac{1}{2}(S_1{}^2 - S_2{}^2)(1 + \sin \frac{1}{2}\Delta_m)}{S_1{}^2} = \frac{n^2 - 1}{n^2 + 1}. \qquad (87\text{–}23)$$

For glass ($n = 1.5$), $\sin \frac{1}{2}\Delta_m = \cos^2 \phi_{1m} = 0.385$ giving $\Delta_m = 45.°3$, $\phi_{1m} = 51.°7$. As Δ_m exceeds $45°$ for ordinary glass, two successive

internal reflections at the proper angle of incidence may be used to produce a phase difference of $\pi/2$. In this way plane polarized light may be converted into circularly polarized light, or *vice versa*. This principle is used in the Fresnel rhomb.

We must not neglect the transmitted wave associated with total internal reflection. As $\cos \phi_2$ is imaginary it can conveniently be represented by $i\gamma$. Then, in the case where $\mathbf{E}$ is perpendicular to the plane of incidence, we find from (87–1) that

$$\mathbf{E}_2 = \boldsymbol{k}A_2 e^{-\omega\gamma S_2 x} e^{i\omega\{S_2 \sin \phi_2 y - t\}}.$$

This represents a wave propagated along the surface with a wave-slowness greater than S_2 and equal to the component parallel to the surface of the wave-slowness S_1 in medium (1). Its amplitude falls off exponentially with the distance x from the surface. The magnetic intensity, as is evident from (87–2), has components in both the X and the Y directions. Therefore $\mathbf{H}_2$ is not wholly perpendicular to the direction of propagation. As the wave moves parallel to the surface, and, in fact, with a wave-slowness equal to the projection on the surface of that in the medium in which total reflection occurs, no net transfer of energy across the surface takes place after the steady state has been established.

Similarly we find from (87–10) for the case where $\mathbf{E}$ lies in the plane of incidence,

$$\mathbf{E}_2 = (-\boldsymbol{i} \sin \phi_2 + \boldsymbol{j} i\gamma) A_2 e^{-\omega\gamma S_2 x} e^{i\omega\{S_2 \sin \phi_2 y - t\}}.$$

In this case $\mathbf{E}_2$ has a longitudinal as well as a transverse component, although $\mathbf{H}_2$ is entirely transverse in accord with (87–11). These surface waves were observed only after the theoretical establishment of their existence.

Next we turn our attention to the phenomenon of metallic reflection, assuming medium (1) in Fig. 102 to be a transparent dielectric with a real permittivity κ_1, and (2) to be a conductor characterized by a complex permittivity $\kappa_2 = \kappa_2' + i\kappa_2''$ and a conductivity σ_2. In the former the wave-slowness S_1 is real, but in the latter the wave-slowness $S_2 = S_2' + iS_2''$ is complex in accord with (83–10).

Evidently $R_\perp = R_\parallel = -1$ at grazing incidence as in the case of non-conducting transparent media. At normal incidence the formulas (87–17) still hold, with n_2 complex. If we put

$$n \equiv \frac{n_2}{n_1} = \nu(1 + i\chi)$$

as in article 83, the common coefficient of reflection for both states of polarization at normal incidence is

$$R = \frac{1 - \nu - i\nu\chi}{1 + \nu + i\nu\chi},$$

and

$$R^2 = \frac{1 + \nu^2(1 + \chi^2) - 2\nu}{1 + \nu^2(1 + \chi^2) + 2\nu} e^{-2i(\theta_1+\theta_2)},$$

where

$$\tan\,\theta_1 = \frac{\nu\chi}{1 + \nu}, \quad \tan\,\theta_2 = \frac{\nu\chi}{1 - \nu}.$$

The reflecting power ρ_n of the surface is evidently given by the modulus of R^2. It is

$$\rho_n = 1 - \frac{4\nu}{1 + \nu^2(1 + \chi^2) + 2\nu}. \tag{87–24}$$

It was shown in article 83 that $\chi \doteqdot 1$ for a good conductor. In such a case (87–24) reduces to the simpler formula

$$\rho_n = 1 - \frac{4\nu}{1 + 2\nu + 2\nu^2}, \tag{87–25}$$

and, as $\nu \doteqdot \sqrt{\sigma_2/2\omega}$ is large compared with unity, ρ_n is very close to unity. Thus the reflecting power of a silver surface at normal incidence may be as high as 95%. Calculation of the conductivity from the observed value of ρ_n by means of (87–25) for long waves in the infra-red region of the spectrum yields results in good agreement with the conductivity measured for steady currents. As ν decreases with increasing frequency, the reflecting power of a metallic surface decreases as we pass from the red to the violet end of the visible spectrum. Thus the reflecting power of silver falls from 95% at 7000 Å to 87% at 4200 Å.

While both change in amplitude and change in phase are the same for the two components at grazing incidence and at normal incidence, this is no longer true at other angles of incidence. In general one component suffers both a different change in amplitude and a different change in phase from the other. Thus, if the incident light is plane polarized at 45° to the plane of incidence so as to contain both components with equal amplitudes and in the same phase, the reflected light is elliptically polarized. We shall now investigate this case.

If we write $\quad R_\perp = r_\perp e^{-i\delta_\perp}, \quad R_\parallel = r_\parallel e^{-i\delta_\parallel},$

and put $r \equiv r_\parallel / r_\perp$, $\Delta \equiv \delta_\parallel - \delta_\perp$, we have, in accord with (87–21),

$$r\,e^{-i\Delta} = \frac{R_\parallel}{R_\perp} = \frac{B^2 - C_1C_2}{B^2 + C_1C_2}. \tag{87–26}$$

Let us construct a figure of the same type as Fig. 104. The present case differs from the previous one in that $S_2{}^2$ is not real, but

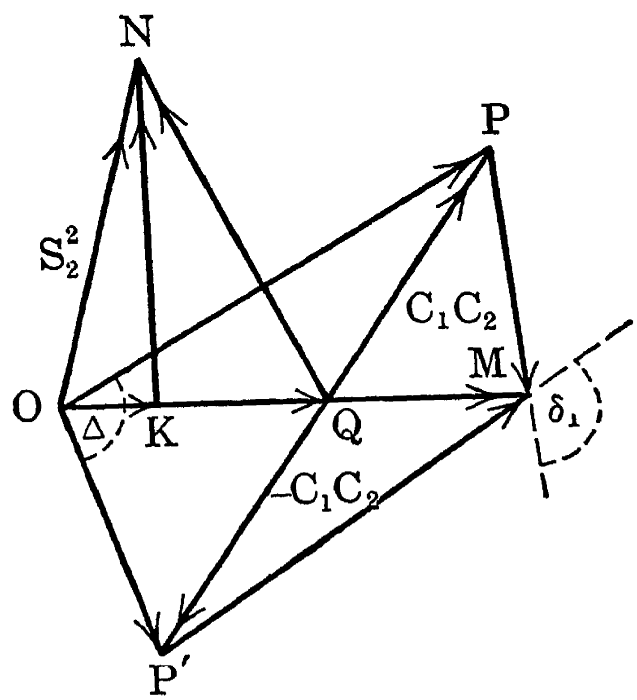

FIG. 105.

consists of both a real and an imaginary part in accord with (83–10). Therefore we lay off (Fig. 105) $\overrightarrow{OK} = \kappa_2' S_0{}^2$ and $\overrightarrow{KN} = (\kappa_2'' + \sigma_2/\omega)S_0{}^2$ making $\overrightarrow{ON} = S_2{}^2$, $\overrightarrow{OM} = S_1{}^2$ and $\overrightarrow{OQ} = B^2$. Then $QM = C_1{}^2$ and $\overrightarrow{QN} = C_2{}^2$. Now if we write the complex quantity $C_2{}^2$ in the polar form $C_2{}^2 = p\,e^{i\theta}$ we see at once that $C_1C_2 = C_1\sqrt{p}\,e^{i\theta/2}$. Therefore $\overrightarrow{QP} = C_1C_2$ bisects the angle NQM, and $\overrightarrow{QP'} = -\,C_1C_2$ is of equal length and oppositely directed. As before $\overrightarrow{OP'} = B^2 - C_1C_2$ and $\overrightarrow{OP} = B^2 + C_1C_2$, and the phase difference Δ of the two components after reflection is the angle $P'OP$. In addition to the relation

$$r\,e^{-i\Delta} = \frac{R_\parallel}{R_\perp} = \frac{\overrightarrow{OP'}}{\overrightarrow{OP}} \tag{87–27}$$

we have also

$$r_{\perp}e^{-i\delta_{\perp}} = R_{\perp} = \frac{\overrightarrow{PM}}{\overrightarrow{P'M}} \tag{87–28}$$

by (87–19), since $\overrightarrow{PM} = C_1^2 - C_1C_2$ and $\overrightarrow{P'M} = C_1^2 + C_1C_2$. Therefore $\delta_{\perp}$ is the angle between the positive senses of these two lines, and $\delta_{\parallel} = \delta_{\perp} + \Delta$.

The optical constants of a metal are usually measured by adjusting the angle of incidence of plane polarized radiation, polarized with the electric vector inclined at 45° to the plane of incidence, until the phase difference Δ of the two reflected components is $\pi/2$. The attainment of this condition can be tested by placing in the path of the elliptically polarized reflected radiation a quarter-wave plate which converts the radiation into plane polarized light with the electric vector making an angle β with the normal to the plane of incidence. Then $\tan \beta = r_{\parallel}/r_{\perp} = r$. This angle β, known as the *principal azimuth*, and the angle of incidence ϕ_{1P} which makes $\Delta = \pi/2$, known as the *principal incidence*, are sufficient to determine the optical constants of the metal, as we shall now show.

Putting $\Delta = \pi/2$ and $r = \tan \beta$ in (87–26) we have

$$\frac{B^2 - C_1C_2}{B^2 + C_1C_2} = -\,i \tan \beta,$$

whence

$$\frac{C_1C_2}{B^2} = \frac{1 + i \tan \beta}{1 - i \tan \beta} = e^{2i\beta}.$$

Now $B/C_1 = \tan \phi_{1P}$ from Fig. 103. Therefore

$$\frac{S_2^2 - B^2}{B^2} = \frac{C_2^2}{B^2} = \tan^2 \phi_{1P}\, e^{4i\beta}.$$

But $B^2 = S_1^2 \sin^2 \phi_{1P}$. Consequently

$$\frac{S_2^2}{S_1^2} = \sin^2 \phi_{1P}\,(1 + \tan^2 \phi_{1P}\, e^{4i\beta}). \tag{87–29}$$

However, in accord with (83–10),

$$\frac{S_2^2}{S_1^2} = \frac{\kappa_2' + i\left(\kappa_2'' + \dfrac{\sigma_2}{\omega}\right)}{\kappa_1}.$$

Comparing with (87–29) we find

$$\left.\begin{aligned} \frac{\kappa_2'}{\kappa_1} &= \sin^2\phi_{1P}\,(1 + \tan^2\phi_{1P}\cos 4\beta), \\ \frac{\kappa_2'' + \dfrac{\sigma_2}{\omega}}{\kappa_1} &= \sin^2\phi_{1P}\tan^2\phi_{1P}\sin 4\beta. \end{aligned}\right\} \qquad (87\text{–}30)$$

As was noted earlier, we cannot distinguish between κ_2'' and σ_2/ω. Evidently a non-conducting dielectric in the region of an absorption band, where κ_2'' is large, will exhibit all the phenomena of metallic reflection.

Now S_2/S_1 is the complex index of refraction $n = \nu(1 + i\chi)$. Since the index of refraction of a good conductor is large compared with unity we can generally neglect the first term on the right of (87–29) compared with the second. Then

$$\left.\begin{aligned} \nu &= \sin\phi_{1P}\tan\phi_{1P}\cos 2\beta, \\ \chi &= \tan 2\beta. \end{aligned}\right\} \qquad (87\text{–}31)$$

As we observed in article 83, χ is of the order of magnitude of unity for a metal of high conductivity. Hence β is in the neighborhood of $22°.5$ and the principal incidence ϕ_{1P} is large.

88. Metallic Reflection in the Presence of External Magnetic Field. — In the ordinary reflection from a metallic surface discussed in the last article, we have seen that plane polarized incident radiation in which the electric vector makes an angle of 45° with the plane of incidence is converted into elliptically polarized radiation on reflection. If, on the other hand, the electric vector in the incident beam is either perpendicular or parallel to the plane of incidence, the reflected light, like the incident light, is plane polarized. Now Kerr noticed, in 1877, that plane polarized light of any orientation gives rise to an elliptically polarized beam when the reflection takes place in the presence of a magnetic field. At normal incidence, however, the short axis of the ellipse is so small compared with the long axis that the light is effectively plane polarized along the long axis of the ellipse. The significant phenomenon in this instance is a slight rotation of the plane of polarization. We shall now investigate the reflection of light from the surface of a good conductor located in an external magnetostatic field.

First we must set up the field equations in a conducting dielectric located in a constant magnetic field $\mathbf{H}_0$. Since the imaginary part of a complex permittivity is indistinguishable from a conductivity we may limit ourselves to a medium with a real permittivity. Then (85–7) gives

$$\mathbf{D} = \kappa\mathbf{E} + \mathrm{i}\alpha\mathbf{H}_0 \times \mathbf{E}, \quad \alpha \equiv \frac{\omega}{Nec}(\kappa - 1)^2, \tag{88–1}$$

where N is the number of bound electrons per unit volume.

As regards the current, we have from the equation (67–14) describing the Hall effect,

$$\rho\mathbf{V} = \sigma\mathbf{E} - \beta\mathbf{H}_0 \times \mathbf{E}, \quad \beta \equiv \frac{3\pi}{8nec}\sigma^2, \tag{88–2}$$

where n is the number of free electrons per unit volume. From the equation of continuity (62–1) we find, with the aid of (82–4) and (82–10),

$$\rho = \mathbf{S}\cdot(\sigma\mathbf{E} - \beta\mathbf{H}_0 \times \mathbf{E}). \tag{88–3}$$

Therefore the field equations (62–12) take the form (82–11) with the effective electric displacement

$$\begin{aligned}\mathbf{D}_e &= \mathbf{D} + \mathrm{i}\left(\frac{\sigma}{\omega}\mathbf{E} - \frac{\beta}{\omega}\mathbf{H}_0 \times \mathbf{E}\right) \\ &= \left(\kappa + \mathrm{i}\frac{\sigma}{\omega}\right)\mathbf{E} + \mathrm{i}\left(\alpha - \frac{\beta}{\omega}\right)\mathbf{H}_0 \times \mathbf{E},\end{aligned} \tag{88–4}$$

in which we may consider $(\alpha - \beta/\omega)H_0$ to be very small compared with κ or σ/ω. As noted in article 82, $\mathbf{D}_e$ and $\mathbf{H}$ lie in the wave-front at right angles to each other, and $\mathbf{E}$ lies in the plane of $\mathbf{S}$ and $\mathbf{D}_e$.

As κ is very small compared with σ/ω for a good conductor such as we are interested in, we may approximate to the extent of writing

$$\mathbf{D}_e = \mathrm{i}\epsilon\{\mathbf{E} + \delta\mathbf{H}_0 \times \mathbf{E}\} \tag{88–5}$$

in place of (88–4), where $\epsilon \equiv \sigma/\omega$ and $\epsilon\delta \equiv \alpha - \beta/\omega$. Solving for $\mathbf{E}$ with neglect of the square and higher powers of δ,

$$\mathbf{E} = -\frac{\mathrm{i}}{\epsilon}\{\mathbf{D}_e - \delta\mathbf{H}_0 \times \mathbf{D}_e\}. \tag{88–6}$$

The wave equation (82–12) then becomes

$$(S^2 - \mathrm{i}\epsilon S_0{}^2)\mathbf{D}_e - \delta(S^2\mathbf{H}_0 \times \mathbf{D}_e - \mathbf{S}\cdot\mathbf{H}_0 \times \mathbf{D}_e\mathbf{S}) = 0. \tag{88–7}$$

Taking, for the moment, the X axis in the direction of propagation, $\mathbf{D}_e = \boldsymbol{j}D_y + \boldsymbol{k}D_z$ and $S^2\mathbf{H}_0 \times \mathbf{D}_e - \mathbf{S}\cdot\mathbf{H}_0 \times \mathbf{D}_e\mathbf{S} = S^2H_{0x}(-\boldsymbol{j}D_z + \boldsymbol{k}D_y)$ as in article 85. Consequently the non-vanishing components of the vector wave equation are

$$\left.\begin{aligned}(S^2 - \mathrm{i}\epsilon S_0{}^2)D_y &= -\ \delta S^2H_{0x}D_z,\\ (S^2 - \mathrm{i}\epsilon S_0{}^2)D_z &= \delta S^2H_{0x}D_y,\end{aligned}\right\} \qquad (88\text{–}8)$$

which give

$$S^2 - \mathrm{i}\epsilon S_0{}^2 = \pm\ \mathrm{i}\delta S^2H_{0x}, \qquad (88\text{–}9)$$

and

$$n = \frac{S}{S_0} = \sqrt{\epsilon}\,\mathrm{e}^{\mathrm{i}\pi/4}(1 \pm \tfrac{1}{2}\delta H_{0x}). \qquad (88\text{–}10)$$

Evidently the upper sign in (88–9) and (88–10) corresponds to the right-circularly polarized wave for which $D_z = -\ \mathrm{i}D_y$, and the lower sign to the left-circularly polarized wave for which $D_z = \mathrm{i}D_y$.

Next we must set up the boundary conditions at the surface of the conducting medium, which is represented by medium (2) of Fig. 102. For simplicity we shall suppose medium (1) to be empty space ($S_1 = S_0$) and shall assume the magnetic field to be perpendicular to the surface ($\mathbf{H}_0 = \boldsymbol{i}H_0$). As the general theory is rather complicated, we shall content ourselves with a discussion of the simple case of normal incidence.

Since the conducting medium can transmit only circularly polarized waves, it is appropriate to treat circularly polarized incident and reflected beams as fundamental, rather than plane polarized beams. Then a plane polarized incident beam can be synthesized by combining a right-circularly and a left-circularly polarized beam of the same amplitude.

First consider the right-circularly polarized incident wave

$$\left.\begin{aligned}\mathbf{E}_1 &= (\boldsymbol{j} - \mathrm{i}\boldsymbol{k})A_1\mathrm{e}^{\mathrm{i}\omega(S_0x - t)},\\ \mathbf{H}_1 &= (\mathrm{i}\boldsymbol{j} + \boldsymbol{k})A_1\mathrm{e}^{\mathrm{i}\omega(S_0x - t)}.\end{aligned}\right\} \qquad (88\text{–}11)$$

This will give rise to a reflected wave, which, if the reflecting medium were a perfect conductor, would obviously be left-circularly polarized. Hence we take for the reflected wave

$$\left.\begin{aligned}\mathbf{E}_1' &= (\boldsymbol{j} - \mathrm{i}\boldsymbol{k})A_1'\mathrm{e}^{\mathrm{i}\omega(-S_0x - t)},\\ \mathbf{H}_1' &= -\ (\mathrm{i}\boldsymbol{j} + \boldsymbol{k})A_1'\mathrm{e}^{\mathrm{i}\omega(-S_0x - t)}.\end{aligned}\right\} \qquad (88\text{–}12)$$

The transmitted wave must be polarized in the same sense as the incident wave. In accord with (85–15) we may write for the effective electric displacement

$$\mathbf{D}_{e2} = (\boldsymbol{j} - \mathrm{i}\boldsymbol{k})n_r^{\,2}A_2 e^{\mathrm{i}\omega(S_r x - t)},$$

where S_r is the wave-slowness and n_r the index of refraction obtained from (88–10) by using the upper sign. The electric intensity can be obtained from (88–6) or more easily from (82–14), and the magnetic intensity from (82–13). They are

$$\left.\begin{aligned} \mathbf{E}_2 &= (\boldsymbol{j} - \mathrm{i}\boldsymbol{k})A_2 e^{\mathrm{i}\omega(S_r x - t)}, \\ \mathbf{H}_2 &= (\mathrm{i}\boldsymbol{j} + \boldsymbol{k})n_r A_2 e^{\mathrm{i}\omega(S_r x - t)}. \end{aligned}\right\} \tag{88–13}$$

Since both **E** and **H** are tangential to the surface of the conductor in all three waves, the only boundary conditions are

$$\left.\begin{aligned} A_1 + A_1' &= A_2, \\ A_1 - A_1' &= n_r A_2. \end{aligned}\right\} \tag{88–14}$$

Eliminating A_2, we find for the coefficient of reflection A_1'/A_1

$$R_r = \frac{1 - n_r}{1 + n_r} \tag{88–15}$$

for the right-circularly polarized wave. To treat the left-circularly polarized wave all we need do is to replace i by $-$i in the vector part of the amplitude of the three waves, and substitute S_l and n_l, obtained from (88–10) by using the lower sign, for S_r and n_r respectively. Thus we get

$$R_l = \frac{1 - n_l}{1 + n_l} \tag{88–16}$$

for the coefficient of reflection of the left-circularly polarized wave.

Plane polarized incident radiation can be resolved into two circularly polarized beams, one right and the other left, of the same amplitude and phase. Thus we may write for an incident train of plane polarized waves

$$\begin{aligned} \mathbf{E}_1 &= \boldsymbol{j}A_1 e^{\mathrm{i}\omega(S_0 x - t)} \\ &= \tfrac{1}{2}\{(\boldsymbol{j} - \mathrm{i}\boldsymbol{k}) + (\boldsymbol{j} + \mathrm{i}\boldsymbol{k})\}A_1 e^{\mathrm{i}\omega(S_0 x - t)}. \end{aligned} \tag{88–17}$$

Then the electric intensity in the reflected train of waves is

$$\mathbf{E}_1' = \tfrac{1}{2}\{R_r(j - ik) + R_l(j + ik)\}A_1 e^{i\omega(-S_0x-t)}$$
$$= \tfrac{1}{2}\{(R_r + R_l)j - i(R_r - R_l)k\}A_1 e^{i\omega(-S_0x-t)}. \quad (88\text{–}18)$$

As $R_r \neq R_l$, the reflected wave has a component of electric intensity in the direction of the Z axis. Now, since $\epsilon \gg 1$ for a good conductor,

$$\frac{R_r - R_l}{R_r + R_l} = -\frac{n_r - n_l}{1 - n_r n_l} = \frac{-\sqrt{\epsilon}\,\delta H_0 e^{i(\pi/4)}}{1 - i\epsilon} \doteq -i\frac{\delta H_0}{\sqrt{2\epsilon}}(1 + i). \quad (88\text{–}19)$$

Therefore the long axis of the ellipse in the reflected light makes an angle $-\delta H_0/\sqrt{2\epsilon}$ with the plane of the electric vector in the incident light. As $\sigma/\omega \gg 1$, α is negligible compared with β/ω, and $\delta = -\beta/(\omega\epsilon)$ is positive since the electronic charge e in (88–2) is negative. Hence the rotation is in the *opposite* sense to that of the current in the solenoid producing the magnetic field. The predicted sense of rotation is observed when light is reflected from iron, nickel and cobalt, but the opposite sense in the case of magnetite.

89. The Zeeman Effect. — In article 65 we saw that the equation of motion of a bound electron in an atom or molecule placed in an external magnetostatic field $\mathbf{H}$ is the same, relative to axes rotating about the magnetic lines of force with angular velocity $\boldsymbol{\omega} = -(e/2mc)\mathbf{H}$, as the equation of motion relative to axes fixed in an inertial system in the absence of the field. If, then, the electron executes linear oscillations with angular frequency ω_0 in a fixed plane in the absence of the magnetic field, it will oscillate with the same frequency in a plane which rotates about the lines of force when the atom which contains it is placed in a magnetic field. Under these circumstances the radiation which it emits, as viewed by an observer in an inertial system, will contain, in addition to the frequency ω_0, frequencies which are combinations of ω_0 and ω. Therefore each line in the spectrum of an incandescent gas, which we may suppose to be due to a simple harmonic linear oscillation of a definite angular frequency ω_0, splits into several components when the source is placed in a magnetic field. We shall now develop the electromagnetic theory of this phenomenon, discovered by Zeeman in 1897.

Let XYZ (Fig. 106) be a set of axes fixed in the observer's inertial system with the X axis in the direction of the magnetic field $\mathbf{H}$. Then an electron which executes oscillations of frequency ω_0 and amplitude A along the fixed line OP in the absence of the field will oscillate with the same frequency and amplitude along a line $\overline{OP}$ in

a plane XOQ, which rotates about the X axis with angular velocity ω, when the field is present. The components of displacement of the electron along the X, Y and Z axes are then

$$x = A \cos \omega_0 t \cos \theta,$$
$$y = A \cos \omega_0 t \sin \theta \cos \omega t = \tfrac{1}{2} A \sin \theta \left\{\cos (\omega_0 + \omega)t + \cos (\omega_0 - \omega)t\right\},$$
$$z = A \cos \omega_0 t \sin \theta \sin \omega t = \tfrac{1}{2} A \sin \theta \left\{\sin (\omega_0 + \omega)t - \sin (\omega_0 - \omega)t\right\}.$$

Of these the part

$$y_1 = \tfrac{1}{2} A \sin \theta \cos (\omega_0 + \omega)t,$$
$$z_1 = \tfrac{1}{2} A \sin \theta \sin (\omega_0 + \omega)t,$$

represents a circular vibration in the YZ plane of angular frequency $\omega_0 + \omega$ in the counter-clockwise sense, and

$$y_2 = \tfrac{1}{2} A \sin \theta \cos (\omega_0 - \omega)t,$$
$$z_2 = - \tfrac{1}{2} A \sin \theta \sin (\omega_0 - \omega)t,$$

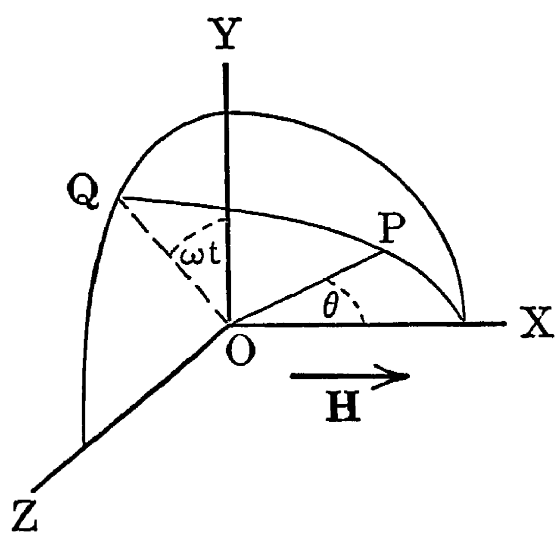

FIG. 106.

a circular vibration of angular frequency $\omega_0 - \omega$ in the clockwise sense. Hence the resultant oscillation consists of a linear oscillation of angular frequency ω_0 along the X axis and two circular oscillations in opposite senses in the YZ plane of angular frequencies $\omega_0 + \omega$ and $\omega_0 - \omega$, respectively. Each of the latter we can resolve, if we choose, into linear oscillations of the same amplitude along the Y and Z axes, differing in phase by a quarter period. Expressing ω in terms of H, the three true frequencies appearing when the source is placed in a magnetic field are

$$\left.\begin{aligned} \nu_0 &= \frac{\omega_0}{2\pi}, \\ \nu_1 &= \frac{\omega_0 + \omega}{2\pi} = \nu_0 - \frac{e}{4\pi mc} H, \\ \nu_2 &= \frac{\omega_0 - \omega}{2\pi} = \nu_0 + \frac{e}{4\pi mc} H. \end{aligned}\right\} \qquad (89\text{–}1)$$

The polarization of the radiation associated with each of these

frequencies depends upon the position of the observer. In *transverse observation* the radiation proceeding from a source located at the origin is viewed from a point in the YZ plane. Let the point of observation be on the Y axis. Then, as found in article 72, no radiation is received from the component oscillations along the Y axis, the component oscillations along the Z axis give rise to plane polarized waves with the electric vector perpendicular to the lines of force of the impressed magnetic field, and the component oscillation along the X axis to plane polarized waves with the electric vector parallel to the magnetic field. Plane polarized radiation with **E** perpendicular to the impressed field is labelled s (senkrecht) and with **E** parallel p (parallel). Hence in transverse observation the spectrum contains,

(a) (b)

FIG. 107.

in place of the single unpolarized line of frequency ν_0 existing in the absence of the magnetic field, the *normal Lorentz triplet* (Fig. 107*a*) consisting of three plane polarized lines of frequencies ν_1, ν_0 and ν_2, the middle one being polarized p and the two outer ones s.

In *longitudinal observation* the radiation proceeding from the origin is viewed from a point on the X axis. Therefore no radiation is received from the component oscillation along the X axis, but all component oscillations in the YZ plane contribute. Consequently no radiation of frequency ν_0 is observed, but two components of frequencies ν_1 and ν_2 (Fig. 107*b*), circularly polarized in contrary senses, are seen. The polarization of these components is labelled c.

From the observed separation

$$\Delta\nu = \frac{e}{4\pi mc} H \tag{89–2}$$

of the lines in the normal Lorentz triplet the ratio e/m may be calculated. This was one of the earliest accurate methods of measuring this important constant.

The normal Lorentz triplet is observed only in the simplest spectra. The anomalous Zeeman pattern more frequently found cannot be explained by electromagnetic theory without the aid of the quantum hypothesis.

CHAPTER 9

FOUR-DIMENSIONAL VECTOR ANALYSIS

90. Vectors and Vector Products. — In Chapter 2 we saw that, if we locate events in space-time by the rectangular coordinates x, y, z, $l \equiv ict$, the Lorentz transformation is represented by a simple rotation of the axes without change of scale. Therefore it is evident that not only the kinematical relations of the special relativity, but also the relations comprised in any theory, such as electrodynamics, which has its origin in the relativity theory, can be represented far more simply in terms of a four-dimensional vector analysis than in terms of the three-dimensional vector analysis employed heretofore. In this chapter we shall develop the four-dimensional vector analysis of a *flat* or *homaloidal* space-time. However, the reader must be warned that, while the four-dimensional representation of physical relations is very elegant in form, it is of very little aid in the solution of electromagnetic problems. Therefore we have not deprived him of a valuable analytical method in deferring until this chapter the development of four-dimensional vector analysis.

In a manifold of four or more dimensions a great deal of writing can be saved by adopting a more symmetrical notation than that commonly employed in three dimensions. We shall confine ourselves entirely to rectangular coordinate systems and shall generally designate the four coordinates by x_1, x_2, x_3, x_4 instead of x, y, z, l, always understanding l by x_4. The unit vectors parallel to the coordinate axes we shall denote by $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, $\boldsymbol{k}_4$, respectively.

In three-dimensional vector analysis we required only a single type of vector, that having the properties of a directed segment of a straight line. This simplicity was made possible by the fact that a plane area had a unique perpendicular, the direction of which we could take as representative of the vectorial properties of the two-dimensional area. Furthermore, as all three-dimensional volumes were vectorially equivalent, we were able to treat them as scalars. In four dimensions these devices are no longer possible. A two-dimensional parallelogram has no unique perpendicular and cannot

be represented vectorially by a directed linear segment. Moreover, three-dimensional parallelopipeds are not vectorially equivalent and must be treated as vectors rather than as scalars. Therefore we recognize four different types of vectors: a vector having the properties of a directed element of arc, a vector having the properties of a directed element of area, a vector having the properties of a directed element of three-dimensional volume and a vector having the properties of a directed element of four-dimensional volume. We designate them as vectors of the first, second, third and fourth ranks, respectively. Vectors of the fourth rank, being equivalent vectorially, are pseudo-scalars.

A vector **P** of the first rank is expressed in terms of its components P_1, P_2, P_3, P_4 along the X_1, X_2, X_3, X_4 axes by $\mathbf{P} = P_1\boldsymbol{k}_1 + P_2\boldsymbol{k}_2 + P_3\boldsymbol{k}_3 + P_4\boldsymbol{k}_4$, exactly as in three-dimensional analysis. We can write this expression more simply as $\mathbf{P} = \sum_{\alpha=1}^{4} P_\alpha \boldsymbol{k}_\alpha$, and, if we understand

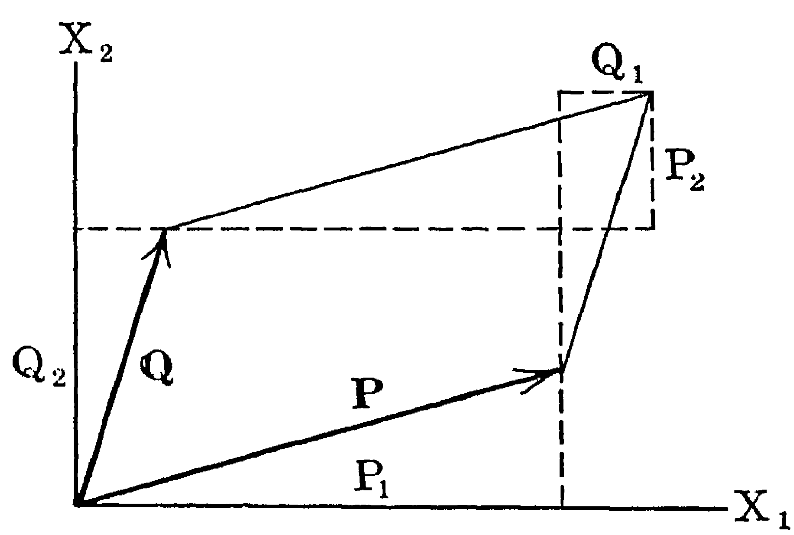

FIG. 108.

that a repeated *literal* suffix is always summed over, we may omit the summation sign and write merely $\mathbf{P} = P_\alpha \boldsymbol{k}_\alpha$. Evidently a repeated literal suffix may be replaced by any other letter. Thus $P_\alpha \boldsymbol{k}_\alpha = P_\beta \boldsymbol{k}_\beta$ since they both represent the same sum of products. We shall employ this summation convention consistently in this chapter, and shall find it very convenient when a number of suffixes are present. Obviously no suffix can appear more than twice in a single product.

Consider two vectors of the first rank, **P** and **Q**, lying in the X_1X_2 coordinate plane. The area of the parallelogram (Fig. 108), of which

$\mathbf{P}$ and $\mathbf{Q}$ are the sides, is clearly $P_1Q_2 - P_2Q_1$. We shall try to represent this area vectorially by a product $\mathbf{P} \times \mathbf{Q}$ obeying the distributive law. Carrying out the indicated multiplication we get

$$\mathbf{P} \times \mathbf{Q} = (P_1\boldsymbol{k}_1 + P_2\boldsymbol{k}_2) \times (Q_1\boldsymbol{k}_1 + Q_2\boldsymbol{k}_2)$$
$$= P_1Q_1\boldsymbol{k}_1 \times \boldsymbol{k}_1 + P_1Q_2\boldsymbol{k}_1 \times \boldsymbol{k}_2 + P_2Q_1\boldsymbol{k}_2 \times \boldsymbol{k}_1 + P_2Q_2\boldsymbol{k}_2 \times \boldsymbol{k}_2,$$

which yields a vector of the correct magnitude only if we make $\boldsymbol{k}_1 \times \boldsymbol{k}_1 = \boldsymbol{k}_2 \times \boldsymbol{k}_2 = 0$, $\boldsymbol{k}_2 \times \boldsymbol{k}_1 = -\boldsymbol{k}_1 \times \boldsymbol{k}_2$, exactly as in the case of the vector product in three-dimensional analysis. Then $\boldsymbol{k}_1 \times \boldsymbol{k}_2$ is a unit vector square in the X_1X_2 plane, and

$$\mathbf{P}\times\mathbf{Q} = (P_1\boldsymbol{k}_1+P_2\boldsymbol{k}_2)\times(Q_1\boldsymbol{k}_1+Q_2\boldsymbol{k}_2) = \begin{vmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{vmatrix} \boldsymbol{k}_1\times\boldsymbol{k}_2. \qquad (90\text{--}1)$$

We shall write the unit vector $\boldsymbol{k}_1 \times \boldsymbol{k}_2$ of the second rank in the abbreviated form $\boldsymbol{k}_1 \times \boldsymbol{k}_2 \equiv \boldsymbol{k}_{12}$. Then $\boldsymbol{k}_{21} = -\boldsymbol{k}_{12}$. As there are six coordinate planes there are six independent pairs of unit vectors of the second rank, $\boldsymbol{k}_{12} = -\boldsymbol{k}_{21}$, $\boldsymbol{k}_{23} = -\boldsymbol{k}_{32}$, $\boldsymbol{k}_{31} = -\boldsymbol{k}_{13}$, $\boldsymbol{k}_{14} = -\boldsymbol{k}_{41}$, $\boldsymbol{k}_{24} = -\boldsymbol{k}_{42}$, $\boldsymbol{k}_{34} = -\boldsymbol{k}_{43}$. Now, in expressing a vector of the first rank as the sum of its components we utilize only the four unit vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, $\boldsymbol{k}_4$ having the positive directions of the coordinate axes, discarding the four related unit vectors $-\boldsymbol{k}_1$, $-\boldsymbol{k}_2$, $-\boldsymbol{k}_3$, $-\boldsymbol{k}_4$ having the negative directions of the axes. So, in expressing a vector of the second rank as the sum of its components, we shall generally employ only the six unit vectors $\boldsymbol{k}_{12}$, $\boldsymbol{k}_{23}$, $\boldsymbol{k}_{31}$, $\boldsymbol{k}_{14}$, $\boldsymbol{k}_{24}$, $\boldsymbol{k}_{34}$, writing $\mathbf{M} = M_{12}\boldsymbol{k}_{12} + M_{23}\boldsymbol{k}_{23} + M_{31}\boldsymbol{k}_{31} + M_{14}\boldsymbol{k}_{14} + M_{24}\boldsymbol{k}_{24} + M_{34}\boldsymbol{k}_{34}$, or more simply $\mathbf{M} = M_{\alpha\beta}\boldsymbol{k}_{\alpha\beta}$, $\alpha\beta = 12, 23, 31, 14, 24, 34$. Nevertheless, we shall not hesitate, when it is convenient, to replace $\boldsymbol{k}_{12}$ by $-\boldsymbol{k}_{21}$, etc. Evidently $\mathbf{P} \times \mathbf{Q}$ obeys the anti-commutative law $\mathbf{Q} \times \mathbf{P} = -\mathbf{P} \times \mathbf{Q}$.

Next consider three vectors of the first rank, $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, all lying in the $X_1X_2X_3$ coordinate *planoid* or three-dimensional *subspace*. The volume of the parallelopiped of which they are the edges is $P_1Q_2R_3 + P_2Q_3R_1 + P_3Q_1R_2 - P_1Q_3R_2 - P_2Q_1R_3 - P_3Q_2R_1$ by (6–3). Although this quantity was represented by the triple scalar product in three-dimensional analysis, we shall try to represent it in the more general vector analysis which we are now developing by a product $\mathbf{P} \times \mathbf{Q} \times \mathbf{R}$ obeying the associative and distributive laws. We observe that

$$\mathbf{P} \times \mathbf{Q} \times \mathbf{R} = (P_1\boldsymbol{k}_1 + P_2\boldsymbol{k}_2 + P_3\boldsymbol{k}_3) \times (Q_1\boldsymbol{k}_1 + Q_2\boldsymbol{k}_2 + Q_3\boldsymbol{k}_3)$$

$$\times (R_1\boldsymbol{k}_1 + R_2\boldsymbol{k}_2 + R_3\boldsymbol{k}_3) = \begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix} \boldsymbol{k}_1 \times \boldsymbol{k}_2 \times \boldsymbol{k}_3 \quad (90\text{–}2)$$

only if $\boldsymbol{k}_\alpha \times \boldsymbol{k}_\beta \times \boldsymbol{k}_\gamma$ vanishes when any two suffixes are the same and changes sign when any two adjacent suffixes are interchanged. Evidently $\boldsymbol{k}_1 \times \boldsymbol{k}_2 \times \boldsymbol{k}_3$ is a unit vector cube in the $X_1X_2X_3$ coordinate planoid. For brevity we write $\boldsymbol{k}_1 \times \boldsymbol{k}_2 \times \boldsymbol{k}_3 \equiv \boldsymbol{k}_{123} = \boldsymbol{k}_{231} = \boldsymbol{k}_{312} = -\boldsymbol{k}_{321} = -\boldsymbol{k}_{213} = -\boldsymbol{k}_{132}$. As there are four coordinate planoids there are four independent sextets of unit vectors of the third rank, of which we shall generally employ only the four unit vectors $\boldsymbol{k}_{123}$, $\boldsymbol{k}_{234}$, $\boldsymbol{k}_{341}$, $\boldsymbol{k}_{412}$, writing $\mathbf{H} = H_{123}\boldsymbol{k}_{123} + H_{234}\boldsymbol{k}_{234} + H_{341}\boldsymbol{k}_{341} + H_{412}\boldsymbol{k}_{412}$ or, more simply, $\mathbf{H} = H_{\alpha\beta\gamma}\boldsymbol{k}_{\alpha\beta\gamma}$, $\alpha\beta\gamma = 123$, 234, 341, 412. Evidently $\boldsymbol{k}_{12} \times \boldsymbol{k}_3 = \boldsymbol{k}_1 \times \boldsymbol{k}_2 \times \boldsymbol{k}_3 = \boldsymbol{k}_{123}$, and $\boldsymbol{k}_3 \times \boldsymbol{k}_{12} = \boldsymbol{k}_{123}$ as well. Therefore the vector product of a vector **M** of the second rank by a vector **P** of the first rank obeys the commutative law $\mathbf{M} \times \mathbf{P} = \mathbf{P} \times \mathbf{M}$. In terms of the components of the two vectors,

$$\begin{aligned} \mathbf{P} \times \mathbf{M} = {} & (P_1M_{23} + P_2M_{31} + P_3M_{12}) \,\boldsymbol{k}_{123} \\ & + (P_2M_{34} - P_3M_{24} + P_4M_{23}) \,\boldsymbol{k}_{234} \\ & + (P_1M_{34} - P_3M_{14} - P_4M_{31}) \,\boldsymbol{k}_{341} \\ & + (P_1M_{24} - P_2M_{14} + P_4M_{12}) \,\boldsymbol{k}_{412}. \quad (90\text{–}3) \end{aligned}$$

Finally, the vector product of four vectors of the first rank, **P**, **Q**, **R** and **S**, gives us the pseudo-scalar

$$\mathbf{P} \times \mathbf{Q} \times \mathbf{R} \times \mathbf{S} = \begin{vmatrix} P_1 & P_2 & P_3 & P_4 \\ Q_1 & Q_2 & Q_3 & Q_4 \\ R_1 & R_2 & R_3 & R_4 \\ S_1 & S_2 & S_3 & S_4 \end{vmatrix} \boldsymbol{k}_1 \times \boldsymbol{k}_2 \times \boldsymbol{k}_3 \times \boldsymbol{k}_4, \quad (90\text{–}4)$$

provided $\boldsymbol{k}_\alpha \times \boldsymbol{k}_\beta \times \boldsymbol{k}_\gamma \times \boldsymbol{k}_\delta$ vanishes when any two suffixes are the same and changes sign when any two adjacent suffixes are interchanged. As before we write $\boldsymbol{k}_1 \times \boldsymbol{k}_2 \times \boldsymbol{k}_3 \times \boldsymbol{k}_4 \equiv \boldsymbol{k}_{1234}$. Evidently the twenty-four unit vectors of the fourth rank, obtained by permuting the suffixes in $\boldsymbol{k}_{1234}$, are all related. Therefore the unit

vector of the fourth rank, representing a unit four-dimensional cube, is a pseudo-scalar. We may interpret the magnitude of the product $\mathbf{P} \times \mathbf{Q} \times \mathbf{R} \times \mathbf{S}$ as the four-dimensional volume of the *hyper-parallelopiped* of which $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ are the edges. Evidently the vector product of a vector of the third rank by a vector of the first rank, or of two vectors of the second rank, is a vector of the fourth rank or a pseudo-scalar.

A vector of the second rank, which is expressible as the vector product of two vectors of the first rank, is said to be *uniplanar*. We shall now show that not all vectors of the second rank are uniplanar. Let, for instance, $\mathbf{M}$ and $\mathbf{N}$ be two uniplanar vectors of the second rank. If their planes intersect in a line, which we can represent by a vector $\mathbf{R}$ of first rank, we can write $\mathbf{M} = \mathbf{P} \times \mathbf{R}$ and $\mathbf{N} = \mathbf{Q} \times \mathbf{R}$, where $\mathbf{P}$ and $\mathbf{Q}$ are vectors of the first rank. Then $\mathbf{M} + \mathbf{N} = (\mathbf{P} + \mathbf{Q}) \times \mathbf{R}$ is a uniplanar vector of second rank. But if the planes of $\mathbf{M}$ and $\mathbf{N}$ intersect only in a point, as may happen in a four-dimensional space, this representation is impossible and $\mathbf{M} + \mathbf{N}$ is not uniplanar.

If $\mathbf{M} = \mathbf{P} \times \mathbf{Q} + \mathbf{R} \times \mathbf{S}$, where the plane of $\mathbf{P}$ and $\mathbf{Q}$ and the plane of $\mathbf{R}$ and $\mathbf{S}$ contain no common line, the vector $\mathbf{M}$ of the second rank is *biplanar*. We shall now prove that every vector of the second rank is at most biplanar. For suppose that $\mathbf{M}$ is *triplanar*, that is, $\mathbf{M} = \mathbf{P} \times \mathbf{Q} + \mathbf{R} \times \mathbf{S} + \mathbf{T} \times \mathbf{U}$. Then, as $\mathbf{P} \times \mathbf{Q}$ and $\mathbf{R} \times \mathbf{S}$ have no common line, we can express $\mathbf{T}$ in the form $\mathbf{T} = a\mathbf{P} + b\mathbf{Q} + c\mathbf{R} + d\mathbf{S}$ and $\mathbf{M}$ becomes $\mathbf{M} = (\mathbf{P} - b\mathbf{U}) \times (\mathbf{Q} + a\mathbf{U}) + (\mathbf{R} - d\mathbf{U}) \times (\mathbf{S} + c\mathbf{U})$, which is biplanar.

Let $\mathbf{M}$ be uniplanar. Then $\mathbf{M} = \mathbf{P} \times \mathbf{Q}$ and $\mathbf{M} \times \mathbf{M} = \mathbf{P} \times \mathbf{Q} \times \mathbf{P} \times \mathbf{Q} = 0$, since the determinant (90–4) has two rows the same. Conversely, if $\mathbf{M} \times \mathbf{M} = 0$, $\mathbf{M}$ is uniplanar. For $\mathbf{M}$ can always be expressed in the form $\mathbf{M} = \mathbf{P} \times \mathbf{Q} + \mathbf{R} \times \mathbf{S}$. Hence $\mathbf{M} \times \mathbf{M} = 2\mathbf{P} \times \mathbf{Q} \times \mathbf{R} \times \mathbf{S}$ and, if this vanishes, the four-dimensional volume of the hyper-parallelopiped of which $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ are the edges is zero. This implies that the planes of $\mathbf{P} \times \mathbf{Q}$ and $\mathbf{R} \times \mathbf{S}$ lie in a three-dimensional sub-space and therefore intersect in a line. Consequently $\mathbf{M}$ is uniplanar.

In any event

$$\mathbf{M} \times \mathbf{M} = 2(M_{12}M_{34} + M_{23}M_{14} + M_{31}M_{24})\, \mathbf{k}_{1234}. \qquad (90\text{–}5)$$

In the applications of four-dimensional vector analysis to electrodynamics we rarely have occasion to use other than vectors of the

first and second ranks. The first, on account of their four components, are commonly called *four-vectors*, and the second, on account of their six components, *six-vectors*.

Finally, we call the reader's attention to the fact that the rank of the vector product of two vectors is, in all instances, equal to the sum of the ranks of the two factors.

91. Transformation of Vectors. — When we pass from one set of rectangular axes $X_1X_2X_3X_4$ to another set $X_1'X_2'X_3'X_4'$ differently oriented the coordinates transform according to the equations

$$x_i' = l_{i\alpha}x_\alpha, \qquad x_\alpha = l'_{\alpha i}x_i', \tag{91–1}$$

as in (8–1), where $l_{i\alpha} = l'_{\alpha i}$ represents the cosine of the angle between the X_α and the X_i' axes. We find immediately, as noted in (8–5), that

$$l_{i\alpha} = l'_{\alpha i} = \frac{\partial x_i'}{\partial x_\alpha} = \frac{\partial x_\alpha}{\partial x_i'}. \tag{91–2}$$

Hence, if $\mathbf{k}_1'$, $\mathbf{k}_2'$, $\mathbf{k}_3'$, $\mathbf{k}_4'$ are unit vectors parallel to the X_1', X_2', X_3', X_4' axes,

$$\mathbf{k}_\alpha = \frac{\partial x_i'}{\partial x_\alpha}\,\mathbf{k}_i'. \tag{91–3}$$

Now, if $\mathbf{P}$ is a vector of the first rank,

$$\mathbf{P} = P_\alpha \mathbf{k}_\alpha = P_i'\mathbf{k}_i'.$$

Using (91–3), then,

$$P_i' = P_\alpha \frac{\partial x_i'}{\partial x_\alpha}, \tag{91–4}$$

and similarly

$$P_\alpha = P_i' \frac{\partial x_\alpha}{\partial x_i'}. \tag{91–5}$$

These are the transformations for the components of a vector of the first rank or four-vector.

Consider, for instance, the rotation of axes corresponding to the transformation (42–5) which takes us from the inertial system S to an inertial system S' moving relative to S in the direction of the X axis with velocity v. Evaluating the derivatives $\dfrac{\partial x_i'}{\partial x_\alpha}$ or $\dfrac{\partial x_\alpha}{\partial x_i'}$ we construct the table:

$$\begin{array}{c|c|c|c|c}
 & P_x & P_y & P_z & P_l \\
\hline
P_x' & k & 0 & 0 & i\beta k \\
P_y' & 0 & 1 & 0 & 0 \\
P_z' & 0 & 0 & 1 & 0 \\
P_l' & -i\beta k & 0 & 0 & k
\end{array} \tag{91-6}$$

giving the components of **P** relative to S' in terms of its components relative to S and *vice versa*. As usual $\beta \equiv v/c$ and $k \equiv 1/\sqrt{1 - \beta^2}$.

In the case of the unit vector of the second rank we have from (91–3)

$$\boldsymbol{k}_{\alpha\beta} = \boldsymbol{k}_\alpha \times \boldsymbol{k}_\beta = \left(\frac{\partial x_i'}{\partial x_\alpha} \boldsymbol{k}_i'\right) \times \left(\frac{\partial x_j'}{\partial x_\beta} \boldsymbol{k}_j'\right).$$

Hence, if we use the common compressed notation

$$\frac{\partial(x_i', x_j')}{\partial(x_\alpha, x_\beta)} \equiv \begin{vmatrix} \dfrac{\partial x_i'}{\partial x_\alpha} & \dfrac{\partial x_i'}{\partial x_\beta} \\[2ex] \dfrac{\partial x_j'}{\partial x_\alpha} & \dfrac{\partial x_j'}{\partial x_\beta} \end{vmatrix},$$

we find from (90–1) that

$$\boldsymbol{k}_{\alpha\beta} = \frac{\partial(x_i', x_j')}{\partial(x_\alpha, x_\beta)} \boldsymbol{k}_{ij}', \quad ij \text{ not permuted,} \tag{91-7}$$

where, by the designation "ij not permuted" we indicate that ij is limited to the values 12, 23, 31, 14, 24, 34.

Hence, if **M** is a vector of the second rank,

$$\mathbf{M} = M_{\alpha\beta}\boldsymbol{k}_{\alpha\beta} = M_{ij}'\boldsymbol{k}_{ij}', \quad \alpha\beta,\ ij \text{ not permuted,}$$

and (91–7) gives

$$M_{ij}' = M_{\alpha\beta} \frac{\partial(x_i', x_j')}{\partial(x_\alpha, x_\beta)}, \quad \alpha\beta \text{ not permuted.} \tag{91-8}$$

Similarly

$$M_{\alpha\beta} = M_{ij}' \frac{\partial(x_\alpha, x_\beta)}{\partial(x_i', x_j')}, \quad ij \text{ not permuted.} \tag{91-9}$$

These, then, are the transformations for the components of a vector of the second rank or six-vector.

Evaluating the determinants $\dfrac{\partial(x_i', x_j')}{\partial(x_\alpha, x_\beta)}$ or $\dfrac{\partial(x_\alpha, x_\beta)}{\partial(x_i', x_j')}$ for the transformation (42–5) we construct the table

	M_{xy}	M_{yz}	M_{zx}	M_{xl}	M_{yl}	M_{zl}
M'_{xy}	k	0	0	0	$-i\beta k$	0
M'_{yz}	0	1	0	0	0	0
M'_{zx}	0	0	k	0	0	$i\beta k$
M'_{xl}	0	0	0	1	0	0
M'_{yl}	$i\beta k$	0	0	0	k	0
M'_{zl}	0	0	$-i\beta k$	0	0	k

(91–10)

which gives the components of $\mathbf{M}$ relative to S' in terms of its components relative to S and *vice versa*.

If we replace M_{xy} by M_{zl}, M_{yz} by M_{xl}, M_{zx} by M_{yl}, M_{xl} by M_{yz}, M_{yl} by M_{zx}, M_{zl} by M_{xy} and similarly with the primed components, we obtain the table

	M_{zl}	M_{xl}	M_{yl}	M_{yz}	M_{zx}	M_{xy}
M'_{zl}	k	0	0	0	$-i\beta k$	0
M'_{xl}	0	1	0	0	0	0
M'_{yl}	0	0	k	0	0	$i\beta k$
M'_{yz}	0	0	0	1	0	0
M'_{zx}	$i\beta k$	0	0	0	k	0
M'_{xy}	0	0	$-i\beta k$	0	0	k

(91–11)

which is identical with (91–10). Therefore, if

$$\mathbf{M} \equiv M_{xy}\mathbf{k}_{12}+M_{yz}\mathbf{k}_{23}+M_{zx}\mathbf{k}_{31}+M_{xl}\mathbf{k}_{14}+M_{yl}\mathbf{k}_{24}+M_{zl}\mathbf{k}_{34} \quad (91–12)$$

is a six-vector,

$$\mathbf{M}^* \equiv M_{zl}\mathbf{k}_{12}+M_{xl}\mathbf{k}_{23}+M_{yl}\mathbf{k}_{31}+M_{yz}\mathbf{k}_{14}+M_{zx}\mathbf{k}_{24}+M_{xy}\mathbf{k}_{34} \quad (91–13)$$

is also. We call $\mathbf{M}^*$ the *associated six-vector* of $\mathbf{M}$.

We continue in the same manner with the vector of the third rank. From (91–3) and (90–2) we find

$$\mathbf{k}_{\alpha\beta\gamma} = \frac{\partial(x_i', x_j', x_k')}{\partial(x_\alpha, x_\beta, x_\gamma)}\,\mathbf{k}'_{ijk}, \quad ijk \text{ not permuted,} \qquad (91–14)$$

where ijk are limited to the values 123, 234, 341, 412. Hence, if $\mathbf{H}$ is a vector of the third rank,

$$\mathbf{H} = H_{\alpha\beta\gamma}\boldsymbol{k}_{\alpha\beta\gamma} = H'_{ijk}\,\boldsymbol{k}'_{ijk}, \quad \alpha\beta\gamma,\ ijk \text{ not permuted,}$$

and

$$H'_{ijk} = H_{\alpha\beta\gamma}\frac{\partial(x_i', x_j', x_k')}{\partial(x_\alpha, x_\beta, x_\gamma)}, \quad \alpha\beta\gamma \text{ not permuted,} \tag{91–15}$$

$$H_{\alpha\beta\gamma} = H'_{ijk}\frac{\partial(x_\alpha, x_\beta, x_\gamma)}{\partial(x_i', x_j', x_k')}, \quad ijk \text{ not permuted.} \tag{91–16}$$

For the transformation (42–5) these yield the table

	H_{xyz}	H_{yzl}	H_{zlx}	H_{lxy}
H'_{xyz}	k	$i\beta k$	0	0
H'_{yzl}	$-i\beta k$	k	0	0
H'_{zlx}	0	0	1	0
H'_{lxy}	0	0	0	1

(91–17)

Finally, for the unit vector of the fourth rank we find

$$\boldsymbol{k}_{\alpha\beta\gamma\delta} = \frac{\partial(x_i', x_j', x_k', x_l')}{\partial(x_\alpha, x_\beta, x_\gamma, x_\delta)}\,\boldsymbol{k}'_{ijkl} = \boldsymbol{k}'_{ijkl}, \quad ijkl \text{ not permuted,} \tag{91–18}$$

since the determinant of an orthogonal transformation is unity. Thus the one component of a vector of the fourth rank is the same relative to all reference systems. For this reason it is called a pseudo-scalar.

Just as in three-dimensional vector analysis, the only scalar and vector functions of the coordinates x_1, x_2, x_3, x_4 and of the components of a number of constant vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n$ which can appear in physical laws are *proper* scalar and vector functions. The square of the element of arc

$$\begin{aligned} d\lambda^2 &= dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \\ &= dx^2 + dy^2 + dz^2 - c^2dt^2 \end{aligned} \tag{91–19}$$

is evidently a proper scalar function or scalar invariant, since, when we transform from x_1, x_2, x_3, x_4 to x_1', x_2', x_3', x_4', it is found to be the same function of the latter variables as it is of the former, in accord with (42–4). Since we can write (91–19) in the form

$d\lambda^2 = -(1 - V^2/c^2)c^2dt^2$, where **V** is the three-dimensional vector velocity of a moving point, it follows that

$$K\frac{d}{dt}, \qquad K \equiv \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \tag{91–20}$$

is an invariant total differential operator. It corresponds to the time differential operator d/dt in three-dimensional analysis.

Evidently the position vector

$$\mathbf{R} = \boldsymbol{k}_1 x + \boldsymbol{k}_2 y + \boldsymbol{k}_3 z + \boldsymbol{k}_4 ict \tag{91–21}$$

of an event is the fundamental proper four-vector or directed linear segment. Operating with (91–20) we get the proper four-vector velocity

$$\mathbf{Q} = K\frac{d\mathbf{R}}{dt} = \boldsymbol{k}_1 KV_x + \boldsymbol{k}_2 KV_y + \boldsymbol{k}_3 KV_z + \boldsymbol{k}_4 iKc. \tag{91–22}$$

Operating again, we find the proper four-vector acceleration

$$\mathbf{S} = K\frac{d\mathbf{Q}}{dt} = \boldsymbol{k}_1 K^2\left(f_x + K^2\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}V_x\right) + \boldsymbol{k}_2 K^2\left(f_y + K^2\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}V_y\right)$$
$$+ \boldsymbol{k}_3 K^2\left(f_z + K^2\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}V_z\right) + \boldsymbol{k}_4 iK^4 c\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}, \tag{91–23}$$

where **f** is the three-dimensional vector acceleration $d\mathbf{V}/dt$, and operating once more we obtain the proper four-vector time rate of change of acceleration

$$\mathbf{T} = K\frac{d\mathbf{S}}{dt} = \boldsymbol{k}_1\left[K^3\left\{\dot{f}_x + K^2\frac{\dot{\mathbf{f}}\cdot\mathbf{V}}{c^2}V_x\right\} + 3K^5\left\{\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}f_x + K^2\left(\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\right)^2 V_x\right\}\right.$$
$$\left. + K^4\left\{\frac{f^2}{c^2} + K^2\left(\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\right)^2\right\}KV_x\right] + \boldsymbol{k}_2[\cdots] + \boldsymbol{k}_3[\cdots]$$
$$+ \boldsymbol{k}_4\left[iK^5 c\left\{\frac{\dot{\mathbf{f}}\cdot\mathbf{V}}{c^2} + \frac{f^2}{c^2} + 4K^2\left(\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\right)^2\right\}\right]. \tag{91–24}$$

Consider a charge distribution of density ρ_0 momentarily at rest in inertial system S_0. Then, if ρ is the charge density in system S and ρ' in system S',

$$\rho_0 = \rho\sqrt{1 - \frac{V^2}{c^2}} = \rho'\sqrt{1 - \frac{V'^2}{c^2}}$$

on account of the Fitzgerald-Lorentz contraction. Therefore ρ/K is a scalar invariant. Multiplying the four-vector velocity (91–22) by this proper scalar and dividing by the constant c, we get the four-vector current

$$\mathbf{P} = \boldsymbol{k}_1\rho\frac{V_x}{c} + \boldsymbol{k}_2\rho\frac{V_y}{c} + \boldsymbol{k}_3\rho\frac{V_z}{c} + \boldsymbol{k}_4\mathrm{i}\rho. \qquad (91\text{–}25)$$

The three space components are the three components of the current density $\rho\mathbf{V}$ divided by c, and the time component is the charge density ρ multiplied by i.

Given a three-dimensional vector function and a knowledge of the manner in which its components transform when we pass from one inertial system to another, we may be able to construct a four-dimensional vector function. Our success or failure is determined by whether or not the proposed four-dimensional vector function transforms in accord with the formulas developed in this article.

Consider, for instance, the three-dimensional vectors $\mathbf{E}$ and $\mathbf{B}$ (or $\mathbf{H}$), $\mathbf{D}$ and $\mathbf{F}$, whose transformation laws are given in (68–10). As $\mathbf{E}$ was defined originally in terms of a discrete number of tubes of force per unit cross-section, we should expect to be able to assemble the components of these vectors in such a way as to form four-dimensional vectors having the properties of directed areas. If we put

$$\mathbf{M} \equiv B_z\boldsymbol{k}_{12} + B_x\boldsymbol{k}_{23} + B_y\boldsymbol{k}_{31} - \mathrm{i}E_x\boldsymbol{k}_{14} - \mathrm{i}E_y\boldsymbol{k}_{24} - \mathrm{i}E_z\boldsymbol{k}_{34} \qquad (91\text{–}26)$$

we find, by comparing (68–10) with (91–10), that $\mathbf{M}$ is a six-vector. The associated six-vector is

$$\mathbf{M}^* \equiv -\mathrm{i}E_z\boldsymbol{k}_{12} - \mathrm{i}E_x\boldsymbol{k}_{23} - \mathrm{i}E_y\boldsymbol{k}_{31} + B_x\boldsymbol{k}_{14} + B_y\boldsymbol{k}_{24} + B_z\boldsymbol{k}_{34}. \qquad (91\text{–}27)$$

Similarly we find that

$$\mathbf{N} \equiv F_z\boldsymbol{k}_{12} + F_x\boldsymbol{k}_{23} + F_y\boldsymbol{k}_{31} - \mathrm{i}D_x\boldsymbol{k}_{14} - \mathrm{i}D_y\boldsymbol{k}_{24} - \mathrm{i}D_z\boldsymbol{k}_{34} \qquad (91\text{–}28)$$

is a six-vector, its associated six-vector being

$$\mathbf{N}^* \equiv -\mathrm{i}D_z\boldsymbol{k}_{12} - \mathrm{i}D_x\boldsymbol{k}_{23} - \mathrm{i}D_y\boldsymbol{k}_{31} + F_x\boldsymbol{k}_{14} + F_y\boldsymbol{k}_{24} + F_z\boldsymbol{k}_{34}. \qquad (91\text{–}29)$$

Since

$$\mathbf{M} \times \mathbf{M} = \mathbf{M}^* \times \mathbf{M}^* = -2\mathrm{i}(E_xB_x + E_yB_y + E_zB_z)\boldsymbol{k}_{1234},$$

$$\mathbf{N} \times \mathbf{N} = \mathbf{N}^* \times \mathbf{N}^* = -2\mathrm{i}(D_xF_x + D_yF_y + D_zF_z)\boldsymbol{k}_{1234},$$

M and **M*** are uniplanar only when **E** and **B** (or **H**) are at right angles, and **N** and **N*** only when **D** and **F** are perpendicular. Thus **M** and **N** in a plane electromagnetic wave in an isotropic medium are uniplanar.

Problem 91a. The transformations for the components of the current density and for the charge density are given in (59–1). Verify from these that (91–25) is a four-vector.

92. Scalar Products.—Let **M** be a six-vector. Since $\mathbf{k}_{21} = -\mathbf{k}_{12}$, $M_{12}\mathbf{k}_{12} = \frac{1}{2}M_{12}\mathbf{k}_{12} - \frac{1}{2}M_{12}\mathbf{k}_{21}$. Hence, if we put $m_{12} \equiv \frac{1}{2}M_{12}$, $m_{21} \equiv -\frac{1}{2}M_{12}$, we can express the six-vector in the more symmetric form

$$\mathbf{M} = M_{\alpha\beta}\mathbf{k}_{\alpha\beta}, \quad \alpha\beta \text{ not permuted,}$$

$$= m_{\alpha\beta}\mathbf{k}_{\alpha\beta}, \quad \alpha\beta \text{ permuted,} \tag{92–1}$$

where, in the last expression, $\alpha\beta$ assumes all the permutations 12, 21, 23, 32, 31, 13, 14, 41, 24, 42, 34, 43. Now the transformation (91–8) may be written

$$M'_{ij} = M_{\alpha\beta}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\beta} - M_{\alpha\beta}\frac{\partial x_i'}{\partial x_\beta}\frac{\partial x_j'}{\partial x_\alpha}, \quad \alpha\beta \text{ not permuted,}$$

$$= 2m_{\alpha\beta}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\beta}, \qquad \alpha\beta \text{ permuted,}$$

which gives

$$m'_{ij} = m_{\alpha\beta}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\beta}, \quad \alpha\beta \text{ permuted,} \tag{92–2}$$

and, similarly,

$$m_{\alpha\beta} = m'_{ij}\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}, \quad ij \text{ permuted.} \tag{92–3}$$

Similarly, if **H** is a vector of third rank, $H_{123}\mathbf{k}_{123} = \frac{1}{6}H_{123}\mathbf{k}_{123} + \frac{1}{6}H_{123}\mathbf{k}_{231} + \frac{1}{6}H_{123}\mathbf{k}_{312} - \frac{1}{6}H_{123}\mathbf{k}_{321} - \frac{1}{6}H_{123}\mathbf{k}_{213} - \frac{1}{6}H_{123}\mathbf{k}_{132}$, and, if we put $h_{123} = h_{231} = h_{312} \equiv \frac{1}{6}H_{123}$, $h_{321} = h_{213} = h_{132} \equiv -\frac{1}{6}H_{123}$,

$$\mathbf{H} = H_{\alpha\beta\gamma}\mathbf{k}_{\alpha\beta\gamma}, \quad \alpha\beta\gamma \text{ not permuted,}$$

$$= h_{\alpha\beta\gamma}\mathbf{k}_{\alpha\beta\gamma}, \quad \alpha\beta\gamma \text{ permuted.} \tag{92–4}$$

From this it follows that the transformation (91–15) may be written

$$H'_{ijk} = H_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\beta}\frac{\partial x_k'}{\partial x_\gamma} + H_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\beta}\frac{\partial x_j'}{\partial x_\gamma}\frac{\partial x_k'}{\partial x_\alpha} + H_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\gamma}\frac{\partial x_j'}{\partial x_\alpha}\frac{\partial x_k'}{\partial x_\beta}$$

$$- H_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\gamma}\frac{\partial x_j'}{\partial x_\beta}\frac{\partial x_k'}{\partial x_\alpha} - H_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\beta}\frac{\partial x_j'}{\partial x_\alpha}\frac{\partial x_k'}{\partial x_\gamma} - H_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\gamma}\frac{\partial x_k'}{\partial x_\beta},$$

$\alpha\beta\gamma$ not permuted,

$$= 6h_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\beta}\frac{\partial x_k'}{\partial x_\gamma}, \quad \alpha\beta\gamma \text{ permuted.}$$

Hence,

$$h'_{ijk} = h_{\alpha\beta\gamma}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\beta}\frac{\partial x_k'}{\partial x_\gamma}, \quad \alpha\beta\gamma \text{ permuted,} \tag{92–5}$$

and, similarly,

$$h_{\alpha\beta\gamma} = h'_{ijk}\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}\frac{\partial x_\gamma}{\partial x_k'}, \quad ijk \text{ permuted.} \tag{92–6}$$

By means of the vector product we have been able to construct a vector of a rank equal to the sum of the ranks of the two factors. Now we shall devise a product, known as the scalar product, whose rank is equal to the difference of the ranks of the two factors. As our geometrical visualization of a four-dimensional manifold is restricted, we must proceed by strictly analytical methods. The fundamental formulas which we require are

$$\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\alpha}{\partial x_j'} = \delta_{ij}, \qquad \frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_i'}{\partial x_\beta} = \delta_{\alpha\beta}, \tag{92–7}$$

where the *Kronecker delta* δ_{ij} is a symbol representing unity when $i = j$ and zero when $i \neq j$, and similarly for $\delta_{\alpha\beta}$.

Remembering that the partial derivatives appearing in (92–7) represent cosines of angles between the primed and unprimed axes in accord with (91–2), we could infer the relations (92–7) as the four-dimensional generalization of the result of problem 5*b*. However we shall give a formal proof. From (91–2)

$$\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\alpha}{\partial x_j'} = \frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial x_j'}$$

$$= \frac{\partial x_i'}{\partial x_1}\frac{\partial x_1}{\partial x_j'} + \frac{\partial x_i'}{\partial x_2}\frac{\partial x_2}{\partial x_j'} + \frac{\partial x_i'}{\partial x_3}\frac{\partial x_3}{\partial x_j'} + \frac{\partial x_i'}{\partial x_4}\frac{\partial x_4}{\partial x_j'}.$$

But this is just the total change in x_i' per unit change in x_j', all the other x''s remaining constant. Consequently, as x_1', x_2', x_3', x_4' are independent coordinates, the sum of products vanishes when $i \neq j$ and equals unity when $i = j$.

The scalar product of the unit vectors $\boldsymbol{k}_\alpha$ and $\boldsymbol{k}_\beta$ of the first rank is defined by the relation.

$$\boldsymbol{k}_\alpha \cdot \boldsymbol{k}_\beta = \delta_{\alpha\beta}. \tag{92–8}$$

To justify this definition, which is identical with that of three-dimensional vector analysis, we must show that when we take the scalar product of two four-vectors $\mathbf{P}$ and $\mathbf{Q}$ we obtain a vector of zero rank or scalar. Using (91–5) we have

$$\mathbf{P} \cdot \mathbf{Q} = P_\alpha Q_\alpha = P_i' Q_j' \frac{\partial x_\alpha}{\partial x_i'} \frac{\partial x_\alpha}{\partial x_j'} = P_i' Q_i'$$

from (92–7), which proves the validity of the definition (92–8). The product

$$\mathbf{P} \cdot \mathbf{P} = P_1^2 + P_2^2 + P_3^2 + P_4^2$$

is called the *square* of the four-vector $\mathbf{P}$.

Next we define the scalar product of the unit vector $\boldsymbol{k}_{\alpha\beta}$ of the second rank by the unit vector $\boldsymbol{k}_\gamma$ of the first rank by the relation

$$\boldsymbol{k}_{\alpha\beta} \cdot \boldsymbol{k}_\gamma = \boldsymbol{k}_\gamma \cdot \boldsymbol{k}_{\alpha\beta} = \boldsymbol{k}_\alpha \delta_{\beta\gamma} - \boldsymbol{k}_\beta \delta_{\alpha\gamma}. \tag{92–9}$$

To justify this, we must show that the scalar product of a six-vector $\mathbf{M}$ by a four-vector $\mathbf{P}$ is a four-vector. Using (92–1) we find

$$\begin{aligned} \mathbf{M} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{M} &= m_{\alpha\beta} P_\beta \boldsymbol{k}_\alpha - m_{\alpha\beta} P_\alpha \boldsymbol{k}_\beta, \quad \alpha\beta \text{ permuted}, \\ &= m_{\alpha\beta} P_\beta \boldsymbol{k}_\alpha - m_{\beta\alpha} P_\beta \boldsymbol{k}_\alpha, \quad \alpha\beta \text{ permuted}, \\ &= 2 m_{\alpha\beta} P_\beta \boldsymbol{k}_\alpha, \qquad\qquad\qquad \alpha\beta \text{ permuted}, \end{aligned}$$

since $m_{\beta\alpha} = -m_{\alpha\beta}$. To complete the proof we must show that the components of the last expression obey the transformation law of a four-vector. From (91–5) and (92–3) we have

$$2 m_{\alpha\beta} P_\beta = 2 m'_{ij} P_k' \frac{\partial x_\alpha}{\partial x_i'} \frac{\partial x_\beta}{\partial x_j'} \frac{\partial x_\beta}{\partial x_k'} = 2 m'_{ij} P_j' \frac{\partial x_\alpha}{\partial x_i'}$$

with the aid of (92–7). But this is the transformation (91–5) for the components of a four-vector.

The formula (92–9) implies that the scalar product vanishes when

the unit vector of second rank does not contain the suffix appearing in the unit vector of first rank. If, however, the former does contain the suffix appearing in the latter, we arrange the product so that the common suffix comes second in $\boldsymbol{k}_{\alpha\beta}$ and then cross it out in both vectors. Thus $\boldsymbol{k}_{34}\cdot\boldsymbol{k}_4 = \boldsymbol{k}_3$ and $\boldsymbol{k}_{34}\cdot\boldsymbol{k}_3 = -\boldsymbol{k}_{43}\cdot\boldsymbol{k}_3 = -\boldsymbol{k}_4$. Consequently, if $\mathbf{M}$ is the six-vector $M_{12}\boldsymbol{k}_{12} + M_{23}\boldsymbol{k}_{23} + M_{31}\boldsymbol{k}_{31} + M_{14}\boldsymbol{k}_{14} + M_{24}\boldsymbol{k}_{24} + M_{34}\boldsymbol{k}_{34}$ and $\mathbf{P}$ the four-vector $P_1\boldsymbol{k}_1 + P_2\boldsymbol{k}_2 + P_3\boldsymbol{k}_3 + P_4\boldsymbol{k}_4$,

$$\begin{aligned}\mathbf{M}\cdot\mathbf{P} = \mathbf{P}\cdot\mathbf{M} = (&\qquad\qquad\; + P_2M_{12} - P_3M_{31} + P_4M_{14})\boldsymbol{k}_1\\ &+(-P_1M_{12} \qquad\qquad + P_3M_{23} + P_4M_{24})\boldsymbol{k}_2\\ &+(P_1M_{31} - P_2M_{23} \qquad\qquad + P_4M_{34})\boldsymbol{k}_3\\ &+(-P_1M_{14} - P_2M_{24} - P_3M_{34} \qquad\qquad)\boldsymbol{k}_4. \qquad (92\text{–}10)\end{aligned}$$

The differential operator in four-dimensional analysis corresponding to ∇ in three dimensions is

$$\diamondsuit = \boldsymbol{k}_\alpha \frac{\partial}{\partial x_\alpha} = \boldsymbol{k}_1 \frac{\partial}{\partial x_1} + \boldsymbol{k}_2 \frac{\partial}{\partial x_2} + \boldsymbol{k}_3 \frac{\partial}{\partial x_3} + \boldsymbol{k}_4 \frac{\partial}{\partial x_4}, \qquad (92\text{–}11)$$

pronounced "lor" after H. A. Lorentz. Since

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial x_i'}{\partial x_\alpha}\frac{\partial}{\partial x_i'} = \frac{\partial x_\alpha}{\partial x_i'}\frac{\partial}{\partial x_i'}$$

by (91–2), we see that the components of $\diamondsuit$ transform in accord with the law (91–5) for a four-vector, and that $\diamondsuit$ is a proper four-vector differential operator.

The scalar product

$$\diamondsuit\cdot\mathbf{P} = \frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \frac{\partial P_3}{\partial x_3} + \frac{\partial P_4}{\partial x_4} \qquad (92\text{–}12)$$

of $\diamondsuit$ with the four-vector $\mathbf{P}$ is the four-dimensional divergence of $\mathbf{P}$, and the vector product

$$\begin{aligned}\diamondsuit\times\mathbf{P} = &\left(\frac{\partial P_2}{\partial x_1} - \frac{\partial P_1}{\partial x_2}\right)\boldsymbol{k}_{12} + \left(\frac{\partial P_3}{\partial x_2} - \frac{\partial P_2}{\partial x_3}\right)\boldsymbol{k}_{23} + \left(\frac{\partial P_1}{\partial x_3} - \frac{\partial P_3}{\partial x_1}\right)\boldsymbol{k}_{31}\\ &+\left(\frac{\partial P_4}{\partial x_1} - \frac{\partial P_1}{\partial x_4}\right)\boldsymbol{k}_{14} + \left(\frac{\partial P_4}{\partial x_2} - \frac{\partial P_2}{\partial x_4}\right)\boldsymbol{k}_{24} + \left(\frac{\partial P_4}{\partial x_3} - \frac{\partial P_3}{\partial x_4}\right)\boldsymbol{k}_{34} \quad (92\text{–}13)\end{aligned}$$

is the four-dimensional curl of **P**. The invariant scalar differential operator

$$\begin{aligned}\Diamond \cdot \Diamond &= \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} + \frac{\partial^2}{\partial x_4{}^2} \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},\end{aligned} \tag{92–14}$$

known as the *d'Alembertian*, is the four-dimensional analog of the Laplacian $\nabla \cdot \nabla$.

Replacing the components of **P** in (92–10) by those of $\Diamond$ we find for the four-dimensional divergence of the six-vector **M**

$$\begin{aligned}\Diamond \cdot \mathbf{M} = &\left(\qquad + \frac{\partial M_{12}}{\partial x_2} - \frac{\partial M_{31}}{\partial x_3} + \frac{\partial M_{14}}{\partial x_4}\right) \boldsymbol{k}_1 \\ &+ \left(-\frac{\partial M_{12}}{\partial x_1} \qquad + \frac{\partial M_{23}}{\partial x_3} + \frac{\partial M_{24}}{\partial x_4}\right) \boldsymbol{k}_2 \\ &+ \left(\frac{\partial M_{31}}{\partial x_1} - \frac{\partial M_{23}}{\partial x_2} \qquad + \frac{\partial M_{34}}{\partial x_4}\right) \boldsymbol{k}_3 \\ &+ \left(-\frac{\partial M_{14}}{\partial x_1} - \frac{\partial M_{24}}{\partial x_2} - \frac{\partial M_{34}}{\partial x_3} \qquad \right) \boldsymbol{k}_4.\end{aligned} \tag{92–15}$$

We note that the scalar $\Diamond \cdot (\Diamond \cdot \mathbf{M})$ vanishes identically.

Similarly, if we replace the components of **P** in (90–3) by those of $\Diamond$ we obtain the explicit form of the four-dimensional curl of **M**. It is

$$\begin{aligned}\Diamond \times \mathbf{M} = &\left(\frac{\partial M_{23}}{\partial x_1} + \frac{\partial M_{31}}{\partial x_2} + \frac{\partial M_{12}}{\partial x_3} \qquad \right) \boldsymbol{k}_{123} \\ &+ \left(\qquad + \frac{\partial M_{34}}{\partial x_2} - \frac{\partial M_{24}}{\partial x_3} + \frac{\partial M_{23}}{\partial x_4}\right) \boldsymbol{k}_{234} \\ &+ \left(\frac{\partial M_{34}}{\partial x_1} \qquad - \frac{\partial M_{14}}{\partial x_3} - \frac{\partial M_{31}}{\partial x_4}\right) \boldsymbol{k}_{341} \\ &+ \left(\frac{\partial M_{24}}{\partial x_1} - \frac{\partial M_{14}}{\partial x_2} \qquad + \frac{\partial M_{12}}{\partial x_4}\right) \boldsymbol{k}_{412}.\end{aligned} \tag{92–16}$$

The scalar product of the unit vector $\boldsymbol{k}_{\alpha\beta\gamma}$ of the third rank by the unit vector $\boldsymbol{k}_\delta$ of the first rank is defined by the relation

$$\boldsymbol{k}_{\alpha\beta\gamma} \cdot \boldsymbol{k}_\delta = \boldsymbol{k}_\delta \cdot \boldsymbol{k}_{\alpha\beta\gamma} = \boldsymbol{k}_{\alpha\beta}\delta_{\gamma\delta} + \boldsymbol{k}_{\gamma\alpha}\delta_{\beta\delta} + \boldsymbol{k}_{\beta\gamma}\delta_{\alpha\delta}. \tag{92–17}$$

Now, if **H** is a vector of the third rank, we have from (92–4)

$$\mathbf{H}\cdot\mathbf{P} = \mathbf{P}\cdot\mathbf{H} = h_{\alpha\beta\gamma}P_\gamma \boldsymbol{k}_{\alpha\beta} + h_{\alpha\beta\gamma}P_\beta \boldsymbol{k}_{\gamma\alpha} + h_{\alpha\beta\gamma}P_\alpha \boldsymbol{k}_{\beta\gamma}, \quad \alpha\beta\gamma \text{ permuted,}$$
$$= h_{\alpha\beta\gamma}P_\gamma \boldsymbol{k}_{\alpha\beta} + h_{\beta\gamma\alpha}P_\gamma \boldsymbol{k}_{\alpha\beta} + h_{\gamma\alpha\beta}P_\gamma \boldsymbol{k}_{\alpha\beta}, \quad \alpha\beta\gamma \text{ permuted,}$$
$$= 3h_{\alpha\beta\gamma}P_\gamma \boldsymbol{k}_{\alpha\beta}, \quad \alpha\beta\gamma \text{ permuted,}$$

since $h_{\gamma\alpha\beta} = h_{\beta\gamma\alpha} = h_{\alpha\beta\gamma}$. But, from (92–6) and (92–7),

$$3h_{\alpha\beta\gamma}P_\gamma = 3h'_{ijk}P_l' \frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}\frac{\partial x_\gamma}{\partial x_k'}\frac{\partial x_\gamma}{\partial x_l'} = 3h'_{ijk}P_k' \frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}.$$

As $h_{\beta\alpha\gamma}P_\gamma = -h_{\alpha\beta\gamma}P_\gamma$ this is the transformation for the components of a six-vector. Therefore the definition (92–17) is justified, since it yields a vector of the second rank for the scalar product of a vector of the third rank by a vector of the first rank.

The formula (92–17) follows the same rule as (92–9). If $\boldsymbol{k}_{\alpha\beta\gamma}$ and $\boldsymbol{k}_\delta$ do not contain a common suffix, the scalar product of the two unit vectors is zero. If, on the other hand, they do contain a common suffix, we arrange the product so that the common suffix comes at the end in the former, and then cross it out in both vectors. Thus $\boldsymbol{k}_{123}\cdot\boldsymbol{k}_3 = \boldsymbol{k}_{12}$ and $\boldsymbol{k}_{123}\cdot\boldsymbol{k}_2 = -\boldsymbol{k}_{132}\cdot\boldsymbol{k}_2 = -\boldsymbol{k}_{13} = \boldsymbol{k}_{31}$. Similar rules hold for the scalar product of other unit vectors, but, as we shall have no occasion to use them, we shall consider no further cases of the single scalar product.

The double scalar product of two unit vectors of the second rank is important, however. We define this product by the relation

$$\boldsymbol{k}_{\alpha\beta} : \boldsymbol{k}_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}. \qquad (92\text{–}18)$$

To justify this definition we must prove that the double scalar product **M** : **N** of two six-vectors **M** and **N** is a scalar. From (92–1) we have

$$\mathbf{M} : \mathbf{N} = \mathbf{N} : \mathbf{M} = m_{\alpha\beta}n_{\alpha\beta} - m_{\alpha\beta}n_{\beta\alpha}, \quad \alpha\beta \text{ permuted,}$$
$$= 2m_{\alpha\beta}n_{\alpha\beta}, \quad \alpha\beta \text{ permuted,}$$

since $n_{\beta\alpha} = -n_{\alpha\beta}$. Now, by (92–3) and (92–7),

$$2m_{\alpha\beta}n_{\alpha\beta} = 2m'_{ij}n'_{kl}\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}\frac{\partial x_\alpha}{\partial x_k'}\frac{\partial x_\beta}{\partial x_l'} = 2m'_{ij}\,n'_{ij}, \quad ij \text{ permuted,}$$

which proves that **M** : **N** is a scalar.

Since we employ only the permutations 12, 23, 31, 14, 24, 34 in the suffices of unit vectors of the second rank, it follows from (92–18)

that $\boldsymbol{k}_{\alpha\beta} : \boldsymbol{k}_{\gamma\delta}$ vanishes unless $\gamma\delta$ is identical with $\alpha\beta$, in which case the product is unity. The product

$$\mathbf{M} : \mathbf{M} = M_{12}^2 + M_{23}^2 + M_{31}^2 + M_{14}^2 + M_{24}^2 + M_{34}^2$$

is known as the *square* of the six-vector **M**.

93. Four-Dimensional Formulation of the Equations of Electromagnetism. — We start by expressing the field equations (62–12*a*) to (62–12*d*) in four-dimensional vector form. From (92–15) we have for the four-dimensional divergence of the field six-vector (91–27)

$$\begin{aligned}\diamondsuit\cdot\mathbf{M}^* = &-i\left(\qquad + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial B_x}{\partial t}\right)\boldsymbol{k}_1\\ &-i\left(-\frac{\partial E_z}{\partial x} \qquad + \frac{\partial E_x}{\partial z} + \frac{1}{c}\frac{\partial B_y}{\partial t}\right)\boldsymbol{k}_2\\ &-i\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \qquad + \frac{1}{c}\frac{\partial B_z}{\partial t}\right)\boldsymbol{k}_3\\ &-\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\qquad\right)\boldsymbol{k}_4. \end{aligned} \tag{93–1}$$

Comparing with (62–12) we observe that the equation $\diamondsuit\cdot\mathbf{M}^* = 0$ expresses, in its three space components, Faraday's law (62–12*c*), and, in its time component, Coulomb's law (62–12*b*).

Again, the four-dimensional divergence of the field six-vector (91–28) is

$$\begin{aligned}\diamondsuit\cdot\mathbf{N} = &\left(\qquad + \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} - \frac{1}{c}\frac{\partial D_x}{\partial t}\right)\boldsymbol{k}_1\\ &+\left(-\frac{\partial F_z}{\partial x} \qquad + \frac{\partial F_x}{\partial z} - \frac{1}{c}\frac{\partial D_y}{\partial t}\right)\boldsymbol{k}_2\\ &+\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \qquad - \frac{1}{c}\frac{\partial D_z}{\partial t}\right)\boldsymbol{k}_3\\ &+i\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\qquad\right)\boldsymbol{k}_4. \end{aligned} \tag{93–2}$$

If we equate this to the current four-vector (91–25) we find that the three space components of the equation $\diamondsuit\cdot\mathbf{N} = \mathbf{P}$ represent the Ampère-Maxwell law (62–12*d*) and the time component represents

Coulomb's law (62–12a). Incidentally we notice that the equation $\diamondsuit \cdot \mathbf{P} = 0$ is the equation of continuity (62–1).

Next we write down the scalar product $\mathbf{P}\cdot\mathbf{M}$ of the current four-vector (91–25) and the field six-vector (91–26). From (92–10)

$$\begin{aligned}\mathbf{P}\cdot\mathbf{M} = &\left(\qquad\qquad\qquad + \frac{1}{c}\rho V_y B_z - \frac{1}{c}\rho V_z B_y + \rho E_x\right)\boldsymbol{k}_1 \\ &+ \left(-\frac{1}{c}\rho V_x B_z \qquad\qquad + \frac{1}{c}\rho V_z B_x + \rho E_y\right)\boldsymbol{k}_2 \\ &+ \left(\frac{1}{c}\rho V_x B_y - \frac{1}{c}\rho V_y B_x \qquad\qquad + \rho E_z\right)\boldsymbol{k}_3 \\ &+ \frac{i}{c}\left(\rho V_x E_x + \rho V_y E_y + \rho V_z E_z \qquad\right)\boldsymbol{k}_4. \qquad (93\text{–}3)\end{aligned}$$

We observe that the three space components of this four-vector are the three components of the force per unit volume (62–12e) on the free charge ρ, the time component being proportional to the rate at which the field does work on the free charge. If we understand by F this four-dimensional force the electromagnetic equations (62–12) assume the simpler form

$$\left.\begin{aligned}\diamondsuit\cdot\mathbf{N} = \mathbf{P}, \quad (a) \qquad \diamondsuit\cdot\mathbf{M}^* = 0, \quad (b) \\ \mathsf{F} = \mathbf{P}\cdot\mathbf{M}, \quad (c)\end{aligned}\right\} \qquad (93\text{–}4)$$

in four-dimensional vector notation.

By forming the double scalar product of pairs of six-vectors selected from (91–26) to (91–29) we obtain a number of invariants of the Lorentz transformation, of which the most important are

$$\left.\begin{aligned}\mathbf{M} : \mathbf{M} &= \mathbf{M}^* : \mathbf{M}^* = B^2 - E^2, \\ \mathbf{N} : \mathbf{N} &= \mathbf{N}^* : \mathbf{N}^* = F^2 - D^2, \\ \mathbf{M} : \mathbf{M}^* &= -2i(E_xB_x + E_yB_y + E_zB_z), \\ \mathbf{N} : \mathbf{N}^* &= -2i(D_xF_x + D_yF_y + D_zF_z).\end{aligned}\right\} \qquad (93\text{–}5)$$

Since $\diamondsuit\cdot(\diamondsuit\cdot\mathbf{N})$ vanishes identically, as proved in article 92, it follows from (93–4a) that the equation of continuity $\diamondsuit\cdot\mathbf{P} = 0$ is satisfied by the electromagnetic equations.

It is very often convenient, in order to save writing all the components in full, to express the three space components of a four-vector

in three-dimensional vector language. Thus we may express the current vector (91–25) in the form

$$\mathbf{P} = \frac{\rho}{c}\mathbf{V} + k_4 i\rho$$

where $\mathbf{V} \equiv k_1 V_x + k_2 V_y + k_3 V_z$. Adopting this notation, (93–1) may be written

$$\diamondsuit \cdot \mathbf{M}^* = -i\nabla \times \mathbf{E} - \frac{i}{c}\dot{\mathbf{B}} - \nabla \cdot \mathbf{B}k_4$$

and (93–2) in the form

$$\diamondsuit \cdot \mathbf{N} = \nabla \times \mathbf{F} - \frac{1}{c}\dot{\mathbf{D}} + i\nabla \cdot \mathbf{D}k_4.$$

In the same way we may replace (93–3) with

$$\mathbf{P}\cdot\mathbf{M} = \rho\left(\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{B}\right) + \frac{i}{c}\rho\mathbf{V}\cdot\mathbf{E}k_4.$$

In this form these expressions have a much more familiar appearance and consume considerably less space.

Now, if we take the scalar product of (93–4*b*) by $\mathbf{N}^*$ we obtain

$$(\diamondsuit \cdot \mathbf{M}^*)\cdot\mathbf{N}^* = -\nabla\cdot\mathbf{B}\,\mathbf{F} - (\nabla \times \mathbf{E}) \times \mathbf{D} + \frac{1}{c}\mathbf{D} \times \dot{\mathbf{B}}$$
$$+ i\left\{\nabla \times \mathbf{E}\cdot\mathbf{F} + \frac{1}{c}\mathbf{F}\cdot\dot{\mathbf{B}}\right\} k_4 = 0,$$

and, if we take the scalar product of (93–4*a*) by $\mathbf{M}$ and make use of (93–4*c*),

$$(\diamondsuit \cdot \mathbf{N})\cdot\mathbf{M} = \nabla\cdot\mathbf{D}\,\mathbf{E} + (\nabla \times \mathbf{F}) \times \mathbf{B} - \frac{1}{c}\dot{\mathbf{D}} \times \mathbf{B}$$
$$+ i\left\{\nabla \times \mathbf{F}\cdot\mathbf{E} - \frac{1}{c}\mathbf{E}\cdot\dot{\mathbf{D}}\right\} k_4 = \mathsf{F}.$$

Subtracting the first of these equations from the second, we have

$$\mathsf{F} = -\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) + \nabla\cdot(\mathbf{DE} + \mathbf{BF}) - \nabla\mathbf{E}\cdot\mathbf{D} - \nabla\mathbf{F}\cdot\mathbf{B}$$
$$- \frac{i}{c}\left\{\mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{F}\cdot\dot{\mathbf{B}} + c\nabla\cdot(\mathbf{E} \times \mathbf{F})\right\}k_4. \tag{93–6}$$

We recognize the space component of this four-vector equation as

the equation (70–1) which formed the basis of our discussion of electromagnetic stresses, and the time component as the energy equation (69–1).

Next we shall look for a four-dimensional representation of the retarded expressions (50–7) and (50–8) for the scalar and vector potentials of a point charge e. Let x, y, z be the coordinates of the point occupied by the charge at the time t, and x_P, y_P, z_P, t_P the coordinates and time at the field-point P. Then

$$\mathbf{R} = \mathbf{k}_1(x - x_P) + \mathbf{k}_2(y - y_P) + \mathbf{k}_3(z - z_P) + \mathbf{k}_4 ic(t - t_P) \quad (93\text{–}7)$$

is the fundamental four-vector expressing the space-time interval between these two events. Operating with the invariant differential operator (91–20) and dividing by the universal constant c we obtain the four-vector

$$\mathbf{G} \equiv \mathbf{k}_1 K\frac{V_x}{c} + \mathbf{k}_2 K\frac{V_y}{c} + \mathbf{k}_3 K\frac{V_z}{c} + \mathbf{k}_4 iK, \quad (93\text{–}8)$$

which is just the four-vector velocity (91–22) divided by c. The scalar product of $\mathbf{R}$ and $\mathbf{G}$ is

$$\mathbf{R}\cdot\mathbf{G} = K\left\{\frac{(x-x_P)V_x}{c} + \frac{(y-y_P)V_y}{c} + \frac{(z-z_P)V_z}{c} - c(t-t_P)\right\} \quad (93\text{–}9)$$

Now, if $\mathbf{r} \equiv \mathbf{k}_1(x_P - x) + \mathbf{k}_2(y_P - y) + \mathbf{k}_3(z_P - z)$ is the three-dimensional position vector of the field-point P relative to the charge, the condition for retardation is $r = c(t_P - t)$. Also $(x - x_P)V_x + (y - y_P)V_y + (z - z_P)V_z = -\mathbf{r}\cdot\mathbf{V}$. Therefore the retarded value of $\mathbf{R}\cdot\mathbf{G}$ is

$$[\mathbf{R}\cdot\mathbf{G}] = \left[K\left(r - \frac{\mathbf{r}\cdot\mathbf{V}}{c}\right)\right] = \left[Kr\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)\right] \quad (93\text{–}10)$$

in terms of the velocity $\mathbf{c}$ of the moving-elements. If, now, we construct the four-vector

$$\left[\frac{e\mathbf{G}}{4\pi\mathbf{R}\cdot\mathbf{G}}\right] = \left[\frac{e}{4\pi r\left(1 - \frac{\mathbf{c}\cdot\mathbf{V}}{c^2}\right)}\left\{\mathbf{k}_1\frac{V_x}{c} + \mathbf{k}_2\frac{V_y}{c} + \mathbf{k}_3\frac{V_z}{c} + \mathbf{k}_4 \mathrm{i}\right\}\right] \quad (93\text{–}11)$$

we notice that its space component is the vector potential $\mathbf{A}$ of a point charge e as specified by (50–8) and its time component is the scalar potential Φ of the point charge as given by (50–7) multiplied by i.

Summing over all the elementary fields which extend to the point P, we have then

$$\mathbf{W} \equiv \sum_i \left[\frac{e_i \mathbf{G}_i}{4\pi \mathbf{R}_i \cdot \mathbf{G}_i}\right] = \boldsymbol{k}_1 A_x + \boldsymbol{k}_2 A_y + \boldsymbol{k}_3 A_z + \boldsymbol{k}_4 \mathrm{i}\Phi. \qquad (93\text{–}12)$$

If, now, we take the four-dimensional curl of the four-vector potential $\mathbf{W}$ we find, by (92–13),

$$\begin{aligned}\Diamond \times \mathbf{W} = &\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{k}_{12} + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{k}_{23} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{k}_{31} \\ &+ \mathrm{i}\left(\frac{\partial \Phi}{\partial x} + \frac{1}{c}\frac{\partial A_x}{\partial t}\right)\boldsymbol{k}_{14} + \mathrm{i}\left(\frac{\partial \Phi}{\partial y} + \frac{1}{c}\frac{\partial A_y}{\partial t}\right)\boldsymbol{k}_{24} + \mathrm{i}\left(\frac{\partial \Phi}{\partial z} + \frac{1}{c}\frac{\partial A_z}{\partial t}\right)\boldsymbol{k}_{34}.\end{aligned} \qquad (93\text{–}13)$$

Referring to (50–5) and (50–6) we observe that the components of the six-vector $\Diamond \times \mathbf{W}$ are H_z, H_x, H_y, $-\mathrm{i}E_x$, $-\mathrm{i}E_y$, $-\mathrm{i}E_z$. But these are the components of the field six-vector $\mathbf{M}$ specified by (91–26), since $\mathbf{B}$ and $\mathbf{H}$ are merely different symbols for the same physical quantity. Hence the three-dimensional vector equations $\mathbf{H} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\Phi - (1/c)\dot{\mathbf{A}}$ are both contained in the single six-vector equation

$$\mathbf{M} = \Diamond \times \mathbf{W}. \qquad (93\text{–}14)$$

Finally we shall express the equation of motion (57–12) of the Lorentz electron in four-dimensional vector notation. If we take the scalar product of the ratio $\mathbf{G}$ of the four-vector velocity to the velocity of light given by (93–8), and the field six-vector $\mathbf{M}$ specified by (91–26) multiplied by the constant charge e of the electron, we get

$$\begin{aligned} e\mathbf{G}\cdot\mathbf{M} = eK\Big\{&\left(\qquad\qquad + \frac{1}{c}V_y H_z - \frac{1}{c}V_z H_y + E_x\right)\boldsymbol{k}_1 \\ &+ \left(-\frac{1}{c}V_x H_z \qquad\qquad + \frac{1}{c}V_z H_x + E_y\right)\boldsymbol{k}_2 \\ &+ \left(\frac{1}{c}V_x H_y - \frac{1}{c}V_y H_x \qquad\qquad + E_z\right)\boldsymbol{k}_3 \\ &+ \frac{\mathrm{i}}{c}\left(V_x E_x + V_y E_y + V_z E_z \qquad\qquad\right)\boldsymbol{k}_4\Big\} \\ = eK&\left\{\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{H} + \frac{\mathrm{i}}{c}\mathbf{V}\cdot\mathbf{E}\boldsymbol{k}_4\right\}, \end{aligned} \qquad (93\text{–}15)$$

provided we replace **B** by **H** in (91–26). We see, therefore, that the left-hand member of (57–12) is not the space component of a four-vector, but that it becomes one if multiplied by K. Multiplying the entire equation by K, then, and expanding the triple vector products in the right-hand members, we have, if we omit the subscripts on **E** and **H**,

$$eK\left\{\mathbf{E}+\frac{1}{c}\mathbf{V}\times\mathbf{H}\right\} = mK^2\left\{\mathbf{f}+K^2\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\mathbf{V}\right\} - nK^3\left\{\dot{\mathbf{f}}+K^2\frac{\dot{\mathbf{f}}\cdot\mathbf{V}}{c^2}\mathbf{V}\right\}$$
$$-3nK^5\left\{\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\mathbf{f}+K^2\left(\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\right)^2\mathbf{V}\right\}+\cdots. \quad (93\text{–}16)$$

The first term on the right is just the space component of the four-vector acceleration **S** given by (91–23), multiplied by the rest mass m. The sum of the two remaining terms, however, is not proportional to the space component of the four-vector time rate of change of acceleration **T** specified by (91–24). Evidently a four-vector proportional to the four-vector velocity **Q** must be subtracted from **T** before we can make use of this vector. As the coefficient of **Q** in the subtrahend must be a scalar invariant involving only the components of **V** and **f**, let us consider the square of **S** as a possible coefficient. We find

$$S^2 = \mathbf{S}\cdot\mathbf{S} = K^4\left(f^2+2K^2\frac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}+K^4\frac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}\frac{V^2}{c^2}\right)-K^8\frac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}$$
$$= K^4\left(f^2+K^2\frac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}\right).$$

Consequently

$$\mathbf{T}-\frac{1}{c_2}S^2\mathbf{Q}=\mathbf{k}_1\left[K^3\left\{\dot{f}_x+K^2\frac{\dot{\mathbf{f}}\cdot\mathbf{V}}{c^2}V_x\right\}+3K^5\left\{\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}f_x+K^2\left(\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\right)^2V_x\right\}\right]$$
$$+\mathbf{k}_2[\cdots]+\mathbf{k}_3[\cdots]+\mathbf{k}_4\left[icK^5\left\{\frac{\dot{\mathbf{f}}\cdot\mathbf{V}}{c^2}+3K^2\left(\frac{\mathbf{f}\cdot\mathbf{V}}{c^2}\right)^2\right\}\right] \quad (93\text{–}17)$$

is the four-vector to the space component of which the sum of the second and third terms on the right of (93–16) is proportional.

The equation of motion (93–16) of the Lorentz electron is given, then, by the space component of the four-vector equation

$$e\mathbf{G}\cdot\mathbf{M} = m\mathbf{S} - n\left(\mathbf{T}-\frac{1}{c^2}S^2\mathbf{Q}\right)+\cdots. \quad (93\text{–}18)$$

If we write $\mathscr{D}$ for the scalar invariant operator $K\dfrac{d}{dt}$ this may be written more significantly in the form

$$\frac{e}{c}(\mathscr{D}\mathbf{R})\cdot\mathbf{M}=m\mathscr{D}^2\mathbf{R}-n\left\{\mathscr{D}^3\mathbf{R}-\frac{1}{c^2}(\mathscr{D}^2\mathbf{R})\cdot(\mathscr{D}^2\mathbf{R})\mathscr{D}\mathbf{R}\right\}+\cdots. \quad (93\text{–}19)$$

The time component of the four-vector equation (93–18) is

$$e\mathbf{E}\cdot\mathbf{V} = mK^3\mathbf{f}\cdot\mathbf{V} - nK^4\left\{\dot{\mathbf{f}}\cdot\mathbf{V} + 3K^2\,\frac{\overline{\mathbf{f}\cdot\mathbf{V}}^2}{c^2}\right\} + \cdots. \quad (93\text{–}20)$$

It is evident from (57–12) that this represents the rate at which work is done by the impressed field.

94. Tensors. — In the language of tensors a scalar invariant is a tensor of zero rank and a four-vector a tensor of first rank. The tensor of second rank, with which we shall be concerned in this article, is the four-dimensional analog of the dyadic in three-dimensional vector analysis and, like the latter, can best be thought of as an operator, which, when multiplied into a four-vector, yields a new four-vector of different magnitude and different orientation, the components of which are homogeneous linear functions of the components of the first four-vector.

The general tensor $\boldsymbol{\Psi}$ of the second rank is represented by

$$\left.\begin{aligned}\boldsymbol{\Psi} &= a_{\alpha\beta}\boldsymbol{k}_\alpha\boldsymbol{k}_\beta, \quad \alpha\beta \text{ permuted,}\\ &= a'_{ij}\boldsymbol{k}_i{}'\boldsymbol{k}_j{}', \quad ij \text{ permuted.}\end{aligned}\right\} \quad (94\text{–}1)$$

From (91–3) we find

$$a'_{ij} = a_{\alpha\beta}\,\frac{\partial x_i{}'}{\partial x_\alpha}\,\frac{\partial x_j{}'}{\partial x_\beta} \quad (94\text{–}2)$$

and similarly

$$a_{\alpha\beta} = a'_{ij}\,\frac{\partial x_\alpha}{\partial x_i{}'}\,\frac{\partial x_\beta}{\partial x_j{}'} \quad (94\text{–}3)$$

for the transformation of the elements of the tensor.

If $a_{\beta\alpha} = a_{\alpha\beta}$ the tensor is said to be symmetric, and if $a_{\beta\alpha} = -a_{\alpha\beta}$, skew-symmetric. The symmetry properties of a tensor are invariant. To prove this, consider a tensor which is symmetric when referred to the unprimed axes. Then $a_{\beta\alpha} = a_{\alpha\beta}$. Now, from (94–2),

$$a'_{ji} = a_{\beta\alpha}\,\frac{\partial x_j{}'}{\partial x_\beta}\,\frac{\partial x_i{}'}{\partial x_\alpha} = a_{\alpha\beta}\,\frac{\partial x_i{}'}{\partial x_\alpha}\,\frac{\partial x_j{}'}{\partial x_\beta} = a'_{ij}.$$

Similarly $a'_{ji} = -a'_{ij}$ if $a_{\beta\alpha} = -a_{\alpha\beta}$.

Taking the scalar product of the tensor $\boldsymbol{\Psi}$ with the four-vector $\mathbf{P}$ we get

$$\boldsymbol{\Psi}\cdot\mathbf{P} = a_{\alpha\beta}P_\beta\boldsymbol{k}_\alpha.$$

To show that this a four-vector we have from (91–5) and (94–3)

$$a_{\alpha\beta}P_\beta = a'_{ij}P_k'\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}\frac{\partial x_\beta}{\partial x_k'} = a'_{ij}P_j'\frac{\partial x_\alpha}{\partial x_i'},$$

which is the transformation (91–5) for the components of a four-vector.

The conjugate $\boldsymbol{\Psi}_c$ of the tensor $\boldsymbol{\Psi}$ is formed by interchanging the antecedents and consequents of $\boldsymbol{\Psi}$. Thus, if $\boldsymbol{\Psi}$ is represented by (94–1),

$$\boldsymbol{\Psi}_c = b_{\alpha\beta}\boldsymbol{k}_\alpha\boldsymbol{k}_\beta, \qquad b_{\alpha\beta} \equiv a_{\beta\alpha}.$$

By (94–2)

$$b'_{ij} = a'_{ji} = a_{\beta\alpha}\frac{\partial x_j'}{\partial x_\beta}\frac{\partial x_i'}{\partial x_\alpha} = b_{\alpha\beta}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_j'}{\partial x_\beta}.$$

Hence, if $\boldsymbol{\Psi}$ is a tensor, $\boldsymbol{\Psi}_c$ is also.

If we construct the array

$$\boldsymbol{\Psi} = m_{\alpha\beta}\boldsymbol{k}_\alpha\boldsymbol{k}_\beta$$

out of the coefficients $m_{\alpha\beta}$ of (92–1), defined in terms of the components $M_{\alpha\beta}$ of a six-vector by the relations $m_{12} = -m_{21} = \frac{1}{2}M_{12}$, etc., $\boldsymbol{\Psi}$ is a skew-symmetric tensor of the second rank since the law (92–2) for the transformation of the quantities $m_{\alpha\beta}$ is identical with the law (94–2) for the transformation of the elements of a tensor, and, in addition, $m_{\beta\alpha} = -m_{\alpha\beta}$.

The contraction of a tensor $\boldsymbol{\Psi}$ is the quantity obtained by putting a dot (scalar product) between the unit vectors in each term. Thus we obtain $a_{\alpha\alpha}$ for the contraction of $\boldsymbol{\Psi} = a_{\alpha\beta}\boldsymbol{k}_\alpha\boldsymbol{k}_\beta$. That this is a scalar invariant is proved very simply as follows:

$$a'_{ii} = a_{\alpha\beta}\frac{\partial x_i'}{\partial x_\alpha}\frac{\partial x_i'}{\partial x_\beta} = a_{\alpha\alpha}.$$

Therefore the sum of the elements on the principal diagonal of any tensor of the second rank is a scalar invariant.

If we take the scalar product of two tensors $\boldsymbol{\Psi} = a_{\alpha\beta}\boldsymbol{k}_\alpha\boldsymbol{k}_\beta$ and $\boldsymbol{\Phi} = b_{\gamma\delta}\boldsymbol{k}_\gamma\boldsymbol{k}_\delta$ we get

$$\boldsymbol{\Psi}\cdot\boldsymbol{\Phi} = a_{\alpha\beta}b_{\beta\delta}\boldsymbol{k}_\alpha\boldsymbol{k}_\delta.$$

To prove that this is a tensor we note that

$$a_{\alpha\beta}b_{\beta\delta} = a'_{ij}b'_{kl}\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\beta}{\partial x_j'}\frac{\partial x_\beta}{\partial x_k'}\frac{\partial x_\delta}{\partial x_l'} = a'_{ij}b'_{jl}\frac{\partial x_\alpha}{\partial x_i'}\frac{\partial x_\delta}{\partial x_l'}.$$

Now we shall construct an important tensor known as the stress-momentum-energy tensor. First we write down the two skew-symmetric tensors built from the components of the six-vectors **M** (91–26) and **M*** (91–27). These are:

$$\begin{aligned}\mathbf{\Theta} = {} & 0\boldsymbol{k}_1\boldsymbol{k}_1 + \tfrac{1}{2}B_z\boldsymbol{k}_1\boldsymbol{k}_2 - \tfrac{1}{2}B_y\boldsymbol{k}_1\boldsymbol{k}_3 - \tfrac{1}{2}\mathrm{i}E_x\boldsymbol{k}_1\boldsymbol{k}_4\\ & - \tfrac{1}{2}B_z\boldsymbol{k}_2\boldsymbol{k}_1 + 0\boldsymbol{k}_2\boldsymbol{k}_2 + \tfrac{1}{2}B_x\boldsymbol{k}_2\boldsymbol{k}_3 - \tfrac{1}{2}\mathrm{i}E_y\boldsymbol{k}_2\boldsymbol{k}_4\\ & + \tfrac{1}{2}B_y\boldsymbol{k}_3\boldsymbol{k}_1 - \tfrac{1}{2}B_x\boldsymbol{k}_3\boldsymbol{k}_2 + 0\boldsymbol{k}_3\boldsymbol{k}_3 - \tfrac{1}{2}\mathrm{i}E_z\boldsymbol{k}_3\boldsymbol{k}_4\\ & + \tfrac{1}{2}\mathrm{i}E_x\boldsymbol{k}_4\boldsymbol{k}_1 + \tfrac{1}{2}\mathrm{i}E_y\boldsymbol{k}_4\boldsymbol{k}_2 + \tfrac{1}{2}\mathrm{i}E_z\boldsymbol{k}_4\boldsymbol{k}_3 + 0\boldsymbol{k}_4\boldsymbol{k}_4,\end{aligned} \tag{94–4}$$

$$\begin{aligned}\mathbf{X} = {} & 0\boldsymbol{k}_1\boldsymbol{k}_1 - \tfrac{1}{2}\mathrm{i}E_z\boldsymbol{k}_1\boldsymbol{k}_2 + \tfrac{1}{2}\mathrm{i}E_y\boldsymbol{k}_1\boldsymbol{k}_3 + \tfrac{1}{2}B_x\boldsymbol{k}_1\boldsymbol{k}_4\\ & + \tfrac{1}{2}\mathrm{i}E_z\boldsymbol{k}_2\boldsymbol{k}_1 + 0\boldsymbol{k}_2\boldsymbol{k}_2 - \tfrac{1}{2}\mathrm{i}E_x\boldsymbol{k}_2\boldsymbol{k}_3 + \tfrac{1}{2}B_y\boldsymbol{k}_2\boldsymbol{k}_4\\ & - \tfrac{1}{2}\mathrm{i}E_y\boldsymbol{k}_3\boldsymbol{k}_1 + \tfrac{1}{2}\mathrm{i}E_x\boldsymbol{k}_3\boldsymbol{k}_2 + 0\boldsymbol{k}_3\boldsymbol{k}_3 + \tfrac{1}{2}B_z\boldsymbol{k}_3\boldsymbol{k}_4\\ & - \tfrac{1}{2}B_x\boldsymbol{k}_4\boldsymbol{k}_1 - \tfrac{1}{2}B_y\boldsymbol{k}_4\boldsymbol{k}_2 - \tfrac{1}{2}B_z\boldsymbol{k}_4\boldsymbol{k}_3 + 0\boldsymbol{k}_4\boldsymbol{k}_4.\end{aligned} \tag{94–5}$$

Next we write down the conjugates of the two skew-symmetric tensors built from the components of the six-vectors **N** (91–28) and **N*** (91–29). They are:

$$\begin{aligned}\mathbf{\Phi} = {} & 0\boldsymbol{k}_1\boldsymbol{k}_1 - \tfrac{1}{2}F_z\boldsymbol{k}_1\boldsymbol{k}_2 + \tfrac{1}{2}F_y\boldsymbol{k}_1\boldsymbol{k}_3 + \tfrac{1}{2}\mathrm{i}D_x\boldsymbol{k}_1\boldsymbol{k}_4\\ & + \tfrac{1}{2}F_z\boldsymbol{k}_2\boldsymbol{k}_1 + 0\boldsymbol{k}_2\boldsymbol{k}_2 - \tfrac{1}{2}F_x\boldsymbol{k}_2\boldsymbol{k}_3 + \tfrac{1}{2}\mathrm{i}D_y\boldsymbol{k}_2\boldsymbol{k}_4\\ & - \tfrac{1}{2}F_y\boldsymbol{k}_3\boldsymbol{k}_1 + \tfrac{1}{2}F_x\boldsymbol{k}_3\boldsymbol{k}_2 + 0\boldsymbol{k}_3\boldsymbol{k}_3 + \tfrac{1}{2}\mathrm{i}D_z\boldsymbol{k}_3\boldsymbol{k}_4\\ & - \tfrac{1}{2}\mathrm{i}D_x\boldsymbol{k}_4\boldsymbol{k}_1 - \tfrac{1}{2}\mathrm{i}D_y\boldsymbol{k}_4\boldsymbol{k}_2 - \tfrac{1}{2}\mathrm{i}D_z\boldsymbol{k}_4\boldsymbol{k}_3 + 0\boldsymbol{k}_4\boldsymbol{k}_4,\end{aligned} \tag{94–6}$$

$$\begin{aligned}\mathbf{\Omega} = {} & 0\boldsymbol{k}_1\boldsymbol{k}_1 + \tfrac{1}{2}\mathrm{i}D_z\boldsymbol{k}_1\boldsymbol{k}_2 - \tfrac{1}{2}\mathrm{i}D_y\boldsymbol{k}_1\boldsymbol{k}_3 - \tfrac{1}{2}F_x\boldsymbol{k}_1\boldsymbol{k}_4\\ & - \tfrac{1}{2}\mathrm{i}D_z\boldsymbol{k}_2\boldsymbol{k}_1 + 0\boldsymbol{k}_2\boldsymbol{k}_2 + \tfrac{1}{2}\mathrm{i}D_x\boldsymbol{k}_2\boldsymbol{k}_3 - \tfrac{1}{2}F_y\boldsymbol{k}_2\boldsymbol{k}_4\\ & + \tfrac{1}{2}\mathrm{i}D_y\boldsymbol{k}_3\boldsymbol{k}_1 - \tfrac{1}{2}\mathrm{i}D_x\boldsymbol{k}_3\boldsymbol{k}_2 + 0\boldsymbol{k}_3\boldsymbol{k}_3 - \tfrac{1}{2}F_z\boldsymbol{k}_3\boldsymbol{k}_4\\ & + \tfrac{1}{2}F_x\boldsymbol{k}_4\boldsymbol{k}_1 + \tfrac{1}{2}F_y\boldsymbol{k}_4\boldsymbol{k}_2 + \tfrac{1}{2}F_z\boldsymbol{k}_4\boldsymbol{k}_3 + 0\boldsymbol{k}_4\boldsymbol{k}_4.\end{aligned} \tag{94–7}$$

The stress-momentum-energy tensor $\mathbf{\Psi}$ is defined as

$$\mathbf{\Psi} \equiv a_{\alpha\beta}\mathbf{k}_\alpha\mathbf{k}_\beta = 2(\mathbf{X}\cdot\mathbf{\Omega} - \mathbf{\Phi}\cdot\mathbf{\Theta}). \tag{94–8}$$

We find for its elements

$$\begin{aligned}
a_{11} &= \tfrac{1}{2}(E_xD_x - E_yD_y - E_zD_z) + \tfrac{1}{2}(F_xB_x - F_yB_y - F_zB_z),\\
a_{12} &= E_yD_x + F_yB_x,\\
a_{21} &= E_xD_y + F_xB_y,\\
&\cdots\cdots\cdots\cdots\\
a_{14} &= -\mathrm{i}(E_yF_z - E_zF_y),\\
a_{41} &= -\mathrm{i}(D_yB_z - D_zB_y),\\
&\cdots\cdots\cdots\cdots\\
a_{44} &= \tfrac{1}{2}(E_xD_x + E_yD_y + E_zD_z) + \tfrac{1}{2}(F_xB_x + F_yB_y + F_zB_z).
\end{aligned}$$

Comparing with (70–8), we recognize the pure space elements a_{11}, a_{12}, a_{13}; a_{21}, a_{22}, a_{23}; a_{31}, a_{32}, a_{33} of this tensor as the electromagnetic stress elements X_x, Y_x, Z_x; X_y, Y_y, Z_y; X_z, Y_z, Z_z of the three-dimensional stress dyadic. Furthermore, reference to (69–5) and to (70–11) shows that the space-time elements a_{14}, a_{24}, a_{34} are the three components of the Poynting flux $\mathbf{s}$ multiplied by $-\mathrm{i}/c$ and that a_{41}, a_{42}, a_{43} are the three components of the linear electromagnetic momentum $\mathbf{g}_l$ per unit volume multiplied by $-\mathrm{i}c$. Finally comparison with (69–3) and (69–4) reveals that the pure time element a_{44} is the electromagnetic energy $u = u_E + u_H$ per unit volume. In terms of these quantities the stress-momentum-energy tensor is

$$\begin{aligned}
\mathbf{\Psi} = {} & X_x\mathbf{k}_1\mathbf{k}_1 + Y_x\mathbf{k}_1\mathbf{k}_2 + Z_x\mathbf{k}_1\mathbf{k}_3 - \frac{\mathrm{i}}{c}s_x\mathbf{k}_1\mathbf{k}_4\\
&+ X_y\mathbf{k}_2\mathbf{k}_1 + Y_y\mathbf{k}_2\mathbf{k}_2 + Z_y\mathbf{k}_2\mathbf{k}_3 - \frac{\mathrm{i}}{c}s_y\mathbf{k}_2\mathbf{k}_4\\
&+ X_z\mathbf{k}_3\mathbf{k}_1 + Y_z\mathbf{k}_3\mathbf{k}_2 + Z_z\mathbf{k}_3\mathbf{k}_3 - \frac{\mathrm{i}}{c}s_z\mathbf{k}_3\mathbf{k}_4\\
&- \mathrm{i}cg_{lx}\mathbf{k}_4\mathbf{k}_1 - \mathrm{i}cg_{ly}\mathbf{k}_4\mathbf{k}_2 - \mathrm{i}cg_{lz}\mathbf{k}_4\mathbf{k}_3 + u\mathbf{k}_4\mathbf{k}_4.
\end{aligned} \tag{94–9}$$

Although the tensor $\mathbf{\Psi}$ is not in general symmetric, it becomes symmetric in empty space where $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{F}$.

If, now, we form the four-vector

$$\Diamond\cdot\Psi = \left\{\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}-\frac{\partial g_{lx}}{\partial t}\right\}\boldsymbol{k}_1$$
$$+\left\{\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}-\frac{\partial g_{ly}}{\partial t}\right\}\boldsymbol{k}_2$$
$$+\left\{\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}-\frac{\partial g_{lz}}{\partial t}\right\}\boldsymbol{k}_3$$
$$-\frac{i}{c}\left\{\frac{\partial s_x}{\partial x}+\frac{\partial s_y}{\partial y}+\frac{\partial s_z}{\partial z}+\frac{\partial u}{\partial t}\right\}\boldsymbol{k}_4,$$

we recognize the three space components as the three components of the electromagnetic force per unit volume and the time component as i/c times the rate at which work is done by the electromagnetic field per unit volume. But these are just the components of the four-vector $\mathbf{P}\cdot\mathbf{M}$ specified by (93–3). So we have the relation

$$\mathsf{F} = \mathbf{P}\cdot\mathbf{M} = \Diamond\cdot\Psi. \qquad (94\text{–}10)$$

CHAPTER 10

GENERAL DYNAMICAL METHODS

95. Equation of Motion. — In this chapter we shall discuss the motion, relative to an inertial system, of a group of charged particles of charges $e_1, e_2, e_3, \cdots$ and rest masses $m_1, m_2, m_3, \cdots$ placed in an external electromagnetic field of scalar potential Φ_0 and vector potential $\mathbf{A}_0$, the potentials Φ_0 and $\mathbf{A}_0$ being, in general, functions of the time as well as of the coordinates x, y, z. For the kinetic reaction of an elementary particle we shall take the mass reaction of the Lorentz electron, neglecting the small higher order terms responsible for the radiation of energy, but taking into account the variation of mass with velocity. In this chapter we shall depart from our previous notation to the extent of designating the velocity of an elementary particle by $\mathbf{v}$ instead of $\mathbf{V}$, since we wish to reserve the letter V for potential energy.

In accord with (57–14), the equation of motion of the ith particle is

$$\frac{d}{dt}(m_{ti}\mathbf{v}_i) = e_i\left(\mathbf{E}_i + \frac{1}{c}\mathbf{v}_i \times \mathbf{H}_i\right), \tag{95–1}$$

where $\mathbf{E}_i$ and $\mathbf{H}_i$ are the electric and magnetic intensities at the point x_i, y_i, z_i, occupied by the particle at time t, due to external causes and to all the other particles of the group, and m_{ti} is the transverse mass

$$m_{ti} \equiv \frac{m_i}{\sqrt{1 - \dfrac{v^2}{c^2}}}. \tag{95–2}$$

If Φ_i and $\mathbf{A}_i$ are the scalar and vector potentials at x_i, y_i, z_i at time t, due to external causes and to all the other particles, we can express $\mathbf{E}_i$ and $\mathbf{H}_i$ as derivatives of Φ_i and $\mathbf{A}_i$ by means of (50–5) and (50–6).

Hence, as $\mathbf{v}_i \times \mathbf{H}_i = \mathbf{v}_i \times (\nabla_i \times \mathbf{A}_i) = \nabla_i \mathbf{A}_i \cdot \mathbf{v}_i - \mathbf{v}_i \cdot \nabla_i \mathbf{A}_i$, the equation of motion (95–1) becomes

$$\frac{d}{dt}(m_{ti}\mathbf{v}_i) = -e_i \nabla_i \Phi_i - \frac{e_i}{c}\left\{\frac{\partial \mathbf{A}_i}{\partial t} + \mathbf{v}_i \cdot \nabla_i \mathbf{A}_i\right\} + \frac{e_i}{c}\nabla_i \mathbf{A}_i \cdot \mathbf{v}_i.$$

But

$$\frac{\partial \mathbf{A}_i}{\partial t} + \mathbf{v}_i \cdot \nabla_i \mathbf{A}_i = \frac{\partial \mathbf{A}_i}{\partial t} + \frac{\partial \mathbf{A}_i}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial \mathbf{A}_i}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial \mathbf{A}_i}{\partial z_i}\frac{dz_i}{dt} = \frac{d\mathbf{A}_i}{dt}. \qquad (95\text{–}3)$$

Consequently the equation of motion may be written in the form

$$\frac{d}{dt}\left\{m_{ti}\mathbf{v}_i + \frac{e_i}{c}\mathbf{A}_i\right\} - \frac{e_i}{c}\nabla_i \mathbf{A}_i \cdot \mathbf{v}_i = -e_i \nabla_i \Phi_i. \qquad (95\text{–}4)$$

The portions of the scalar and the vector potentials due to other particles of the group are given by (56–4) and (56–5), in which it must be remembered that the operator d/dt acts on the coordinates and velocity components of the particle producing the field. We shall retain the first three terms in (56–4) and only the first term in (56–5). As $\mathbf{A}_i$ is everywhere multiplied by $1/c$ in (95–4), this means that we are including in the equation of motion all terms in $(1/c)^2$ and lower powers. Since we are neglecting the dissipative terms responsible for the radiation of energy, we may anticipate that the approximate theory which we are developing will be that of a conservative system when the external field is static.

Putting $\mathbf{r}_{ij} = \boldsymbol{i}(x_j - x_i) + \boldsymbol{j}(y_j - y_i) + \boldsymbol{k}(z_j - z_i)$ for the position vector of the jth particle relative to the ith, we have for the scalar potential at x_i, y_i, z_i,

$$\Phi_i = \Phi_{0i} + \sum_j \left\{\frac{e_j}{4\pi r_{ij}} + \frac{e_j}{8\pi c^2}\frac{\partial^2 r_{ij}}{\partial t^2}\right\}, \quad j \neq i, \qquad (95\text{–}5)$$

since the time derivative in the third term of (56–4), acting only on the coordinates x_j, y_j, z_j contained implicitly in the scalar r_{ij}, is equivalent to the partial derivative with respect to the time in our present analysis. The vector potential at x_i, y_i, z_i is

$$\mathbf{A}_i = \mathbf{A}_{0i} + \sum_j \left\{\frac{e_j \mathbf{v}_j}{4\pi c r_{ij}}\right\}, \quad j \neq i. \qquad (95\text{–}6)$$

Now

$$\frac{\partial r_{ij}}{\partial t} = \frac{\mathbf{v}_j \cdot \mathbf{r}_{ij}}{r_{ij}}$$

and

$$-\nabla_i \frac{\partial r_{ij}}{\partial t} = \frac{\mathbf{v}_j}{r_{ij}} - \frac{\mathbf{v}_j \cdot \mathbf{r}_{ij}\mathbf{r}_{ij}}{r_{ij}^3}.$$

Hence

$$-\nabla_i \frac{\partial^2 r_{ij}}{\partial t^2} = \frac{\partial}{\partial t}\left\{\frac{\mathbf{v}_j}{r_{ij}} - \frac{\mathbf{v}_j \cdot \mathbf{r}_{ij}\mathbf{r}_{ij}}{r_{ij}^3}\right\}$$

$$= \frac{d}{dt}\left\{\frac{\mathbf{v}_j}{r_{ij}} - \frac{\mathbf{v}_j \cdot \mathbf{r}_{ij}\mathbf{r}_{ij}}{r_{ij}^3}\right\} - \mathbf{v}_i \cdot \nabla_i \left\{\frac{\mathbf{v}_j}{r_{ij}} - \frac{\mathbf{v}_j \cdot \mathbf{r}_{ij}\mathbf{r}_{ij}}{r_{ij}^3}\right\}$$

as in (95–3). So, if we put

$$\Psi_i \equiv \Phi_{0i} + \sum_j \frac{e_j}{4\pi r_{ij}}, \quad j \neq i, \tag{95–7}$$

$$\mathbf{B}_i \equiv -\sum_j \frac{e_j}{8\pi c}\left\{\frac{\mathbf{v}_j}{r_{ij}} - \frac{\mathbf{v}_j \cdot \mathbf{r}_{ij}\mathbf{r}_{ij}}{r_{ij}^3}\right\}$$

$$= \nabla_i \sum_j \frac{e_j}{8\pi c}\frac{\mathbf{v}_j \cdot \mathbf{r}_{ij}}{r_{ij}}, \quad j \neq i, \tag{95–8}$$

the equation of motion (95–4) becomes

$$\frac{d}{dt}\left\{m_{ti}\mathbf{v}_i + \frac{e_i}{c}(\mathbf{A}_i + \mathbf{B}_i)\right\} - \frac{e_i}{c}\{\nabla_i \mathbf{A}_i \cdot \mathbf{v}_i + \mathbf{v}_i \cdot \nabla_i \mathbf{B}_i\} = -e_i \nabla_i \Psi_i. \tag{95–9}$$

We can, however, put this in simpler form. For $\mathbf{v}_i \times (\nabla_i \times \mathbf{B}_i)$ vanishes since $\mathbf{B}_i$ is the gradient of a scalar function by (95–8). Therefore, expanding the triple vector product, we find that $\mathbf{v}_i \cdot \nabla_i \mathbf{B}_i = \nabla_i \mathbf{B}_i \cdot \mathbf{v}_i$. Consequently (95–9) may be written

$$\frac{d}{dt}\left\{m_{ti}\mathbf{v}_i + \frac{e_i}{c}(\mathbf{A}_i + \mathbf{B}_i)\right\} - \nabla_i\left\{\frac{e_i}{c}\mathbf{v}_i \cdot (\mathbf{A}_i + \mathbf{B}_i) - e_i \Psi_i\right\} = 0. \tag{95–10}$$

This is the form of the equation of motion on which our subsequent analysis will be based.

The similarity of equation (95–10) to the Newtonian equation of motion of a particle in a conservative field should be noted, the expression in braces in the first term of (95–10) corresponding to the linear momentum of the particle, and the negative of that in braces in the second term to its potential energy.

96. Lagrange's Equations. — If the group of charged particles under consideration is subject to constraints, the totality of rectangular coordinates of the particles of the group are not all independent. In such an event it is convenient to introduce a set of independent generalized coordinates $q_1, q_2, \cdots q_f$ equal in number to the f degrees of freedom of the dynamical system. Even if we are dealing with a single particle subject to no constraints, and therefore there is no question of a reduction in the number of coordinates, we may find it necessary to employ spherical or other coordinates in place of rectangular coordinates. Such a purpose is equally well served by the introduction of generalized coordinates, which may represent angles or quantities of quite involved physical dimensions in place of distances.

Since the configuration of the dynamical system is completely specified by the generalized coordinates $q_1, q_2, \cdots q_f$, we may express the position vector $\mathbf{r}_i = ix_i + jy_i + kz_i$ of the ith particle as a function of the generalized coordinates, writing

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \cdots q_f). \tag{96–1}$$

Then the velocity of the particle is

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \mathbf{r}_i}{\partial q_f}\dot{q}_f \tag{96–2}$$

and, regarding this as a function of the q's and the $\dot{q}$'s, we find

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} = \frac{\partial \mathbf{r}_i}{\partial q_s}. \tag{96–3}$$

Furthermore,

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_s}\right) = \frac{\partial^2 \mathbf{r}_i}{\partial q_1 \partial q_s}\dot{q}_1 + \frac{\partial^2 \mathbf{r}_i}{\partial q_2 \partial q_s}\dot{q}_2 + \cdots + \frac{\partial^2 \mathbf{r}_i}{\partial q_f \partial q_s}\dot{q}_f = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_s} \tag{96–4}$$

from (96–2).

Now we take the scalar product of the equation of motion (95–10) by $\partial \mathbf{r}_i/\partial q_s$ and sum up over all the particles in the group. Writing $\dot{\mathbf{r}}_i$ for $\mathbf{v}_i$ and using (96–3) and (96–4) this gives

$$\begin{aligned} &\frac{d}{dt}\left[\sum_i \left\{ m_{ti}\dot{\mathbf{r}}_i + \frac{e_i}{c}(\mathbf{A}_i + \mathbf{B}_i)\right\} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s}\right] \\ &\quad - \sum_i \left\{ m_{ti}\dot{\mathbf{r}}_i + \frac{e_i}{c}(\mathbf{A}_i + \mathbf{B}_i)\right\} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_s} \\ &\quad - \sum_i \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i \left\{\frac{e_i}{c}\dot{\mathbf{r}}_i \cdot (\mathbf{A}_i + \mathbf{B}_i) - e_i \Psi_i\right\} = 0. \end{aligned} \tag{96–5}$$

In all, we have f equations of this form, one for each of the f degrees of freedom of the dynamical system.

The sum of the intrinsic kinetic energies of the individual particles is, in accord with (57–18),

$$T_v = \sum_i m_{ti}c^2 = \sum_i \frac{m_i c^2}{\sqrt{1 - \dfrac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i}{c^2}}}. \tag{96–6}$$

We need here a related function

$$T_v' \equiv - \sum_i m_{ti}c^2 \left(1 - \frac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i}{c^2}\right) = - \sum_i m_i c^2 \sqrt{1 - \frac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i}{c^2}}. \tag{96–7}$$

Evidently

$$T_v + T_v' = \sum_i m_{ti}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i. \tag{96–8}$$

Now

$$\frac{\partial T_v'}{\partial \dot{q}_s} = \sum_i \frac{m_i}{\sqrt{1 - \dfrac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i}{c^2}}} \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} = \sum_i m_{ti}\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s}, \tag{96–9}$$

which is just the first part of the expression in brackets in the first term of (96–5). Also

$$- \frac{\partial T_v'}{\partial q_s} = - \sum_i \frac{m_i}{\sqrt{1 - \dfrac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i}{c^2}}} \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_s} = - \sum_i m_{ti}\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_s} \tag{96–10}$$

is the first part of the second term in (96–5).

From (95–6) and (95–8)

$$\mathbf{A}_i + \mathbf{B}_i = \mathbf{A}_{0i} + \sum_j \frac{e_j}{8\pi c} \left\{ \frac{\dot{\mathbf{r}}_j}{r_{ij}} + \frac{\dot{\mathbf{r}}_j \cdot \mathbf{r}_{ij}\mathbf{r}_{ij}}{r_{ij}^3} \right\}, \; j \neq i, \tag{96–11}$$

where $\mathbf{A}_{0i}$ is the vector potential of the external field. Hence

$$\begin{aligned} \sum_i \frac{e_i}{c}(\mathbf{A}_i + \mathbf{B}_i) \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} &= \sum_i \frac{e_i}{c} \mathbf{A}_{0i} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} \\ &\quad + \sum_{ij} \frac{e_i e_j}{8\pi c^2} \left\{ \frac{1}{r_{ij}} \dot{\mathbf{r}}_j \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} + \frac{1}{r_{ij}^3} \mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_j \mathbf{r}_{ij} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} \right\}, \; j \neq i, \\ &= \frac{\partial}{\partial \dot{q}_s} \left[\sum_i \frac{e_i}{c} \mathbf{A}_{0i} \cdot \dot{\mathbf{r}}_i + \sum_{ij} \frac{e_i e_j}{16\pi c^2} \left(\frac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j}{r_{ij}} + \frac{\mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_i \mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_j}{r_{ij}^3} \right) \right], \; j \neq i, \end{aligned}$$

since $\mathbf{A}_{0i}$ is not a function of the $\dot{q}$'s. We call

$$T_H \equiv \sum_i \frac{e_i}{c} \mathbf{A}_{0i} \cdot \dot{\mathbf{r}}_i \tag{96-12}$$

the *magnetic kinetic energy* of the group of particles, since the vector potential $\mathbf{A}_0$ of the external field is responsible for it, and

$$T_m \equiv \sum_{ij} \frac{e_i e_j}{16\pi c^2} \left(\frac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j}{r_{ij}} + \frac{\mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_i \mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_j}{r_{ij}{}^3} \right), \quad j \neq i, \tag{96-13}$$

the *mutual kinetic energy*, since it is due to the interactions of the particles in the group with one another. Hence the remainder of the expression in brackets in the first term of (96–5) can be expressed in the form

$$\sum_i \frac{e_i}{c} (\mathbf{A}_i + \mathbf{B}_i) \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} = \frac{\partial}{\partial \dot{q}_s} (T_H + T_m). \tag{96-14}$$

Now we shall consider the contributions to the second and third terms in (96–5) made by the first term $\mathbf{A}_{0i}$ in the expression (96–11) for $\mathbf{A}_i + \mathbf{B}_i$. In so doing we must remember that ∇_i in the last term of (96–5) operates only on the coordinates and not on the velocities $\dot{\mathbf{r}}_i$. As

$$\frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i = \frac{\partial x_i}{\partial q_s} \frac{\partial}{\partial x_i} + \frac{\partial y_i}{\partial q_s} \frac{\partial}{\partial y_i} + \frac{\partial z_i}{\partial q_s} \frac{\partial}{\partial z_i}, \tag{96-15}$$

it follows that

$$\begin{aligned} -\sum_i \frac{e_i}{c} \mathbf{A}_{0i} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_s} - \sum_i \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i \frac{e_i}{c} \mathbf{A}_{0i} \cdot \dot{\mathbf{r}}_i &= -\frac{\partial}{\partial q_s} \left\{ \sum_i \frac{e_i}{c} \mathbf{A}_{0i} \cdot \dot{\mathbf{r}}_i \right\} \\ &= -\frac{\partial T_H}{\partial q_s}, \end{aligned} \tag{96-16}$$

where, in the right-hand member, the q_s contained implicitly in the $\dot{\mathbf{r}}$'s is varied as well as the q_s in the $\mathbf{r}$'s.

Next we must evaluate the contributions made to these two terms in (96–5) by the rest of the expression (96–11). As an aid to this calculation we note that

$$\frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i\left(\frac{1}{r_{ij}}\right) + \frac{\partial \mathbf{r}_j}{\partial q_s} \cdot \nabla_j\left(\frac{1}{r_{ij}}\right)$$

$$= \frac{(x_j - x_i)\dfrac{\partial x_i}{\partial q_s} + (y_j - y_i)\dfrac{\partial y_i}{\partial q_s} + (z_j - z_i)\dfrac{\partial z_i}{\partial q_s}}{r_{ij}^3}$$

$$- \frac{(x_j - x_i)\dfrac{\partial x_j}{\partial q_s} + (y_j - y_i)\dfrac{\partial y_j}{\partial q_s} + (z_j - z_i)\dfrac{\partial z_j}{\partial q_s}}{r_{ij}^3}$$

$$= \frac{\partial}{\partial q_s}\left(\frac{1}{r_{ij}}\right), \tag{96–17}$$

and similarly

$$\frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i\left(\frac{1}{r_{ij}^3}\right) + \frac{\partial \mathbf{r}_j}{\partial q_s} \cdot \nabla_j\left(\frac{1}{r_{ij}^3}\right) = \frac{\partial}{\partial q_s}\left(\frac{1}{r_{ij}^3}\right). \tag{96–18}$$

Also, remembering that the ∇'s do not act on the $\dot{\mathbf{r}}$'s,

$$\frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i(\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_i\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j) + \frac{\partial \mathbf{r}_j}{\partial q_s} \cdot \nabla_j(\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_i\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j)$$

$$= \left(-\dot{x}_i\frac{\partial x_i}{\partial q_s} - \dot{y}_i\frac{\partial y_i}{\partial q_s} - \dot{z}_i\frac{\partial z_i}{\partial q_s}\right)\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j$$

$$+ \mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_i\left(-\dot{x}_j\frac{\partial x_i}{\partial q_s} - \dot{y}_j\frac{\partial y_i}{\partial q_s} - \dot{z}_j\frac{\partial z_i}{\partial q_s}\right)$$

$$+ \left(\dot{x}_i\frac{\partial x_j}{\partial q_s} + \dot{y}_i\frac{\partial y_j}{\partial q_s} + \dot{z}_i\frac{\partial z_j}{\partial q_s}\right)\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j$$

$$+ \mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_i\left(\dot{x}_j\frac{\partial x_j}{\partial q_s} + \dot{y}_j\frac{\partial y_j}{\partial q_s} + \dot{z}_j\frac{\partial z_j}{\partial q_s}\right)$$

$$= \frac{\partial}{\partial q_s}(\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_i\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j), \tag{96–19}$$

provided $\dot{\mathbf{r}}_i$ and $\dot{\mathbf{r}}_j$ are not varied. Consequently the contribution to (96–5) which we are calculating becomes

$$-\sum_{ij}\frac{e_ie_j}{8\pi c^2}\left\{\frac{1}{r_{ij}}\dot{\mathbf{r}}_j \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_s} + \frac{1}{r_{ij}^3}\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j\mathbf{r}_{ij} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_s}\right\}$$

$$-\sum_{ij}\frac{e_ie_j}{8\pi c^2}\frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i\left\{\frac{\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j}{r_{ij}} + \frac{\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_i\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j}{r_{ij}^3}\right\}$$

$$= -\frac{\partial}{\partial q_s}\left\{\sum_{ij}\frac{e_ie_j}{16\pi c^2}\left(\frac{\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j}{r_{ij}} + \frac{\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_i\mathbf{r}_{ij}\cdot\dot{\mathbf{r}}_j}{r_{ij}^3}\right)\right\} = -\frac{\partial T_m}{\partial q_s}, \tag{96–20}$$

where the q_s contained implicitly in the $\dot{\mathbf{r}}$'s is varied as well as the q_s in the $\mathbf{r}$'s.

Finally, making use of (96–17) again,

$$\sum_i \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i(e_i\Psi_i) = \sum_i e_i \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i \Phi_{0i} + \sum_{ij} \frac{e_i e_j}{4\pi} \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i \left(\frac{1}{r_{ij}}\right)$$

$$= \frac{\partial}{\partial q_s}\left\{\sum_i e_i \Phi_{0i} + \sum_{ij} \frac{e_i e_j}{8\pi r_{ij}}\right\}. \tag{96–21}$$

We call

$$V \equiv \sum_i e_i \Phi_{0i} + \sum_{ij} \frac{e_i e_j}{8\pi r_{ij}}, \quad j \neq i, \tag{96–22}$$

the *potential energy* of the group of particles. Then

$$\sum_i \frac{\partial \mathbf{r}_i}{\partial q_s} \cdot \nabla_i(e_i\Psi_i) = \frac{\partial V}{\partial q_s}. \tag{96–23}$$

We define the *kinetic potential* or *Lagrangian function* $\mathscr{L}$ by

$$\mathscr{L} \equiv T_v' + T_m + T_H - V. \tag{96–24}$$

While T_v', T_m and T_H are functions of both the q's and the $\dot{q}$'s, V is a function of the q's alone. Consequently $\partial V/\partial \dot{q}_s = 0$ and equations (96–5) become

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_s}\right) - \frac{\partial \mathscr{L}}{\partial q_s} = 0, \quad s = 1, 2, \cdots f. \tag{96–25}$$

These second order differential equations of motion, of which there are as many as there are degrees of freedom, are known as *Lagrange's equations*.

For reference we shall set down the explicit expression for the kinetic potential. From (96–7), (96–13), (96–12) and (96–22), we obtain

$$\mathscr{L} = -\sum_i m_i c^2 \sqrt{1 - \frac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i}{c^2}} + \sum_{ij} \frac{e_i e_j}{16\pi c^2}\left(\frac{\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j}{r_{ij}} + \frac{\mathbf{r}_{ij}\cdot \dot{\mathbf{r}}_i \mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_j}{r_{ij}^3}\right)$$

$$+ \sum_i \frac{e_i}{c} \mathbf{A}_{0i} \cdot \dot{\mathbf{r}}_i - \sum_i e_i \Phi_{0i} - \sum_{ij} \frac{e_i e_j}{8\pi r_{ij}}. \tag{96–26}$$

Obviously the kinetic potential must be expressed as a function of the generalized coordinates $q_1, q_2, \cdots q_f$ and the generalized velocities

$\dot{q}_1, \dot{q}_2, \cdots \dot{q}_f$ before the partial derivatives appearing in Lagrange's equations are formed. When necessary to avoid ambiguity we shall draw attention to this fact by writing $\mathscr{L}(q, \dot{q})$ for this function. In cases where the external electromagnetic field varies with the time, Φ_{0i} and $\mathbf{A}_{0i}$ are functions of the time, and $\mathscr{L}$ contains t explicitly as well as $q_1, q_2, \cdots q_f$ and $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_f$.

Consider the quantity

$$\sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} - \mathscr{L} = \sum_s \dot{q}_s \frac{\partial T_v'}{\partial \dot{q}_s} - T_v' + \sum_s \dot{q}_s \frac{\partial T_m}{\partial \dot{q}_s} - T_m$$
$$+ \sum_s \dot{q}_s \frac{\partial T_H}{\partial \dot{q}_s} - T_H - \sum_s \dot{q}_s \frac{\partial V}{\partial \dot{q}_s} + V. \quad (96\text{–}27)$$

From (96–9) and (96–3)

$$\sum_s \dot{q}_s \frac{\partial T_v'}{\partial \dot{q}_s} = \sum_i m_{ti} \dot{\mathbf{r}}_i \cdot \sum_s \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_s} \dot{q}_s = \sum_i m_{ti} \dot{\mathbf{r}}_i \cdot \sum_s \frac{\partial \mathbf{r}_i}{\partial q_s} \dot{q}_s$$
$$= \sum_i m_{ti} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i.$$

This is just the expression (96–8). Hence

$$\sum_s \dot{q}_s \frac{\partial T_v'}{\partial \dot{q}_s} - T_v' = T_v,$$

which represents the total intrinsic kinetic energy of the particles.

To evaluate the remaining terms on the right of (96–27) we make use of Euler's theorem for homogeneous functions. This theorem states that, if F is a homogeneous function of $x_1, x_2, \cdots x_f$ of degree p, then

$$\sum_s x_s \frac{\partial F}{\partial x_s} = pF.$$

Turning to (96–13), (96–12) and (96–22) we observe that T_m, T_H and V are homogeneous functions of $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_f$ of degree two, one and zero, respectively. Consequently

$$\sum_s \dot{q}_s \frac{\partial T_m}{\partial \dot{q}_s} - T_m = T_m,$$

$$\sum_s \dot{q}_s \frac{\partial T_H}{\partial \dot{q}_s} - T_H = 0,$$

$$-\sum_s \dot{q}_s \frac{\partial V}{\partial \dot{q}_s} + V = V.$$

Hence

$$\sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} - \mathscr{L} = T_v + T_m + V. \tag{96–28}$$

This quantity is the sum of the intrinsic kinetic energy of the particles, the mutual kinetic energy and the potential energy. We shall prove next that it is conserved when the external field is static. Then we shall be entitled to refer to it as the *total dynamical energy* of the group of particles.

Taking the total derivative of the kinetic potential with respect to the time we have

$$\begin{aligned} \frac{d\mathscr{L}}{dt} &= \sum_s \ddot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} + \sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial q_s} + \frac{\partial \mathscr{L}}{\partial t} \\ &= \sum_s \ddot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} + \sum_s \dot{q}_s \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_s}\right) + \frac{\partial \mathscr{L}}{\partial t} \end{aligned}$$

from Lagrange's equations (96–25). Consequently

$$\frac{d}{dt}\left\{\sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} - \mathscr{L}\right\} + \frac{\partial \mathscr{L}}{\partial t} = 0. \tag{96–29}$$

If the external electromagnetic field is static, Φ_0 and $\mathbf{A}_0$, and therefore $\mathscr{L}$, do not contain the time explicitly. In such a case $\partial \mathscr{L}/\partial t = 0$ and (96–29) can be integrated, giving

$$\sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} - \mathscr{L} = U, \tag{96–30}$$

where U is a constant independent of the time. Hence the sum

$$U = T_v + T_m + V \tag{96–31}$$

remains constant during the motion. This is the *law of conservation of dynamical energy*.

On the contrary, if the external electromagnetic field is not static, energy is being supplied to, or taken away from, the group of particles under consideration, the energy added per unit time from outside sources being $-\partial \mathscr{L}/\partial t$ in accord with (96–29). Referring to (96–26) we find that

$$\begin{aligned} -\frac{\partial \mathscr{L}}{\partial t} &= \sum_i e_i \frac{\partial \Phi_{0i}}{\partial t} - \sum_i \frac{e_i}{c} \frac{\partial \mathbf{A}_{0i}}{\partial t} \cdot \dot{\mathbf{r}}_i \\ &= \frac{d}{dt}\left(\sum_i e_i \Phi_{0i}\right) - \sum_i e_i \left(\nabla_i \Phi_{0i} + \frac{1}{c}\frac{\partial \mathbf{A}_{0i}}{\partial t}\right) \cdot \dot{\mathbf{r}}_i. \end{aligned}$$

As the electric intensity $\mathbf{E}_0$ of the external electromagnetic field is given by

$$\mathbf{E}_0 = -\nabla\Phi_0 - \frac{1}{c}\frac{\partial \mathbf{A}_0}{\partial t},$$

we can write this in the form

$$-\frac{\partial \mathscr{L}}{\partial t} = \frac{d}{dt}(\sum_i e_i \Phi_{0i}) + \sum_i e_i \mathbf{E}_{0i}\cdot \dot{\mathbf{r}}_i. \tag{96–32}$$

We recognize in the first term on the right the time rate of increase of the portion of the potential energy (96–22) due to the external field, and in the second the time rate at which the external field does work on the group of particles. Therefore the rate at which the intrinsic kinetic energy, mutual kinetic energy, and that portion of the potential energy of the group of particles which is due to their mutual interactions, increases, is equal to the rate at which work is done on the particles by the external field. This, of course, is what we should expect.

97. Applications to Circuit Theory. — We shall make some applications of the theory developed in the last article to closed circuits carrying currents which are either steady or varying so slowly with the time that we can consider each current to be the same at any one instant throughout its length.

The intrinsic masses of the free electrons constituting a current are so small compared with the masses of the atom cores forming the conducting wire in which they move that we need consider only the latter in evaluating T_v and T_v'. Furthermore the greatest velocities acquired by the conductors are so small compared with the velocity of light that we may neglect the variation of mass with velocity. This enables us to replace both T_v and T_v' by the Newtonian expression

$$T_0 = \sum_i \tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \tag{97–1}$$

for the intrinsic kinetic energy, where the velocities are only those of the gross matter constituting the conductors. The fact that T_v and T_v', when we retain no terms in powers of $\dot{\mathbf{r}}_i/c$ higher than the second in the binomial expansion of the radical, differ from T_0 by the large constant $\sum_i m_i c^2$, need cause no concern since this constant disappears when we form the derivatives (96–9) and (96–10). Finally, as T_0 is a homogeneous quadratic function of $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_f$, we get the same

expression (96–28) obtained before for the total dynamical energy with T_0 replacing T_v.

The significant part of the energy of a system of current circuits is the mutual kinetic energy T_m. First we shall calculate the mutual energy of two linear circuits in which currents i_1 and i_2 are flowing. If a current i is due to a charge de moving with velocity $\mathbf{v}$ in an element of length $d\lambda$ of a circuit, $\mathbf{v}de = id\boldsymbol{\lambda}$. Consequently the part of (96–13) representing the mutual kinetic energy of the two circuits is

$$T_{mM} = \frac{i_1 i_2}{8\pi c^2}\left\{\oint\oint\frac{d\boldsymbol{\lambda}_1\cdot d\boldsymbol{\lambda}_2}{r_{12}} + \oint\oint\frac{\mathbf{r}_{12}\cdot d\boldsymbol{\lambda}_1\mathbf{r}_{12}\cdot d\boldsymbol{\lambda}_2}{{r_{12}}^3}\right\}, \qquad (97\text{–}2)$$

the doubling of the numerical coefficient being due to the fact that i, j in (96–13) take on the values 2, 1 as well as 1, 2.

Consider the second integral in (97–2). Keeping x_1, y_1, z_1 constant, we shall integrate first around circuit (2). Then $d\boldsymbol{\lambda}_2 = d\mathbf{r}_{12}$, and

$$\oint\frac{\mathbf{r}_{12}\cdot d\boldsymbol{\lambda}_1\mathbf{r}_{12}\cdot d\mathbf{r}_{12}}{{r_{12}}^3} = \oint\frac{d\boldsymbol{\lambda}_1\cdot d\mathbf{r}_{12}}{r_{12}} - \oint d\left(\frac{\mathbf{r}_{12}\cdot d\boldsymbol{\lambda}_1}{r_{12}}\right)$$

identically. The second loop integral on the right vanishes since the integrand is an exact differential. Hence, restoring $d\boldsymbol{\lambda}_2$ for $d\mathbf{r}_{12}$, (97–2) assumes the simpler form

$$T_{mM} = \frac{i_1 i_2}{4\pi c^2}\oint\oint\frac{d\boldsymbol{\lambda}_1\cdot d\boldsymbol{\lambda}_2}{r_{12}}, \qquad (97\text{–}3)$$

valid for closed circuits.

That this expression is proportional to the magnetic flux through either circuit due to a unit current in the other is easily seen from (64–1). In accord with this equation, the vector potential due to a unit current in (2) is

$$\mathbf{A}_1 = \frac{1}{4\pi c}\oint\frac{d\boldsymbol{\lambda}_2}{r_{12}}.$$

Consequently the magnetic flux N_1 through (1) is given by the surface integral

$$N_1 = \int_{\sigma_1}\nabla\times\mathbf{A}_1\cdot d\boldsymbol{\sigma}_1 = \oint\mathbf{A}_1\cdot d\boldsymbol{\lambda}_1 = \frac{1}{4\pi c}\oint\oint\frac{d\boldsymbol{\lambda}_1\cdot d\boldsymbol{\lambda}_2}{r_{12}}.$$

The *mutual-inductance* M_{12} of circuit (1) relative to (2) is defined as the flux through (1) due to a unit current in (2) divided by c.

Evidently $M_{12} = M_{21} \equiv M$ on account of symmetry, where

$$M = \frac{1}{4\pi c^2} \oint \oint \frac{d\boldsymbol{\lambda}_1 \cdot d\boldsymbol{\lambda}_2}{r_{12}}, \tag{97-4}$$

and

$$T_{mM} = Mi_1 i_2 = \tfrac{1}{2}\{M_{12} i_1 i_2 + M_{21} i_2 i_1\}. \tag{97-5}$$

To find the mutual kinetic energy of the electrons constituting the current i in a single circuit we resolve the current into a large number of current filaments $i_1, i_2, \cdots i_n$, such that $i = \sum_k i_k$. The mutual energy of these filaments is

$$\begin{aligned} T_{mL} &= \frac{1}{4\pi c^2}\left\{\sum_{l=2}^{n} \oint \oint \frac{i_1 i_l d\boldsymbol{\lambda}_1 \cdot d\boldsymbol{\lambda}_l}{r_{1l}} + \sum_{l=3}^{n} \oint \oint \frac{i_2 i_l d\boldsymbol{\lambda}_2 \cdot d\boldsymbol{\lambda}_l}{r_{2l}} + \cdots\right\} \\ &= \frac{1}{8\pi c^2} \sum_{k,\,l} \oint \oint \frac{i_k i_l d\boldsymbol{\lambda}_k \cdot d\boldsymbol{\lambda}_l}{r_{kl}}, \quad k \neq l, \end{aligned} \tag{97-6}$$

where, in the last expression, k, l assume all permutations for which $k \neq l$.

Now the magnetic flux N_k through the kth current filament due to *all* the others, per unit total current i, is

$$N_k = \frac{1}{4\pi c i} \sum_{l} \oint \oint \frac{i_l d\boldsymbol{\lambda}_k \cdot d\boldsymbol{\lambda}_l}{r_{kl}}, \quad l \neq k,$$

and the mean value of all such fluxes, weighted in accord with the magnitude of the current filament i_k, is

$$\overline{N} = \frac{1}{4\pi c i^2} \sum_{k,\,l} \oint \oint \frac{i_k i_l d\boldsymbol{\lambda}_k \cdot d\boldsymbol{\lambda}_l}{r_{kl}}, \quad k \neq l.$$

The *self-inductance* L of the circuit is defined as the mean flux $\overline{N}$ divided by c. Consequently

$$L = \frac{1}{4\pi c^2 i^2} \sum_{k,\,l} \oint \oint \frac{i_k i_l d\boldsymbol{\lambda}_k \cdot d\boldsymbol{\lambda}_l}{r_{kl}}, \quad k \neq l, \tag{97-7}$$

and

$$T_{mL} = \tfrac{1}{2} L i^2. \tag{97-8}$$

If we take the current filaments all of the same magnitude, $i_k = i/n$ and

$$L = \frac{1}{4\pi c^2 n^2} \sum_{k,\,l} \oint \oint \frac{d\boldsymbol{\lambda}_k \cdot d\boldsymbol{\lambda}_l}{r_{kl}}, \quad k \neq l. \tag{97-9}$$

Evidently the finite cross-section of the circuit must be taken into account in calculating its self-inductance, for the self-inductance of a linear circuit is infinite on account of the factor $1/r_{kl}$ in the integrand of (97–7) or (97–9). The self-inductance of a circuit depends not alone on its geometry, but also on the distribution of current through its cross-section. In the case of a steady or slowly varying current, the current density is constant over the cross-section of the conductor and L is a function only of the shape of the circuit. This is the case which we are considering.

Finally we must evaluate the kinetic energy of a current circuit relative to the external magnetic field. From (96–12) we have

$$T_H = \frac{i}{c}\oint \mathrm{A}_0 \cdot d\lambda = \frac{i}{c}\int_\sigma \nabla \times \mathrm{A}_0 \cdot d\sigma = \frac{i}{c} N, \qquad (97\text{–}10)$$

where N is the flux of the magnetic field through the circuit. This expression does not agree in sign with (64–18), since our present result represents work done *by* the electromagnetic forces whereas the earlier expression represented work done *against* these forces.

Now we are ready to consider a group of n separate rigid circuits carrying currents $i_1, i_2, \cdots i_n$. The total mutual kinetic energy of the group is

$$\begin{aligned} T_m = \tfrac{1}{2}\{ & L_1 i_1^2 + M_{12} i_1 i_2 + \cdots + M_{1n} i_1 i_n \\ & + M_{21} i_2 i_1 + L_2 i_2^2 \quad + \cdots + M_{2n} i_2 i_n \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + M_{n1} i_n i_1 + M_{n2} i_n i_2 + \cdots + L_n i_n^2 \} \end{aligned} \qquad (97\text{–}11)$$

from (97–5) and (97–8), and the kinetic energy due to the external magnetic field is

$$T_H = \frac{1}{c}\{N_1 i_1 + N_2 i_2 + \cdots + N_n i_n\} \qquad (97\text{–}12)$$

in accord with (97–10), where the N's are the fluxes due to the external field.

The kinetic potential (96–24) is

$$\mathscr{L} = T_0 + T_m + T_H - V \qquad (97\text{–}13)$$

with the Newtonian intrinsic kinetic energy T_0 replacing T_v'. The

first n of the generalized velocities $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_n, \cdots \dot{q}_f$ may be taken as the currents $i_1, i_2, \cdots i_n$, the conjugate coordinates $q_1, q_2, \cdots q_n$ not appearing in the kinetic potential. The remaining degrees of freedom are those of the geometrical configuration of the circuits. The $f - n$ generalized coordinates corresponding to these degrees of freedom we shall designate by $\xi_1, \xi_2 \cdots \xi_{f-n}$. Then T_0 is a function of the ξ's and the $\dot{\xi}$'s alone; T_m is a function of the i's and, through the M's, of the ξ's, but not of the $\dot{\xi}$'s; T_H is a function of the i's and, through the N's, of the ξ's, and also, if the external field is not static, of the time t; and V is a function of the ξ's alone, with t as an added variable if the external field is changing.

Now we can determine the equations of motion from Lagrange's equations (96–25). These equations separate into two groups:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial i_s}\right) = 0, \quad s = 1, 2, \cdots n, \tag{97–14}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\xi}_r}\right) - \frac{\partial \mathscr{L}}{\partial \xi_r} = 0, \quad r = 1, 2, \cdots f - n. \tag{97–15}$$

The first group specifies how the currents change with the time, the second determines the electromagnetic forces acting on the conductors.

Explicitly (97–14) yields the equations

$$\frac{d}{dt}\left\{M_{s1}i_1 + \cdots + M_{s,s-1}i_{s-1} + L_s i_s + M_{s,s+1}i_{s+1} + \cdots \right.$$
$$\left. + M_{sn}i_n + \frac{1}{c}N_s\right\} = 0, \quad s = 1, 2, \cdots n, \tag{97–16}$$

which tell us that in the non-dissipative system which we are considering the algebraic sum of the electromotive forces in each circuit vanishes at every instant. Equivalently, the integral of this equation states that the total magnetic flux through each circuit remains constant. If the flux due to exterior causes increases, the current in the circuit must decrease sufficiently to compensate for that increase.

Next consider a geometrical coordinate, say ξ_r, which specifies the position, either linear or angular, of circuit s. Then $\dot{\xi}_r$ and ξ_r appear in T_0 and ξ_r in V. Also ξ_r is contained in all the mutual inductances having s as one subscript and in N_s. Consequently

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\xi}_r}\right) - \frac{\partial T_0}{\partial \xi_r} = -\frac{\partial V}{\partial \xi_r}$$
$$+ i_s \left\{ i_1 \frac{\partial M_{s1}}{\partial \xi_r} + \cdots + i_{s-1} \frac{\partial M_{s,s-1}}{\partial \xi_r} + i_{s+1} \frac{\partial M_{s,s+1}}{\partial \xi_r} + \cdots \right.$$
$$\left. + i_n \frac{\partial M_{sn}}{\partial \xi_r} + \frac{1}{c}\frac{\partial N_s}{\partial \xi_r} \right\}, \quad s = 1, 2, \cdots n. \qquad (97\text{–}17)$$

Evidently the right-hand side of (97–17) represents the electromagnetic force or torque, according as ξ_r is a linear or angular coordinate, on the circuit. The expression in the brace is the total change in flux through the circuit per unit change in ξ_r, divided by c, in agreement with (64–13). So far as the magnetic field is concerned, the circuit tends to move in such a way as to increase the flux through it.

Finally, as an example in which the coordinate conjugate to the current i is not absent from the kinetic potential, we shall consider a single circuit containing capacity C as well as self-inductance L, which is subject to an external electromotive force due to changing magnetic flux. If q is the charge on the condenser, $V = \frac{1}{2}\frac{q^2}{C}$, and, as before, $T_m = \frac{1}{2}Li^2$ and $T_H = \frac{1}{c}Ni$. Therefore the kinetic potential is

$$\mathscr{L} = T_0 + \tfrac{1}{2}Li^2 + \frac{1}{c}Ni - \frac{1}{2}\frac{q^2}{C}. \qquad (97\text{–}18)$$

Evidently q is the coordinate conjugate to i since $i = \dot{q}$. Consequently, as T_0 contains neither i nor q, Lagrange's equation for these conjugate variables is

$$\frac{d}{dt}\left\{Li + \frac{1}{c}N\right\} + \frac{q}{C} = 0. \qquad (97\text{–}19)$$

Rearranging terms, we have the familiar equation

$$L\frac{di}{dt} + \frac{q}{C} = -\frac{1}{c}\frac{dN}{dt} \qquad (97\text{–}20)$$

of a circuit with self-inductance and capacity subject to an external electromotive force due to changing magnetic flux.

98. Hamilton's Principle and the Principle of Least Action. — In the f-dimensional configuration space of which $q_1, q_2, \cdots q_f$ are the

coordinates we can represent the geometrical state of a given dynamical system of f degrees of freedom at any one instant by a single point P (Fig. 109). As time goes on, this point describes a curve known as a *trajectory* or *path*. Let AB be a path described in accord with Lagrange's equations (96–25), the point A being occupied by the generating point P at the time t_A and the point B at the later time t_B, and let CD be a nearby path not necessarily described in accord with the equations of motion, the point C being reached by the generating point Q of this path at the time $t_A + \Delta t_A$ and the point D at the time $t_B + \Delta t_B$. The first path we call a *dynamical path* and the second a *varied path*. Points on the two paths which are occupied by their respective generating points at the same time are known as *corresponding points*. If F is a function of the coordinates $q_1, q_2, \cdots q_f$ and the velocities $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_f$ of the generating point and perhaps also of the time t, we indicate by δF the excess of F at a point on the varied path over its value at the corresponding point on the dynamical path. Since corresponding points are reached at the same time, explicit t is constant for this variation.

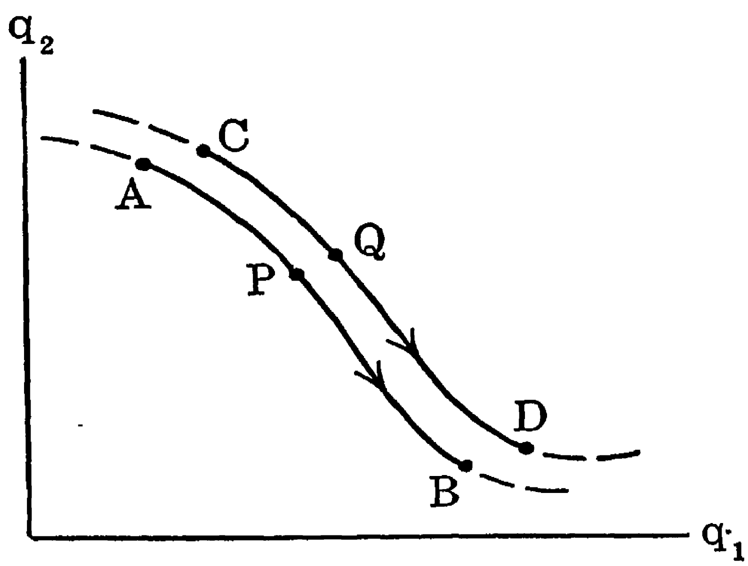

FIG. 109.

If, now, we compare the time integral of the kinetic potential $\mathscr{L}(q_1, \cdots q_f, \dot{q}_1, \cdots \dot{q}_f, t)$ taken over the varied path CD with that over the dynamical path AB we find

$$\int_C^D \mathscr{L}dt - \int_A^B \mathscr{L}dt = \mathscr{L}_B \Delta t_B - \mathscr{L}_A \Delta t_A + \int_{t_A}^{t_B} \delta \mathscr{L} dt$$

$$= \mathscr{L}_B \Delta t_B - \mathscr{L}_A \Delta t_A + \int_{t_A}^{t_B} \sum_s \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_s} \delta \dot{q}_s + \frac{\partial \mathscr{L}}{\partial q_s} \delta q_s \right) dt.$$

As AB is a dynamical path,

$$\frac{\partial \mathscr{L}}{\partial q_s} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_s}\right), \quad s = 1, 2 \cdots f,$$

everywhere along the path from (96–25). Therefore

$$\int_C^D \mathscr{L}dt - \int_A^B \mathscr{L}dt = \mathscr{L}_B\Delta t_B - \mathscr{L}_A\Delta t_A + \int_{t_A}^{t_B} \frac{d}{dt}\left(\sum_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s}\delta q_s\right) dt$$

$$= \mathscr{L}_B\Delta t_B + \left(\sum_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s}\delta q_s\right)_B - \mathscr{L}_A\Delta t_A - \left(\sum_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s}\delta q_s\right)_A.$$

But, if $(\Delta q_s)_B$ denotes the increment in q_s in passing from B to D,

$$(\Delta q_s)_B = (\delta q_s)_B + (\dot{q}_s)_B\Delta t_B,$$

and similarly, if $(\Delta q_s)_A$ is the increment in q_s in passing from A to C,

$$(\Delta q_s)_A = (\delta q_s)_A + (\dot{q}_s)_A\Delta t_A.$$

Consequently

$$\int_C^D \mathscr{L}dt - \int_A^B \mathscr{L}dt = \left|\sum_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s}\Delta q_s + \left(\mathscr{L} - \sum_s \dot{q}_s\frac{\partial \mathscr{L}}{\partial \dot{q}_s}\right)\Delta t\right|_A^B. \quad (98\text{–}1)$$

This is known as the *principle of varying action.* Hamilton's principle and the principle of least action are special cases of this more general result.

(I) *Hamilton's Principle.* If the termini of the varied and dynamical paths coincide and are occupied by their respective generating points at the same time, $\Delta q_1 = \Delta q_2 = \cdots = \Delta q_s = \Delta t = 0$ at both A and B. Then

$$\int_C^D \mathscr{L}dt - \int_A^B \mathscr{L}dt = 0$$

or, more compactly,

$$\delta\int \mathscr{L}dt = 0. \quad (98\text{–}2)$$

This is Hamilton's principle. It states that the time integral of the kinetic potential along a dynamical path has a stationary value as compared with this integral along all nearby varied paths which (*a*) have the same termini and (*b*) are described in the same time. It is equivalent to the Lagrangian equations of motion (96–25), which can be deduced from it.

(II) *Principle of Least Action.* The principle of least action is less general than Hamilton's principle in that it deals only with conservative systems for which

$$\sum_s \dot{q}_s\frac{\partial \mathscr{L}}{\partial \dot{q}_s} - \mathscr{L} = U \text{ (a constant)}. \quad (98\text{–}3)$$

We compare with the dynamical path only those varied paths (*a*) which have the same termini as the dynamical path and (*b*) for which the energy U is the same. Under these conditions (98–3) gives

$$\int_C^D \sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} dt - \int_A^B \sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} dt = \int_C^D \mathscr{L} dt - \int_A^B \mathscr{L} dt + \left| U \Delta t \right|_A^B.$$

But, under the same conditions, (98–1) becomes

$$\int_C^D \mathscr{L} dt - \int_A^B \mathscr{L} dt = - \left| U \Delta t \right|_A^B.$$

Consequently

$$\int_C^D \sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} dt - \int_A^B \sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} dt = 0. \tag{98–4}$$

This is the *principle of least action*, the time integral

$$S \equiv \int \sum_s \dot{q}_s \frac{\partial \mathscr{L}}{\partial \dot{q}_s} dt \tag{98–5}$$

being defined as the *action* along the path. In terms of S we can write (98–4) in the simple form

$$\delta S = 0. \tag{98–6}$$

The principle of least action states that the action has a stationary value along a dynamical path as compared with all nearby varied paths which have the same termini and the same constant energy. It should be noted that the varied paths are not, in general, described in the same time as the dynamical path. In spite of the adjective "least" in the name of the principle, the action along a dynamical path is not necessarily a minimum.

99. The Canonical Equations. — The generalized momentum p_s conjugate to the generalized coordinate q_s is defined in terms of the kinetic potential by the relation

$$p_s \equiv \frac{\partial \mathscr{L}(q, \dot{q})}{\partial \dot{q}_s}, \quad s = 1, 2, \cdots f. \tag{99–1}$$

Evidently there are as many generalized momenta as there are degrees of freedom. Solving the f equations (99–1) for the $\dot{q}_s$'s, we are able to express each generalized velocity as a function $\dot{q}_s(q, p)$ of the coordinates and the momenta. Substituting these expressions

for the $\dot{q}_s$'s appearing in $\mathscr{L}(q, \dot{q})$ we can express the kinetic potential as a function $\mathscr{L}(q, p)$ of the coordinates and the momenta.

The *Hamiltonian function* $\mathscr{H}(p, q)$ is defined by the relation

$$\mathscr{H}(p, q) \equiv \sum_s p_s \dot{q}_s(q, p) - \mathscr{L}(q, p). \tag{99–2}$$

As $p_s = \partial\mathscr{L}(q, \dot{q})/\partial\dot{q}_s$ by definition, it follows from (96–28) that $\mathscr{H}(p, q)$ is the total dynamical energy, that is, the sum of the intrinsic kinetic energy, the mutual kinetic energy and the potential energy of the group of particles, expressed as a function of the generalized momenta and the generalized coordinates and, if the external field is not static, of the time as well.

Let us find the partial derivatives of $\mathscr{H}(p, q)$ with respect to the coordinates and the momenta for a dynamical system obeying Lagrange's equations (96–25) which may be written

$$\dot{p}_r = \frac{\partial\mathscr{L}(q, \dot{q})}{\partial q_r}, \quad r = 1, 2, \cdots f, \tag{99–3}$$

in terms of the generalized momenta. The relations so obtained are known as the *canonical equations*.

Differentiating with respect to q_r we have

$$\frac{\partial\mathscr{H}(p, q)}{\partial q_r} = \sum_s p_s \frac{\partial\dot{q}_s(q, p)}{\partial q_r} - \frac{\partial\mathscr{L}(q, p)}{\partial q_r}.$$

But

$$\frac{\partial\mathscr{L}(q, p)}{\partial q_r} = \frac{\partial\mathscr{L}(q, \dot{q})}{\partial q_r} + \sum_s \frac{\partial\mathscr{L}(q, \dot{q})}{\partial\dot{q}_s}\frac{\partial\dot{q}_s(q, p)}{\partial q_r}$$

$$= \dot{p}_r + \sum_s p_s \frac{\partial\dot{q}_s(q, p)}{\partial q_r}$$

by (99–3) and (99–1). Consequently

$$\dot{p}_r = -\frac{\partial\mathscr{H}(p, q)}{\partial q_r}, \quad r = 1, 2, \cdots f. \tag{99–4}$$

Next, differentiating $\mathscr{H}(p, q)$ with respect to p_r,

$$\frac{\partial\mathscr{H}(p, q)}{\partial p_r} = \dot{q}_r + \sum_s p_s \frac{\partial\dot{q}_s(q, p)}{\partial p_r} - \frac{\partial\mathscr{L}(q, p)}{\partial p_r}.$$

Now
$$\frac{\partial \mathscr{L}(q, p)}{\partial p_r} = \sum_s \frac{\partial \mathscr{L}(q, \dot{q})}{\partial \dot{q}_s} \frac{\partial \dot{q}_s(q, p)}{\partial p_r}$$
$$= \sum_s p_s \frac{\partial \dot{q}_s(q, p)}{\partial p_r}$$
by (99–1). Therefore
$$\dot{q}_r = \frac{\partial \mathscr{H}(p, q)}{\partial p_r}, \quad r = 1, 2, \cdots f. \tag{99–5}$$

The f pairs of first order equations, (99–4) and (99–5), are equivalent to Lagrange's equations. Evidently the group (99–5) are merely definitions, since we used (99–1) alone in obtaining them. The equations of motion are contained in (99–4). In future we shall write $\mathscr{H}$ in place of $\mathscr{H}(p, q)$, always understanding by the Hamiltonian the total energy expressed as a function of the coordinates and the momenta.

The total derivative of $\mathscr{H}$ with respect to the time is
$$\frac{d\mathscr{H}}{dt} = \sum_r \left\{ \frac{\partial \mathscr{H}}{\partial q_r} \dot{q}_r + \frac{\partial \mathscr{H}}{\partial p_r} \dot{p}_r \right\} + \frac{\partial \mathscr{H}}{\partial t}$$
$$= \frac{\partial \mathscr{H}}{\partial t} \tag{99–6}$$
by the aid of the canonical equations (99–4) and (99–5). Since $\mathscr{H}$ can contain explicit t only through the potentials of the external field, this constitutes another proof of the conclusion reached in (96–29) that the dynamical energy of the group of particles remains constant if the external field is static.

100. The Hamilton-Jacobi Equation. — Lagrange's equations or the canonical equations merely enable us to set up, in convenient form, the differential equations of motion. Now we shall develop a method of obtaining the integrated equations of motion of a conservative dynamical system. In doing so we shall make use of Hamilton's principle (98–2). Although Hamilton's principle requires the time integral of the kinetic potential to have a stationary value along a dynamical path as compared with *all* nearby varied paths which have the same q's at the two termini and which are described in the same time, we shall consider here only those varied paths which have the same p's in addition to the same q's as the dynamical path at the two termini.

As the system is conservative—which means that the external electromagnetic field is static—the total dynamical energy remains constant during the motion, and therefore

$$\mathscr{H}(p_1, \cdots p_f, \quad q_1, \cdots q_f) = \alpha_1 \text{ (a constant)} \qquad (100\text{–}1)$$

along a dynamical path.

The integral which appears in Hamilton's principle is

$$\int \mathscr{L} dt = \int (\sum_s p_s \dot{q}_s - \mathscr{H})dt = \int (\sum_s p_s dq_s - \mathscr{H} dt) \qquad (100\text{–}2)$$

by (99–2), where we use dq_s to indicate a differential taken *along* a path, either dynamical or varied, in contrast to δq_s, which takes us from a point on a dynamical path to the corresponding point on a varied path.

The success of our method depends upon our ability to express $\sum_s p_s dq_s$ as the differential of some functon S of $q_1, \cdots q_f$ and f parameters $\alpha_1, \cdots \alpha_f$ which remain constant along a dynamical path but which we can vary to form the varied paths needed to make use of Hamilton's principle. Now if

$$\sum_s p_s dq_s = dS(q_1, \cdots q_f, \quad \alpha_1, \cdots \alpha_f) \qquad (100\text{–}3)$$

along a dynamical path, it follows that

$$p_s = \frac{\partial S}{\partial q_s}, \quad s = 1, 2, \cdots f, \qquad (100\text{–}4)$$

and consequently S must satisfy the partial differential equation

$$\mathscr{H}\left(\frac{\partial S}{\partial q_1}, \cdots \frac{\partial S}{\partial q_f}, q_1, \cdots q_f\right) = \alpha_1 \qquad (100\text{–}5)$$

of the first order, in accord with (100–1). This is known as the *Hamilton-Jacobi partial differential equation.* Its complete solution contains f arbitrary constants of which one must be additive since S itself does not appear in the differential equation. Discarding the additive constant, we have, with the inclusion of α_1, just the required number of parameters $\alpha_1, \alpha_2, \cdots \alpha_f$. Incidentally, as

$$S = \int \sum_s p_s dq_s = \int \sum_s \dot{q}_s \frac{\partial \mathscr{L}(q, \dot{q})}{\partial \dot{q}_s} dt \qquad (100\text{–}6)$$

from (100–3) and (99–1), we observe that the function S is just the action defined by (98–5).

Let us suppose, then, that we have found the desired solution $S(q_1, \cdots q_f, \alpha_1, \cdots \alpha_f)$ of (100–5). For convenience in notation we shall put

$$Q_s \equiv \frac{\partial S}{\partial \alpha_s}, \quad s = 1, 2, \cdots f. \tag{100–7}$$

Then, if we vary both the q's and the α's,

$$dS = \sum_s p_s dq_s + \sum_s Q_s d\alpha_s \tag{100–8}$$

in general, the α's being constants and therefore the $d\alpha$'s zero along a dynamical path, but not so along a varied path.

Putting the value of $\sum_s p_s dq_s$ obtained from (100–8) in (100–2) and replacing $\mathscr{H}$ by its equal α_1, we have, for any path,

$$\begin{aligned}\int_A^B \mathscr{L} dt &= -\int_A^B \left(\sum_s Q_s d\alpha_s + \alpha_1 dt\right) + \int_A^B dS \\ &= -\int_A^B \left(\sum_s Q_s \dot{\alpha}_s + \alpha_1\right) dt + S_B - S_A,\end{aligned}$$

where A and B are the termini of the path, as in article 98.

Now we apply Hamilton's principle (98–2). Since we are resricting the varied paths to those which have the same p's in addition to the same q's as the dynamical path at the two termini, it follows from (100–4) that the α's are the same at the termini for the varied paths as for the dynamical path. Therefore $\delta S_B = \delta S_A = 0$ and Hamilton's principle reduces to

$$\delta \int_A^B \left(-\sum_s Q_s \dot{\alpha}_s - \alpha_1\right) dt = 0.$$

Carrying out the variation,

$$\int_A^B \left\{-\sum_s \dot{\alpha}_s \delta Q_s - \sum_s Q_s \delta \dot{\alpha}_s - \delta \alpha_1\right\} dt = 0 \tag{100–9}$$

as dt is unvaried. Now

$$-\int_A^B Q_s\delta\dot{\alpha}_s dt = -\int_A^B \frac{d}{dt}(Q_s\delta\alpha_s)dt + \int_A^B \dot{Q}_s\delta\alpha_s dt$$
$$= -\left| Q_s\delta\alpha_s \right|_A^B + \int_A^B \dot{Q}_s\delta\alpha_s dt$$
$$= \int_A^B \dot{Q}_s\delta\alpha_s dt$$

since $\delta\alpha_s$ vanishes at both ends of the path. Moreover each α_s is a constant along the dynamical path over which we are integrating, and therefore each $\dot{\alpha}_s = 0$. So (100–9) becomes

$$\int_A^B \{(\dot{Q}_1 - 1)\delta\alpha_1 + \dot{Q}_2\delta\alpha_2 + \cdots + \dot{Q}_f\delta\alpha_f\}dt = 0. \quad (100\text{–}10)$$

As the variations $\delta\alpha_1, \delta\alpha_2, \cdots \delta\alpha_f$ are independent, we may give them such signs that $(\dot{Q}_1 - 1)\delta\alpha_1, \dot{Q}_2\delta\alpha_2, \cdots \dot{Q}_f\delta\alpha_f$ are each positive everywhere along the path. Then (100–10) can be true only if

$$\dot{Q}_1 = 1,$$
$$\dot{Q}_s = 0, \quad s = 2, 3, \cdots f.$$

Integrating these f equations and restoring the expressions for the Q's as derivatives of S in accord with (100–7) we have

$$\left.\begin{aligned} \frac{\partial S}{\partial \alpha_1} &= t + \beta_1, \\ \frac{\partial S}{\partial \alpha_s} &= \beta_s, \qquad s = 2, 3, \cdots f, \end{aligned}\right\} \quad (100\text{–}11)$$

where the β's are constants in the time. As S is a function of $q_1, \cdots q_f$ as well as of $\alpha_1, \cdots \alpha_f$ we have here f relations between the coordinates and the time. Solving for the q's we obtain

$$q_s = q_s(\alpha_1, \cdots \alpha_f, \beta_1, \cdots \beta_f, t), \qquad s = 1, 2, \cdots f. \quad (100\text{–}12)$$

These can be none other than the integrated equations of motion. We note that the equations (100–11) for $s = 2, 3, \cdots f$ do not contain t. Therefore they constitute the equations of the trajectory.

The solution of the Hamilton-Jacobi differential equation is effected by separating the variables so as to resolve it into f ordinary

differential equations. Therefore the success of the method developed in this article depends upon our ability to find coordinates in which the variables are separable. The parameters $\alpha_2, \cdots \alpha_f$ appear as constants of separation, one being introduced by each of the $f - 1$ separations.

We note that if one of the coordinates, say q_1, does not appear explicitly in the equation, we can always perform one separation of variables by solving (100–5) for $\partial S/\partial q_1$. Then $\partial S/\partial q_1 = \alpha_2$, a constant. Such a coordinate is called *ignorable*. In accord with (100–4) the momentum conjugate to an ignorable coordinate is a constant in the time. For example, in the case of a single particle subject to a central force, the azimuth ϕ does not appear explicitly in the Hamiltonian function. Consequently the conjugate momentum, which is the angular momentum about the axis around which ϕ is measured, does not change with the time.

101. Motion of a Particle in a Static Electromagnetic Field. — In this article we shall discuss, first in general, and then with reference to specific fields, the motion of a single particle of charge e and mass m in an external static electromagnetic field.

(I) *Rectangular Coordinates.* Using rectangular coordinates x, y, z the function T_v' defined by (96–7) is

$$T_v' = -mc^2\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}}. \qquad (101\text{–}1)$$

Evidently the mutual kinetic energy T_m vanishes since only a single particle is present. In accord with (96–12) the magnetic energy is

$$T_H = \frac{e}{c}(A_{0x}\dot{x} + A_{0y}\dot{y} + A_{0z}\dot{z}), \qquad (101\text{–}2)$$

and in accord with (96–22) the potential energy is

$$V = e\Phi_0. \qquad (101\text{–}3)$$

Differentiating $\mathscr{L} = T_v' + T_H - V$ partially with respect to $\dot{x}$ we find for the conjugate momentum

$$p_x = \frac{m\dot{x}}{\sqrt{1 - \beta^2}} + \frac{e}{c}A_{0x} = m_t\dot{x} + \frac{e}{c}A_{0x}, \qquad (101\text{–}4)$$

where $\beta^2 \equiv (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2$ as usual and m_t is the transverse mass.

Similar expressions hold for p_y and p_z. Combining the three we get

$$\frac{1}{m^2c^2}\left(p^2 - 2\frac{e}{c}\mathbf{p}\cdot\mathbf{A}_0 + \frac{e^2}{c^2}A_0^{\ 2}\right) = \frac{\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} - 1$$

in terms of $\mathbf{p} = \boldsymbol{i}p_x + \boldsymbol{j}p_y + \boldsymbol{k}p_z$ and $\mathbf{A}_0 = \boldsymbol{i}A_{0x} + \boldsymbol{j}A_{0y} + \boldsymbol{k}A_{0z}$. Consequently the kinetic energy is

$$T_v = \frac{mc^2}{\sqrt{1-\beta^2}} = mc^2\sqrt{1 + \frac{1}{m^2c^2}\left|\mathbf{p} - \frac{e}{c}\mathbf{A}_0\right|^2}. \quad (101\text{–}5)$$

This gives for the Hamiltonian function, $\mathscr{H} = T_v + V$,

$$\mathscr{H} = mc^2\sqrt{1 + \frac{1}{m^2c^2}\left|\mathbf{p} - \frac{e}{c}\mathbf{A}_0\right|^2} + e\Phi_0, \quad (101\text{–}6)$$

where $\mathbf{A}_0$ and Φ_0 are functions of the coordinates. As the dynamical system is conservative, we can equate $\mathscr{H}$ to a constant α_1 representing the total energy. Then, rearranging terms,

$$\left(p_x - \frac{e}{c}A_{0x}\right)^2 + \left(p_y - \frac{e}{c}A_{0y}\right)^2 + \left(p_z - \frac{e}{c}A_{0z}\right)^2 - \left(\frac{\alpha_1}{c} - \frac{e}{c}\Phi_0\right)^2 + m^2c^2 = 0. \quad (101\text{–}7)$$

Now A_{0x}, A_{0y}, A_{0z}, $\mathrm{i}\Phi_0$ are the four components of the four-vector potential $\mathbf{W}$ in accord with (93–12), and p_x, p_y, p_z, $p_l \equiv \mathrm{i}\alpha_1/c$ are the components of the four-vector linear momentum of the particle. In terms of these four-dimensional vectors we can write (101–7) in the more symmetrical form

$$\left(p_x - \frac{e}{c}W_x\right)^2 + \left(p_y - \frac{e}{c}W_y\right)^2 + \left(p_z - \frac{e}{c}W_z\right)^2 + \left(p_l - \frac{e}{c}W_l\right)^2 + m^2c^2 = 0. \quad (101\text{–}8)$$

As an application of (101–7) we shall investigate the motion of an ion in crossed uniform electric and magnetic fields $\mathbf{E}_0$ and $\mathbf{H}_0$, the first being parallel to the Y axis and the second to the Z axis. Then $\Phi_0 = -E_0y$ and for $\mathbf{A}_0$ we can take any one of the three vector functions $-\boldsymbol{i}H_0y$, $\boldsymbol{j}H_0x$, and $\frac{1}{2}H_0(-\boldsymbol{i}y + \boldsymbol{j}x)$, since the curl of each of them is equal to $\boldsymbol{k}H_0$. But, as Φ_0 is a function of y, we choose

$\mathbf{A}_0 = -iH_0y$ so as to make possible the separation of variables in the Hamilton-Jacobi equation. Then (101–7) becomes

$$\left(p_x + \frac{eH_0}{c}y\right)^2 + p_y^2 + p_z^2 - \left(\frac{\alpha_1}{c} + \frac{eE_0}{c}y\right)^2 + m^2c^2 = 0. \qquad (101\text{–}9)$$

Since x and z are ignorable coordinates, p_x and p_z are constants in the time. Therefore

$$\frac{\partial S}{\partial x} = p_x = \frac{m\dot{x}}{\sqrt{1-\beta^2}} - \frac{eH_0}{c}y = \frac{\alpha_2}{c}, \qquad (101\text{–}10)$$

$$\frac{\partial S}{\partial z} = p_z = \frac{m\dot{z}}{\sqrt{1-\beta^2}} = \frac{\alpha_3}{c}, \qquad (101\text{–}11)$$

where α_2 and α_3 are constants. Consequently we are left with

$$\frac{\partial S}{\partial y} = p_y = \frac{m\dot{y}}{\sqrt{1-\beta^2}}$$
$$= \frac{1}{c}\sqrt{(\alpha_1 + eE_0y)^2 - (\alpha_2 + eH_0y)^2 - (m^2c^4 + \alpha_3^2)}. \qquad (101\text{–}12)$$

Hence the action is

$$S = \frac{1}{c}\left\{\alpha_2 x + \int\sqrt{(\alpha_1 + eE_0y)^2 - (\alpha_2 + eH_0y)^2 - (m^2c^4 + \alpha_3^2)}\,dy + \alpha_3 z\right\}. \qquad (101\text{–}13)$$

The integrated equations of motion are obtained from this expression by differentiation in accord with (100–11). We shall limit our further investigation to the determination of the projection of the trajectory on the XY plane. The equation of this projection is obtained by equating $\partial S/\partial\alpha_2$ to a constant. Differentiating under the sign of integration we find

$$x + \beta_2 = \int\frac{(\alpha_2 + eH_0y)dy}{\sqrt{(\alpha_1 + eE_0y)^2 - (\alpha_2 + eH_0y)^2 - (m^2c^4 + \alpha_3^2)}}. \qquad (101\text{–}14)$$

Put $\gamma \equiv E_0/H_0$, and, as the position of the origin is a matter of no importance, define new coordinates ξ and η by the relations

$$\xi \equiv x + \beta_2, \quad \eta \equiv y + \frac{\alpha_2}{eH_0}.$$

Then

$$\xi = \int \frac{\eta d\eta}{\sqrt{A + B\eta - C\eta^2}}, \tag{101–15}$$

where

$$A \equiv \frac{(\alpha_1 - \gamma\alpha_2)^2 - \alpha_3{}^2 - m^2c^4}{e^2H_0{}^2}, \quad B \equiv 2\gamma\frac{\alpha_1 - \gamma\alpha_2}{eH_0}, \quad C \equiv 1 - \gamma^2.$$

The integral (101–15) takes different forms according as C is positive or negative. The case of greatest interest is that where $E_0 < H_0$ and therefore C is positive. Then

$$\xi = -\frac{\sqrt{A + B\eta - C\eta^2}}{C} + \frac{B}{2C\sqrt{C}}\cos^{-1}\left(\frac{B - 2C\eta}{\sqrt{B^2 + 4AC}}\right). \tag{101–16}$$

Let us introduce the parameter θ by means of the relations

$$\cos\theta = \frac{B - 2C\eta}{\sqrt{B^2 + 4AC}}, \quad \sin\theta = \frac{2\sqrt{C}\sqrt{A + B\eta - C\eta^2}}{\sqrt{B^2 + 4AC}}. \tag{101–18}$$

In terms of this parameter (101–16) may be written

$$\sqrt{C}\,\xi = \frac{B}{2C}\theta - \frac{\sqrt{B^2 + 4AC}}{2C}\sin\theta. \tag{101–19}$$

This equation, together with the equation defining θ, which can be written in the form

$$\eta = \frac{B}{2C} - \frac{\sqrt{B^2 + 4AC}}{2C}\cos\theta, \tag{101–20}$$

constitute the parametric equations of the projection of the trajectory on the XY plane. If, instead of plotting η against ξ, we plot η against $\sqrt{C}\xi$, the curve is a cycloid, which is curtate, common or prolate according as A is positive, zero or negative.

This is shown very easily by comparing (101–19) and (101–20) with the parametric equations of the cycloid. Let O (Fig. 110) be the center of a circle of radius a rolling along the X axis, and let P be

the point attached to the circle which generates the cycloid PP_1Q. When the circle rolls through an angle θ, O moves to O_1 and P to P_1. If $\overline{OP} = b$,

$$x = a\theta - b \sin \theta,$$

$$y = a - b \cos \theta.$$

Therefore the radius of the rolling circle which generates the cycloid in $\sqrt{C}\xi$ and η described by (101–19) and (101–20) is $B/2C$ and the distance of the point P from the center of the circle is $\sqrt{B^2 + 4AC}/2C$. Since $C = 1 - E_0^2/H_0^2$, the actual projection of

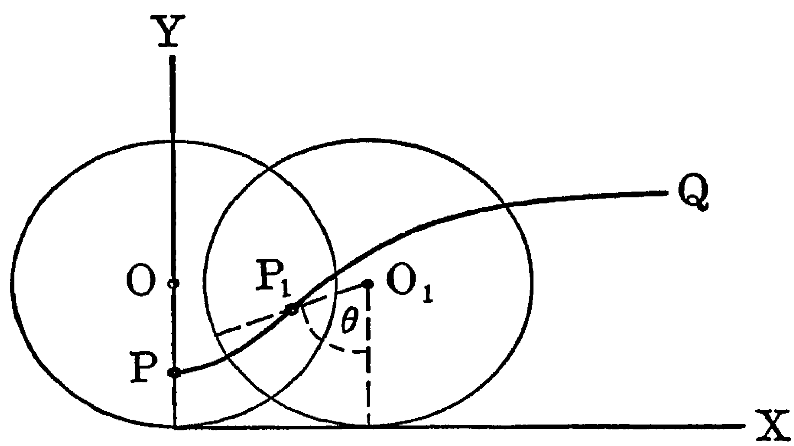

FIG. 110.

the trajectory in the rectangular coordinates ξ and η is a cycloid whose ξ dimensions have been stretched in the ratio $1 : \sqrt{1 - (u/c)^2}$, where $u \equiv cE_0/H_0$ is the mean velocity of progression in the X direction noted in article 66.

Next consider the special case where $E_0 = 0$. Then $B = 0$ and $C = 1$. Consequently $\xi = -\sqrt{A} \sin \theta$ and $\eta = -\sqrt{A} \cos \theta$, giving the circle

$$\xi^2 + \eta^2 = \frac{\alpha_1{}^2 - \alpha_3{}^2 - m^2c^4}{e^2H_0{}^2}. \tag{101–21}$$

As $\alpha_1{}^2 = m^2c^4/(1 - \beta^2)$ and $\alpha_3{}^2 = m^2c^2\dot{z}^2/(1 - \beta^2)$, the square of the radius a of the circular path is

$$a^2 = \frac{\dot{x}^2 + \dot{y}^2}{(1 - \beta^2)\dfrac{e^2H_0{}^2}{m^2c^2}} \tag{101–22}$$

in agreement with (66–14).

Another case of interest is that in which $E_0 = H_0$ and therefore

$C = 0$. It is convenient to shift the origin by the substitution $\eta = \eta' - A/B$. Then (101–15) becomes

$$\sqrt{B}\,\xi = \int \sqrt{\eta'}\,d\eta' - \frac{A}{B}\int \frac{d\eta'}{\sqrt{\eta'}}$$

$$= \tfrac{2}{3}\sqrt{\eta'}\left(\eta' - 3\frac{A}{B}\right). \qquad (101\text{–}23)$$

A typical trajectory for A positive is shown in Fig. 111. This path corresponds to the curtate cycloid for $\gamma < 1$, the electric field being too strong to permit the ion to make more than a single loop under the transverse force exerted by the magnetic field. If $A = 0$ the loop

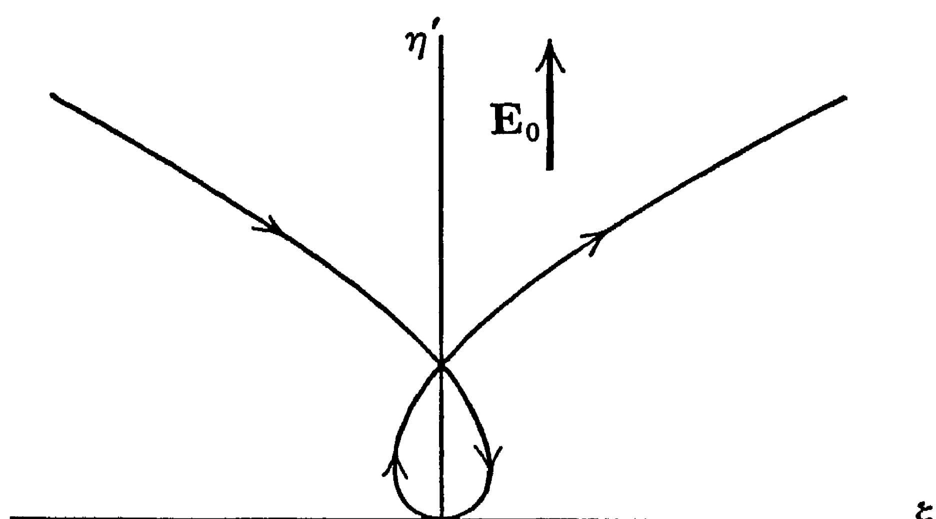

FIG. 111.

degenerates into a cusp, the trajectory being the analog of the common cycloid for $\gamma < 1$, and if A is negative the curve is tangent to the X axis at the origin without any reversal in sense of the X component of velocity, as in the case of the prolate cycloid for $\gamma < 1$.

Finally, if $B^2 + 4AC = 0$, it follows from (101–20) that $\eta = B/2C$ and from (101–12) that $\dot{y} = 0$. The projection of the trajectory on the XY plane is a straight line parallel to the X axis, the force due to the magnetic field being exactly equal and opposite to that due to the electric field. Both $\dot{x}$ and $\dot{z}$ are constants, and $\dot{x}/c = E_0/H_0$.

(II) *Orthogonal Curvilinear Coordinates.* Using the notation of article 19 we have for the square of the velocity of a particle in orthogonal curvilinear coordinates

$$a_u{}^2\dot{u}^2 + a_v{}^2\dot{v}^2 + a_w{}^2\dot{w}^2 \qquad (101\text{–}24)$$

from (19–13). Therefore

$$T_v' = -mc^2\sqrt{1 - \frac{a_u{}^2\dot{u}^2 + a_v{}^2\dot{v}^2 + a_w{}^2\dot{w}^2}{c^2}}, \qquad (101\text{–}25)$$

and

$$T_H = \frac{e}{c}\left(A_{0u}a_u\dot{u} + A_{0v}a_v\dot{v} + A_{0w}a_w\dot{w}\right), \qquad (101\text{–}26)$$

the expression for the potential energy remaining as in (101–3). Consequently

$$p_u = \frac{ma_u{}^2\dot{u}}{\sqrt{1-\beta^2}} + \frac{e}{c}a_uA_{0u} = m_ta_u{}^2\dot{u} + \frac{e}{c}a_uA_{0u}, \qquad (101\text{–}27)$$

and similar expressions hold for p_v and p_w. Hence

$$\frac{1}{m^2c^2}\left\{\left(\frac{p_u}{a_u} - \frac{e}{c}A_{0u}\right)^2 + \left(\frac{p_v}{a_v} - \frac{e}{c}A_{0v}\right)^2 + \left(\frac{p_w}{a_w} - \frac{e}{c}A_{0w}\right)^2\right\} = \frac{1}{1-\beta^2} - 1$$

and the kinetic energy is

$$T_v = mc^2\sqrt{1 + \frac{1}{m^2c^2}\left\{\left(\frac{p_u}{a_u} - \frac{e}{c}A_{0u}\right)^2 + \left(\frac{p_v}{a_v} - \frac{e}{c}A_{0v}\right)^2 + \left(\frac{p_w}{a_w} - \frac{e}{c}A_{0w}\right)^2\right\}}. \qquad (101\text{–}28)$$

Equating the sum of the kinetic and potential energies to a constant α_1, we obtain in place of (101–7)

$$\left(\frac{p_u}{a_u} - \frac{e}{c}A_{0u}\right)^2 + \left(\frac{p_v}{a_v} - \frac{c}{c}A_{0v}\right)^2 + \left(\frac{p_w}{a_w} - \frac{e}{c}A_{0w}\right)^2 - \left(\frac{\alpha_1}{c} - \frac{e}{c}\Phi_0\right)^2 + m^2c^2 = 0. \qquad (101\text{–}29)$$

We shall confine our attention here to a uniform magnetic field of intensity $\mathbf{H}_0$ and to an electric field which is symmetrical about an axis through the origin parallel to $\mathbf{H}_0$. Then one of the coordinates, say w, must be taken as the azimuth ϕ measured about the axis of symmetry. For the vector potential we can take $\mathbf{A}_0 = \frac{1}{2}\mathbf{H}_0 \times \mathbf{r}$, where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, since

$$\nabla \times (\tfrac{1}{2}\mathbf{H}_0 \times \mathbf{r}) = \tfrac{1}{2}(\nabla\cdot\mathbf{r}\mathbf{H}_0 - \mathbf{H}_0\cdot\nabla\mathbf{r}) = \tfrac{3}{2}\mathbf{H}_0 - \tfrac{1}{2}\mathbf{H}_0 = \mathbf{H}_0.$$

If, then, ρ is the perpendicular distance of the particle from the axis

of symmetry, $A_{0u} = A_{0v} = 0$, $A_{0\phi} = \frac{1}{2}H_0\rho$, and $a_\phi = \rho$. Hence (101–29) becomes

$$\left(\frac{p_u}{a_u}\right)^2 + \left(\frac{p_v}{a_v}\right)^2 + \left(\frac{p_\phi}{\rho} - \frac{eH_0}{2c}\rho\right)^2 - \left(\frac{\alpha_1}{c} - \frac{e}{c}\Phi_0\right)^2 + m^2c^2 = 0. \tag{101–30}$$

(III) *Spherical Coordinates.* We continue to confine our attention to a uniform magnetic field and an axially symmetrical electric field. Then, taking the axis of spherical coordinates r, θ, ϕ parallel to $\mathbf{H}_0$, $a_r = 1$, $a_\theta = r$, $\rho = r \sin\theta$ and (101–30) becomes

$$p_r^2 + \frac{1}{r^2}p_\theta^2 + \left(\frac{p_\phi}{r\sin\theta} - \frac{eH_0}{2c}r\sin\theta\right)^2 - \left(\frac{\alpha_1}{c} - \frac{e}{c}\Phi_0\right)^2 + m^2c^2 = 0. \tag{101–31}$$

We shall investigate the motion of an electron subject to the attraction of a massive atomic nucleus located at the origin, the entire structure being placed in an external magnetic field $\mathbf{H}_0$. Then

$$\frac{e}{c}\Phi_0 = -\frac{e^2Z}{4\pi cr}, \tag{101–32}$$

where Z is the atomic number of the nucleus. In the cases of interest $eH_0r\sin\theta/2c$ is so small compared with $e\Phi_0/c$ that we are justified in neglecting its square. Then (101–31) reduces to

$$p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2 - \frac{eH_0}{c}p_\phi - \left(\frac{\alpha_1}{c} + \frac{e^2Z}{4\pi cr}\right)^2 + m^2c^2 = 0. \tag{101–33}$$

Since ϕ is ignorable,

$$\frac{\partial S}{\partial\phi} = p_\phi = r^2\sin^2\theta\left\{\frac{m\dot{\phi}}{\sqrt{1-\beta^2}} + \frac{eH_0}{2c}\right\} = \alpha_2, \tag{101–34}$$

where α_2 is a constant in the time. This statement, we may point out, is true even when we retain all the terms in the Hamiltonian function.

We can separate the remaining variables in (101–33), obtaining

$$\frac{\partial S}{\partial\theta} = p_\theta = \frac{mr^2\dot{\theta}}{\sqrt{1-\beta^2}} = \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\sin^2\theta}}, \tag{101–35}$$

$$\frac{\partial S}{\partial r} = p_r = \frac{m\dot{r}}{\sqrt{1-\beta^2}}$$

$$= \sqrt{\frac{\alpha_1{}^2 - m^2c^4 + eH_0\alpha_2 c}{c^2} + \frac{e^2Z\alpha_1}{2\pi c^2}\frac{1}{r} + \left(\frac{e^4Z^2}{16\pi^2c^2} - \alpha_3{}^2\right)\frac{1}{r^2}}. \quad (101\text{–}36)$$

Consequently the action is

$$S = \int \sqrt{A + \frac{B}{r} + \frac{C}{r^2}}\, dr + \int \sqrt{\alpha_3{}^2 - \frac{\alpha_2{}^2}{\sin^2\theta}}\, d\theta + \alpha_2\phi, \quad (101\text{–}37)$$

where

$$A \equiv \frac{\alpha_1{}^2 - m^2c^4 + eH_0\alpha_2 c}{c^2}, \quad B \equiv \frac{e^2Z\alpha_1}{2\pi c^2}, \quad C \equiv \frac{e^4Z^2}{16\pi^2c^2} - \alpha_3{}^2.$$

We shall investigate the orbit. Equating the derivatives of S with respect to α_2 and α_3 to constants, we have

$$\frac{eH_0}{2c}\int \frac{r\, dr}{\sqrt{C + Br + Ar^2}} - \alpha_2 \int \frac{d\theta}{\sin\theta\sqrt{\alpha_3{}^2\sin^2\theta - \alpha_2{}^2}} + \phi = \beta_2. \quad (101\text{–}38)$$

$$-\alpha_3 \int \frac{dr}{r\sqrt{C + Br + Ar^2}} + \alpha_3 \int \frac{\sin\theta\, d\theta}{\sqrt{\alpha_3{}^2\sin^2\theta - \alpha_2{}^2}} = \beta_3. \quad (101\text{–}39)$$

Closed orbits occur when A and C are negative. We shall consider only such cases. Then, if we carry out the integration indicated,

$$\frac{eH_0}{2c}\left\{\frac{\sqrt{C + Br + Ar^2}}{A} - \frac{B}{2A\sqrt{-A}}\cos^{-1}\left(\frac{B + 2Ar}{\sqrt{B^2 - 4AC}}\right)\right\} - \cos^{-1}\left(\frac{\alpha_2\cot\theta}{\sqrt{\alpha_3{}^2 - \alpha_2{}^2}}\right) + \phi = \beta_2, \quad (101\text{–}40)$$

$$\frac{\alpha_3}{\sqrt{-C}}\cos^{-1}\left(\frac{2C + Br}{r\sqrt{B^2 - 4AC}}\right) + \cos^{-1}\frac{\alpha_3\cos\theta}{\sqrt{\alpha_3{}^2 - \alpha_2{}^2}} = \beta_3, \quad (101\text{–}41)$$

Consider equation (101–40) first, writing it in the more significant form

$$\cot\theta = \frac{\sqrt{\alpha_3{}^2 - \alpha_2{}^2}}{\alpha_2}\cos\left[\phi - \beta_2 - \frac{eH_0}{2c}\left\{\frac{B}{2A\sqrt{-A}}\cos^{-1}\left(\frac{B + 2Ar}{\sqrt{B^2 - 4AC}}\right) - \frac{\sqrt{C + Br + Ar^2}}{A}\right\}\right]. \quad (101\text{–}42)$$

Now the relation between θ and ϕ which defines a plane through the origin whose normal makes an angle γ with the polar axis is

$$\cot\theta = \tan\gamma\cos(\phi - \beta'). \qquad (101\text{–}43)$$

If β' as well as γ is a constant this represents a fixed plane; if, however, β' increases with the time, it represents a plane whose normal precesses around the polar axis in the positive sense while maintaining the constant angle γ with this axis. Comparing (101–42) with (101–43) we see that the orbit lies in a plane whose normal makes the angle

$$\gamma = \cos^{-1}\frac{\alpha_2}{\alpha_3} \qquad (101\text{–}44)$$

with the lines of magnetic force and precesses around them at such a rate that, when r goes through a cycle of values, β' increases by

$$\Delta\beta' = \frac{eH_0}{2c}\,\frac{2\pi B}{2A\sqrt{-A}}. \qquad (101\text{–}45)$$

That this precession is the Larmor precession discussed in article 65 is easily shown. If we neglect β^2 as compared with unity and omit the term in H_0, we find, to a first approximation, that $A = 2mU$, where U is the sum of the kinetic and potential energies on the Newtonian dynamics. Also $B = mc^2Z/2\pi$. Now, for motion under an inverse square force of attraction on the Newtonian dynamics,[1] it is well known that $U = -\,c^2Z/8\pi a$, where a is the semi-major axis of the elliptical orbit. Substituting these values of A and B in (101–45), and using the familiar expression $P = 2\pi a^{3/2}\sqrt{4\pi m/c^2Z}$ for the period, we find

$$\Delta\beta' = -\,\frac{eH_0}{2mc}\,P, \qquad (101\text{–}46)$$

agreeing exactly with (65–13) for the Larmor precession.

Next we investigate the orbit of the electron in the precessing plane. If ψ is the azimuth in this plane, measured from the line of intersection of the perpendicular plane through the origin parallel to the magnetic field,

$$\cos\psi = \sin\theta\cos(\phi - \beta')\cos\gamma + \cos\theta\sin\gamma$$

$$= \frac{\alpha_3}{\sqrt{\alpha_3{}^2 - \alpha_2{}^2}}\cos\theta \qquad (101\text{–}47)$$

[1] L. Page, *Theoretical Physics*, 2nd Edit. p. 95.

with the aid (101–43) and (101–44). Hence (101–41) becomes

$$\frac{1}{r} = -\frac{B}{2C} + \frac{\sqrt{B^2 - 4AC}}{2C} \cos \frac{\sqrt{-C}}{\alpha_3}(\psi - \beta_3). \quad (101\text{–}48)$$

Now the equation of an ellipse of semi-major axis a and eccentricity ϵ in polar coordinates r and χ, referred to a focus as origin, may be written [2]

$$\frac{1}{r} = \frac{1 - \epsilon \cos (\chi - \delta)}{a(1 - \epsilon^2)}. \quad (101\text{–}49)$$

Remembering that A and C are negative constants, while B is positive, we observe that (101–48) is the equation of an ellipse of semi-major axis $a = -B/2A$ and eccentricity $\epsilon = \sqrt{1 - 4AC/B^2}$ in the variables r and $\sqrt{-C}\,\psi/\alpha_3$. Therefore, when r goes through a cycle of values, the increase in ψ is

$$\Delta\psi = 2\pi \frac{\alpha_3}{\sqrt{-C}} = 2\pi \bigg/ \sqrt{1 - \frac{e^4 Z^2}{16\pi^2 c^2 \alpha_3{}^2}}. \quad (101\text{–}50)$$

As this increase is greater than 2π, the orbit precesses in the plane in which it lies, as indicated in Fig. 112. This precession, which

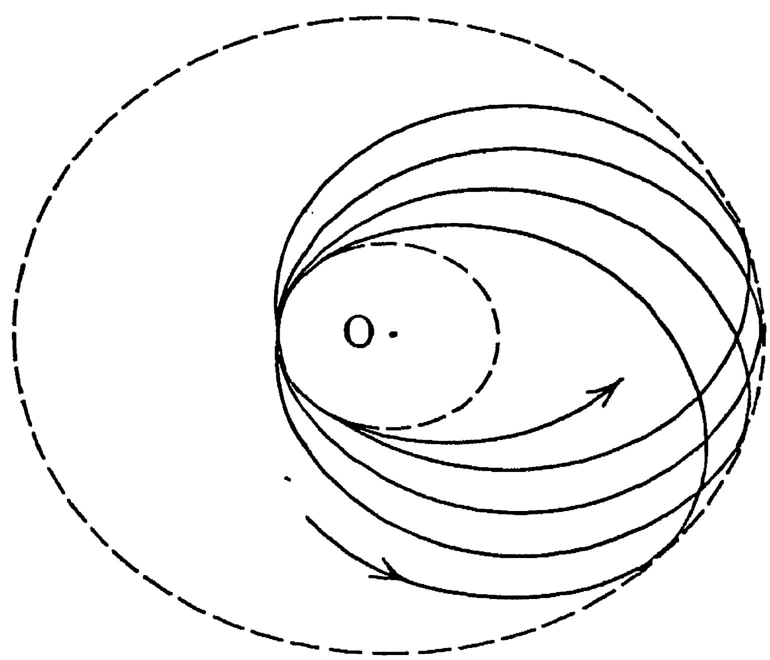

FIG. 112.

is entirely independent of the presence of the magnetic field, is due to the variation of mass with velocity. Had we solved the problem

[2] L. Page, *Theoretical Physics*, 2nd Edit. p. 91.

on the basis of the Newtonian mechanics with constant mass, no such precession would have been found.

102. The Magnetron. — This instrument consists of a straight filament of radius a coaxial with an evacuated cylindrical tube of much larger radius b, placed in a uniform magnetic field H_0 parallel to the common axis of the two cylinders. A current i through the filament raises it to a temperature high enough to produce emission of electrons which are driven toward the outer cylinder by a potential difference $\Phi_{ab} \equiv \Phi_b - \Phi_a$ maintained between the two electrodes.

Taking the common axis of the two cylinders as the Z coordinate axis, it is evident that cylindrical coordinates ρ, ϕ, z are indicated for the solution of the problem. We have two magnetic fields to consider; the uniform external field of vector potential $\frac{1}{2}H_0\rho\boldsymbol{\phi}_1$ and the field due to the current along the Z axis of vector potential $-\{(i/2\pi c)\log\rho\}\boldsymbol{k}$. Since the space charge due to the electrons between the two electrodes is unknown, all we can say of the scalar potential Φ_0 is that it is a function of ρ only. Consequently (101–29) becomes

$$p_\rho{}^2 + \left(\frac{p_\phi}{\rho} - \frac{eH_0}{2c}\rho\right)^2 + \left(p_z + \frac{ei}{2\pi c^2}\log\rho\right)^2 - \left(\frac{\alpha_1}{c} - \frac{e}{c}\Phi_0\right)^2 + m^2c^2 = 0. \quad (102\text{–}1)$$

Since both ϕ and z are ignorable,

$$\frac{\partial S}{\partial \phi} = p_\phi = \frac{m\rho^2\dot{\phi}}{\sqrt{1-\beta^2}} + \frac{eH_0}{2c}\rho^2 = \alpha_2, \quad (102\text{–}2)$$

$$\frac{\partial S}{\partial z} = p_z = \frac{m\dot{z}}{\sqrt{1-\beta^2}} - \frac{ei}{2\pi c^2}\log\rho = \alpha_3, \quad (102\text{–}3)$$

where α_2 and α_3 are constants. Consequently

$$\frac{\partial S}{\partial \rho} = p_\rho = \frac{m\dot{\rho}}{\sqrt{1-\beta^2}}$$
$$= \sqrt{\left(\frac{\alpha_1}{c} - \frac{e}{c}\Phi_0\right)^2 - \left(\frac{\alpha_2}{\rho} - \frac{eH_0}{2c}\rho\right)^2 - \left(\alpha_3 + \frac{ei}{2\pi c^2}\log\rho\right)^2 - m^2c^2}. \quad (102\text{–}4)$$

As the external magnetic field is increased the ion paths become more curved until finally the electrons are unable to reach the outer

electrode. The condition for cut-off is that $\dot{\rho} = 0$ for $\rho = b$. Then

$$\left(\frac{\alpha_1}{c} - \frac{e}{c}\Phi_b\right)^2 = m^2c^2 + \left(\frac{\alpha_2}{b} - \frac{eH_0}{2c}b\right)^2 + \left(\alpha_3 + \frac{ei}{2\pi c^2}\log b\right)^2. \quad (102\text{–}5)$$

The constants $\alpha_1, \alpha_2, \alpha_3$ must be determined by the initial conditions. In actuality the electrons leave the filament with negligibly small velocities. Therefore $\alpha_1 = mc^2 + e\Phi_a$, $\alpha_2 = (eH_0/2c)a^2$, $\alpha_3 = -(ei/2\pi c^2)\log a$. Consequently the condition for cut-off becomes

$$\left(mc - \frac{e}{c}\Phi_{ab}\right)^2 = m^2c^2 + \frac{e^2H_0^2}{4c^2b^2}(b^2 - a^2)^2 + \frac{e^2i^2}{4\pi^2c^4}\log^2\frac{b}{a}. \quad (102\text{–}6)$$

The magnetic flux between the electrodes is $N = \pi(b^2 - a^2)H_0$. Hence, if we solve (102–6) for Φ_{ab},

$$\Phi_{ab} = \frac{m}{e}c^2\left\{1 - \sqrt{1 + \frac{1}{4\pi^2c^4}\frac{e^2}{m^2}\left(\frac{N^2}{b^2} + \frac{i^2}{c^2}\log^2\frac{b}{a}\right)}\right\}. \quad (102\text{–}7)$$

Obviously this condition is independent of the presence of space charge.

By measuring Φ_{ab}, N and i at cut-off the ratio of charge to mass of the electron can be determined from the formula

$$\frac{e}{m} = \frac{2\Phi_{ab}}{\dfrac{\Phi_{ab}^2}{c^2} - \dfrac{1}{4\pi^2c^2}\left(\dfrac{N^2}{b^2} + \dfrac{i^2}{c^2}\log^2\dfrac{b}{a}\right)}. \quad (102\text{–}8)$$

103. Cosmic Ray Trajectories. — In this article we shall investigate the motion of an ion in the field of a uniformly magnetized sphere at rest in the observer's inertial system. This is essentially the problem of the deflection of cosmic rays by the earth's magnetic field. Unfortunately we cannot separate the variables completely, but we can carry the analysis far enough to obtain some interesting results.

The vector potential was obtained in article 58. Using spherical coordinates r, θ, ϕ with the polar axis along the magnetic axis of the sphere, we have

$$A_{0r} = A_{0\theta} = 0, \quad A_{0\phi} = \frac{p_H \sin\theta}{4\pi r^2}, \quad (103\text{–}1)$$

from (58–10), where p_H is the magnetic moment of the sphere. Therefore (101–29) takes the form

$$p_r^2 + \frac{p_\theta^2}{r^2} + \left(\frac{p_\phi}{r\sin\theta} - \frac{ep_H\sin\theta}{4\pi cr^2}\right)^2 - \frac{\alpha_1^2}{c^2} + m^2c^2 = 0, \quad (103\text{–}2)$$

where the total energy α_1 consists solely of the kinetic energy

$$\alpha_1 = \frac{mc^2}{\sqrt{1 - \beta^2}}. \tag{103-3}$$

As ϕ is an ignorable variable,

$$\frac{\partial S}{\partial \phi} = p_\phi = \sin^2\theta \left\{ \frac{mr^2\dot{\phi}}{\sqrt{1 - \beta^2}} + \frac{ep_H}{4\pi cr} \right\} = \alpha_2, \tag{103-4}$$

where α_2 is a constant in the time. If we put $\xi \equiv r\sqrt{\alpha_1{}^2/c^2 - m^2c^2}$, the Hamilton-Jacobi equation in the remaining variables is

$$\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\alpha_2{}^2}{\sin^2\theta} + \xi^2\left\{\left(\frac{\partial S}{\partial \xi}\right)^2 - 1 - 2\frac{k\alpha_2}{\xi^3}\right\} + \frac{k^2\sin^2\theta}{\xi^2} = 0, \tag{103-5}$$

where $k \equiv (ep_H/4\pi c)\sqrt{\alpha_1{}^2/c^2 - m^2c^2}$.

On account of the last term in (103–5) we cannot separate the variables. However, if we omit this term we can get a solution which is valid at distances from the sphere for which $\xi^2 \gg k$. Unfortunately this solution fails to describe the motion of cosmic rays approaching the earth at distances less than several hundred earth radii, since k/ξ^2 at the surface of the earth is of the order of magnitude of 30 even for rays with energies as high as $(10)^{10}$ electron-volts.

Neglecting the last term in (103–5) we get

$$\frac{\partial S}{\partial \theta} = p_\theta = \frac{mr^2\dot{\theta}}{\sqrt{1 - \beta^2}} = \sqrt{\alpha_3{}^2 - \frac{\alpha_2{}^2}{\sin^2\theta}}, \tag{103-6}$$

$$\frac{\partial S}{\partial \xi} = p_\xi = \frac{m\dot{\xi}}{\sqrt{1 - \beta^2}} = \sqrt{1 - \frac{\alpha_3{}^2}{\xi^2} + 2\frac{k\alpha_2}{\xi^3}}. \tag{103-7}$$

The constants α_2 and α_3 admit of a simple interpretation in terms of the motion at great distances from the sphere. From (103–4) it is clear that α_2 is the component

$$P_\phi \equiv \frac{m}{\sqrt{1 - \beta^2}} r^2 \sin^2\theta\dot{\phi}$$

parallel to the polar axis of the mechanical angular momentum at infinity. Now the mechanical linear momentum at infinity is

$$\mathbf{P}_l = \frac{m}{\sqrt{1 - \beta^2}} \{\mathbf{r}_1\dot{r} + \boldsymbol{\theta}_1 r\dot{\theta} + \boldsymbol{\phi}_1 r\sin\theta\,\dot{\phi}\}$$

and the mechanical angular momentum is

$$\mathbf{P}_a = \mathbf{r} \times \mathbf{P}_l = \frac{mr}{\sqrt{1-\beta^2}}\{-\boldsymbol{\theta}_1 r \sin\theta\, \dot{\phi} + \boldsymbol{\phi}_1 r\dot{\theta}\}.$$

Hence the square of the total mechanical angular momentum is

$$P_a^2 = \frac{m^2r^2}{1-\beta^2}\{r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\}$$

$$= \frac{m^2r^4\dot{\theta}^2}{1-\beta^2} + \frac{\alpha_2^2}{\sin^2\theta}.$$

Comparing with (103–6) we see that $\alpha_3 = P_a$.

From (103–4), (103–6) and (103–7) we find for the action

$$S = \alpha_2\phi + \int\sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\sin^2\theta}}\,d\theta + \int\sqrt{1 - \frac{\alpha_3^2}{\xi^2} + 2\frac{k\alpha_2}{\xi^3}}\,d\xi. \quad (103\text{–}8)$$

Hence the equations of the orbit are

$$k\int\frac{d\xi}{\xi^2\sqrt{\xi^2-\alpha_3^2}} - \alpha_2\int\frac{d\theta}{\sin\theta\sqrt{\alpha_3^2\sin^2\theta - \alpha_2^2}} + \phi = \beta_2, \quad (103\text{–}9)$$

$$-\alpha_3\int\frac{d\xi}{\xi\sqrt{\xi^2 - \alpha_3^2 + 2\dfrac{k\alpha_2}{\xi}}} + \alpha_3\int\frac{\sin\theta\,d\theta}{\sqrt{\alpha_3^2\sin^2\theta - \alpha_2^2}} = \beta_3, \quad (103\text{–}10)$$

where we have omitted the term in k in the integrand of the first integral in (103–9) since the entire integral is multiplied by the small quantity k. Carrying out the integration of (103–9) we get

$$\cot\theta = \frac{\sqrt{\alpha_3^2-\alpha_2^2}}{\alpha_2}\cos\left(\phi - \beta_2 + \frac{k}{\alpha_3^2}\sqrt{1-\frac{\alpha_3^2}{\xi^2}}\right). \quad (103\text{–}11)$$

This equation is similar in form to (101–42), and signifies, as was shown in detail in article 101, that the orbit lies in a plane whose normal makes an angle $\cos^{-1}(\alpha_2/\alpha_3)$ with the polar axis. In this case the phase angle β' of (101–43) is

$$\beta' = \beta_2 - \frac{k}{\alpha_3^2}\sqrt{1-\frac{\alpha_3^2}{\xi^2}}. \quad (103\text{–}12)$$

For positive k, β' increases as the particle approaches the origin and

the normal to the orbital plane precesses in the positive sense about the polar axis, and *vice versa* as the particle recedes.

The second integral in (103–10) was evaluated in article 101. Put J for the first. Then, making the substitutions $\xi \equiv \alpha_3/x$, $k \equiv l\alpha_3{}^3/\alpha_2$,

$$J \equiv -\alpha_3 \int \frac{d\xi}{\xi\sqrt{\xi^2 - \alpha_3{}^2 + 2\dfrac{k\alpha_2}{\xi}}} = \int \frac{dx}{\sqrt{1 - x^2 + 2lx^3}}$$

$$= \int \frac{dx}{\sqrt{1 - (x - lx^2)^2}}$$

through terms in the first power of l. Continuing,

$$J = \int \frac{d(x - lx^2)}{\sqrt{1 - (x - lx^2)^2}} + l \int \frac{d(x^2)}{\sqrt{1 - x^2}}$$

provided we omit terms in l in the integrand of the second integral. This is justified as the integral is already multiplied by l. Therefore

$$J = -\cos^{-1}(x - lx^2) - l\sqrt{1 - x^2},$$

and (103–10) leads to

$$-\cos^{-1}\left\{\frac{\alpha_3}{\xi}\left(1 - \frac{k\alpha_2}{\alpha_3{}^2\xi}\right)\right\} - \frac{k\alpha_2}{\alpha_3{}^3}\sqrt{1 - \frac{\alpha_3{}^2}{\xi^2}}$$

$$+ \cos^{-1}\frac{\alpha_3 \cos\theta}{\sqrt{\alpha_3{}^2 - \alpha_2{}^2}} = \beta_3. \quad (103\text{–}13)$$

Now, if ψ is the azimuth measured in the orbital plane, we find, with the aid of (101–47),

$$\frac{\alpha_3}{\xi}\left(1 - \frac{k\alpha_2}{\alpha_3{}^2\xi}\right) = \cos\left(\psi - \beta_3 - \frac{k\alpha_2}{\alpha_3{}^3}\sqrt{1 - \frac{\alpha_3{}^2}{\xi^2}}\right). \quad (103\text{–}14)$$

To a first approximation $\alpha_3/\xi = \cos(\psi - \beta_3)$, the equation of a straight line. Using this relation to express the coefficients of k in (103–14) in terms of $\cos(\psi - \beta_3)$, we obtain

$$\frac{1}{\xi} = k\frac{\alpha_2}{\alpha_3{}^4} + \frac{1}{\alpha_3}\cos(\psi - \beta_3) \qquad (103\text{–}15)$$

for the equation of the orbit. This is the equation of a hyperbola

referred to one focus as origin. For positive ξ, which is all that is physically allowable, the equation represents the near branch of the hyperbola for positive $k\alpha_2$ and the far branch for negative $k\alpha_2$. This is in accord with qualitative expectations. For, if p_H and e are positive, a particle approaching the sphere with a positive component of angular momentum about the magnetic axis is subject to a force $(e/c)\mathbf{v} \times \mathbf{H}_0$ tending to deflect it *toward* the sphere, whereas one possessing a negative component of angular momentum is subject to a force tending to deflect it *away from* the sphere.

From (103–15) we find for the nearest distance of approach of the ion to the center of the sphere

$$\xi_0 = \alpha_3\left(1 - \frac{k\alpha_2}{\alpha_3{}^3}\right) \tag{103–16}$$

for any given values of the constants α_2 and α_3. The effect of the magnetic field is to diminish this distance when $k\alpha_2$ is positive, and to increase it when $k\alpha_2$ is negative. For our approximation to be valid, it is evident that we must have $|\, k/\alpha_3{}^2 \,| \ll 1$.

Next we shall consider trajectories lying in the equatorial plane. In this case we can separate the variables in the Hamilton-Jacobi equation without making any approximation. As $\partial S/\partial\theta$ vanishes and $\sin\theta = 1$, (103–5) reduces to

$$\frac{\partial S}{\partial \xi} = \sqrt{1 - \left(\frac{\alpha_2}{\xi} - \frac{k}{\xi^2}\right)^2}, \tag{103–17}$$

and the action is

$$S = \alpha_2\phi + \int\sqrt{1 - \left(\frac{\alpha_2}{\xi} - \frac{k}{\xi^2}\right)^2}\,d\xi. \tag{103–18}$$

This gives for the equation of the orbit

$$\phi - \beta_2 = \int\frac{(\alpha_2\xi - k)d\xi}{\xi\sqrt{\xi^4 - (\alpha_2\xi - k)^2}}, \tag{103–19}$$

or, in differential form,

$$\frac{d\xi}{d\phi} = \frac{\xi\sqrt{\xi^4 - (\alpha_2\xi - k)^2}}{\alpha_2\xi - k}. \tag{103–20}$$

We have three cases to consider, depending on the magnitude of α_2. Throughout we shall suppose the polar axis to be so directed as to make k positive. This involves no loss of generality.

Case I. The constant α_2 is positive. Put $y \equiv \xi/\alpha_2$, $\epsilon \equiv k/\alpha_2{}^2$. Then (103–20) becomes

$$\frac{dy}{d\phi} = \frac{y\sqrt{y^4 - (y - \epsilon)^2}}{y - \epsilon}. \qquad (103\text{–}21)$$

The zeros of the function $f(y) \equiv y^4 - (y - \epsilon)^2$ appearing under the square root sign are $y_1 = \frac{1}{2}(-1 - \sqrt{1 + 4\epsilon})$, $y_2 = \frac{1}{2}(-1 + \sqrt{1 + 4\epsilon})$, $y_3 = \frac{1}{2}(1 - \sqrt{1 - 4\epsilon})$, $y_4 = \frac{1}{2}(1 + \sqrt{1 - 4\epsilon})$ and the zero of the denominator is $y_0 = \epsilon$. Hence we can write (103–21) in the form

$$\frac{dy}{d\phi} = \frac{y\sqrt{(y - y_1)(y - y_2)(y - y_3)(y - y_4)}}{y - y_0}. \qquad (103\text{–}22)$$

As ξ is essentially positive, only positive values of y have physical significance. The zero y_1 of $f(y)$ is negative, the zero y_2 is positive, and y_3 and y_4 are positive or complex according as $\epsilon < \frac{1}{4}$ or $> \frac{1}{4}$. Also $f(0) = -\epsilon^2$ and $f(\infty) = \infty$. So, if we plot $f(y)$ against y we get

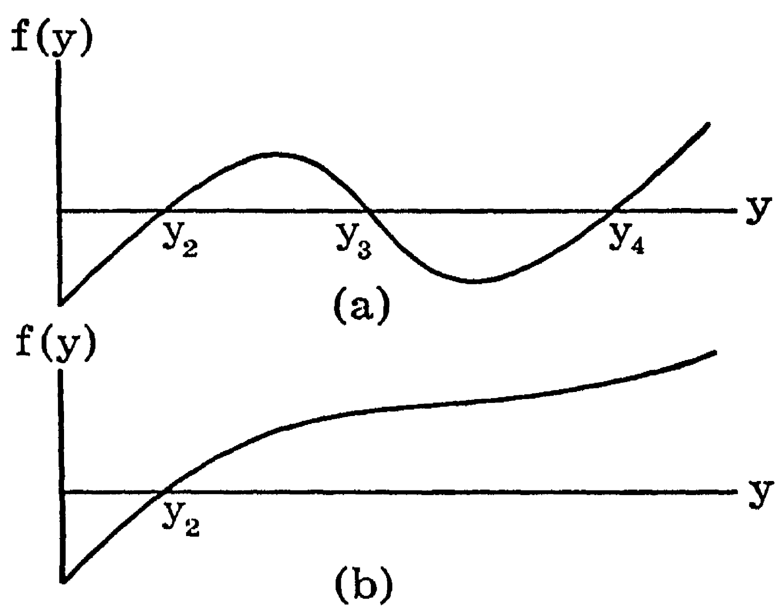

FIG. 113.

the curve shown in Fig. 113*a* if $\epsilon < \frac{1}{4}$ or that in Fig. 113*b* if $\epsilon > \frac{1}{4}$. Since $f(y)$ must be positive in order that $dy/d\phi$ may be real, we have in the first case both the closed orbit extending between the libration limits y_2 and y_3 and the open orbit reaching from y_4 to ∞. As $y_2 < y_0 < y_3$ the derivative $d\phi/dy$ vanishes during the course of y from y_2 to y_3 and again during the return of y from y_3 to y_2. Therefore the closed orbit does not, in general, encircle the sphere until it

has made a number of loops. In the second case only the open orbit extending from y_2 to ∞ exists.

In connection with cosmic rays it is important to find the nearest distance to the sphere which can be attained by a particle of given energy α_1 coming from infinity. When $\epsilon \leqslant \frac{1}{4}$ this nearest distance is given by $\xi_4 = \frac{1}{2}\alpha_2 + \frac{1}{2}\sqrt{\alpha_2^2 - 4k}$. As $d\xi_4/d\alpha_2$ is positive for all values of α_2, the smallest ξ_4 occurs when α_2^2 has its minimum value $4k$. Then, however, ξ_3 and ξ_4 coalesce, and the particle can proceed to the nearer distance given by ξ_2. So the nearest approach occurs for $\epsilon > \frac{1}{4}$, being given by $\xi_2 = -\frac{1}{2}\alpha_2 + \frac{1}{2}\sqrt{\alpha_2^2 + 4k}$. Since $d\xi_2/d\alpha_2$ is negative for all values of α_2, the smallest ξ_2 corresponds to the greatest α_2, namely that for which $\alpha_2^2 = 4k$. Then $\xi_2 = (\sqrt{2} - 1)\sqrt{k}$, or the nearest distance to the sphere which can be attained by a particle of velocity v is

$$r_{\min} = (\sqrt{2} - 1)\sqrt{\frac{ep_H\sqrt{1 - \beta^2}}{4\pi cmv}}. \qquad (103\text{–}23)$$

Next we shall integrate the equation (103–21) of the path. First consider the closed orbit extending from y_2 to y_3 for $\epsilon < \frac{1}{4}$. Making the substitution $y \equiv \frac{1}{2}(y_3 + y_2) + z$, (103–22) gives

$$\phi - \beta_2 = \int \frac{(z + d)dz}{(z+b)\sqrt{(a^2 - z^2)(e+z)(f-z)}}, \quad -a \leqslant z \leqslant a, \quad (103\text{–}24)$$

where

$$a \equiv \tfrac{1}{2}(y_3 - y_2) = \tfrac{1}{4}\{2 - \sqrt{1 + 4\epsilon} - \sqrt{1 - 4\epsilon}\},$$

$$b \equiv \tfrac{1}{2}(y_3 + y_2) = \tfrac{1}{4}\{\sqrt{1 + 4\epsilon} - \sqrt{1 - 4\epsilon}\},$$

$$d \equiv \tfrac{1}{2}(y_3 + y_2) - y_0 = \tfrac{1}{4}\{\sqrt{1 + 4\epsilon} - \sqrt{1 - 4\epsilon}\} - \epsilon,$$

$$e \equiv \tfrac{1}{2}(y_3 + y_2) - y_1 = \tfrac{1}{2} + \tfrac{3}{4}\sqrt{1 + 4\epsilon} - \tfrac{1}{4}\sqrt{1 - 4\epsilon},$$

$$f \equiv y_4 - \tfrac{1}{2}(y_3 + y_2) = \tfrac{1}{2} + \tfrac{3}{4}\sqrt{1 - 4\epsilon} - \tfrac{1}{4}\sqrt{1 + 4\epsilon}.$$

Evidently e and f are of order unity, b of order ϵ, a of order ϵ^2, and d of order ϵ^3. We shall carry the integration only through terms of order ϵ^2. As z is never greater than a in magnitude, we can consider it to be of order ϵ^2. Hence we can neglect z as compared with e

and f in the last two factors in the denominator, and express the factor $z + b$ as a power series in z/b, obtaining

$$\phi - \beta_2 = \frac{1}{b\sqrt{ef}} \int \left\{ d + z - \frac{1}{b} z^2 \right\} \frac{dz}{\sqrt{a^2 - z^2}}$$

$$= \frac{1}{b\sqrt{ef}} \left\{ \left(d - \frac{a^2}{2b} \right) \sin^{-1} \frac{z}{a} - \left(1 - \frac{z}{2b} \right) \sqrt{a^2 - z^2} \right\}. \qquad (103\text{–}25)$$

Therefore, when z runs through a cycle of values, the azimuth increases by the amount

$$\Delta\phi = \frac{2\pi}{b\sqrt{ef}} \left(d - \frac{a^2}{2b} \right) \qquad (103\text{–}26)$$

of order ϵ^2.

If, for instance, $\epsilon = 2/17 = 0.1176$, then $\Delta\phi = 8.^\circ 3$ and $\phi - \beta_2 = 0.023 \sin^{-1}(67z) - 8.65(1 - 4.12z)\sqrt{0.000222 - z^2}$ when z is increasing with ϕ. When z is decreasing the sign of the right-hand side must be changed and the constant β_2 adjusted to make the curve

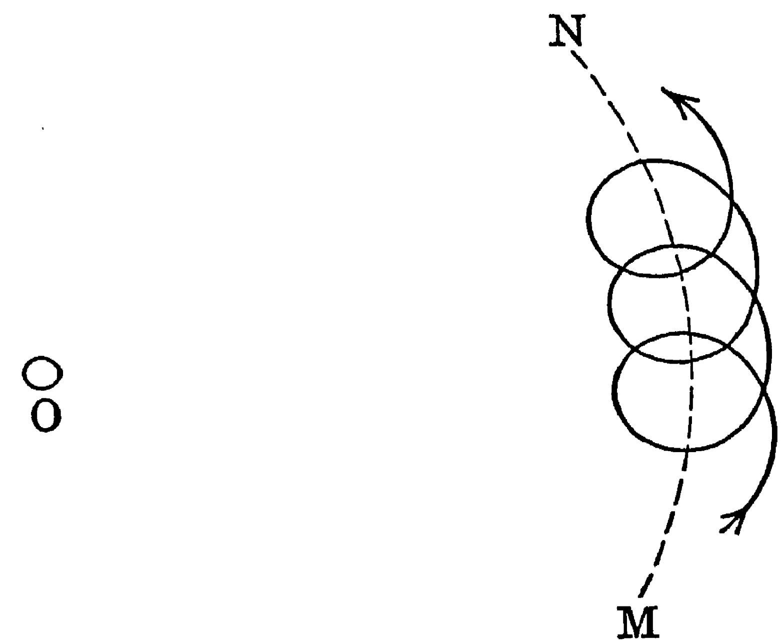

FIG. 114.

continuous. The same purpose is served by allowing the arc sine to increase continuously and changing the sign of the radical at each turning point of z. A few loops of the trajectory are drawn to scale in Fig. 114, where the sphere is at O and MN is the circle about which the ion oscillates at the same time that it revolves around the sphere. It should be noted that the curvature of the path increases as the ion comes into the more intense field nearer to the sphere.

Next consider the open orbit extending from y_4 to ∞ for $\epsilon < \frac{1}{4}$. As y_0, y_2 and y_3 in (103–22) are of the order of ϵ, we shall neglect them as compared with y_1 and y_4 which are of the order of unity. Then (103–22) gives

$$\phi - \beta_2 = \int \frac{dy}{y\sqrt{y^2 + 2b - g^2}}, \tag{103–27}$$

where

$$b \equiv \tfrac{1}{4}(\sqrt{1 + 4\epsilon} - \sqrt{1 - 4\epsilon}),$$
$$g^2 \equiv \tfrac{1}{4}(1 + \sqrt{1 + 4\epsilon})(1 + \sqrt{1 - 4\epsilon}).$$

Evidently b is of order ϵ and g of order unity but slightly larger than y_4. Integrating (103–27) we find the trajectory to be the hyperbola

$$\frac{1}{y} = \frac{b}{g^2} + \frac{\sqrt{b^2 + g^2}}{g^2} \cos g(\phi - \beta_2) \tag{103–28}$$

in y and $g\phi$, of eccentricity $\sqrt{1 + g^2/b^2}$.

We have investigated the trajectories for which $\epsilon < \frac{1}{4}$. Now let us consider the case $\epsilon = \frac{1}{4}$. Then $y_3 = y_4 = \frac{1}{2}$ and the curve in Fig. 113a touches the axis of abscissas tangentially without crossing it at this double zero. From (103–22) we have for the equation of that portion of the orbit along which y increases with ϕ, for values of y less than $\frac{1}{2}$,

$$\begin{aligned}\phi - \beta_2 &= -\int \frac{(y - \frac{1}{4})dy}{y(y - \frac{1}{2})\sqrt{y^2 + y - \frac{1}{4}}} \\ &= -\tfrac{1}{2}\int \frac{dy}{y\sqrt{y^2 + y - \frac{1}{4}}} - \tfrac{1}{2}\int \frac{dy}{(y - \frac{1}{2})\sqrt{y^2 + y - \frac{1}{4}}}.\end{aligned} \tag{103–29}$$

We can integrate this exactly, obtaining,

$$\begin{aligned}\phi - \beta_2 &= \cos^{-1}\left(\frac{1}{2\sqrt{2y}} - \frac{1}{\sqrt{2}}\right) \\ &\quad + \frac{\sqrt{2}}{2}\log\left\{\frac{\sqrt{y^2 + y - \frac{1}{4}} + \sqrt{\frac{1}{2}}}{\frac{1}{2} - y} - \sqrt{2}\right\}.\end{aligned} \tag{103–30}$$

When $y = y_2 = \frac{1}{2}(-1 + \sqrt{2})$, $\phi - \beta_2 = 0$, and, when $y = y_3 = \frac{1}{2}$, $\phi - \beta_2 = \infty$.

The portion of the orbit along which y decreases as ϕ increases is given by (103–30) with the sign of the right-hand member changed. Evidently the ion spirals inward from the circle of radius given by y_3 to the circle of radius given by y_2, and then spirals outward to the first circle, which it approaches asymptotically but never surpasses.

Similarly, for values of y greater than $\frac{1}{2}$, the portion of the orbit along which y decreases with ϕ is given by

$$\phi - \beta_2 = \cos^{-1}\left(\frac{1}{2\sqrt{2y}} - \frac{1}{\sqrt{2}}\right) + \frac{\sqrt{2}}{2}\log\left\{\frac{\sqrt{y^2+y-\frac{1}{4}}+\sqrt{\frac{1}{2}}}{y-\frac{1}{2}}+\sqrt{2}\right\}. \qquad (103\text{–}31)$$

Here $\phi - \beta_2$ is finite when $y = \infty$ but increases without limit as y approaches $\frac{1}{2}$. The ion, therefore, spirals in from infinity, its path approaching asymptotically the circle of radius given by y_3.

This circular path of radius specified by y_3 is one in which the ion can remain indefinitely. In fact we can easily show from first principles that it is a possible orbit. Since the magnetic intensity

$$H_0 = -\frac{p_H}{4\pi r^3} \qquad (103\text{–}32)$$

is a function of r only, it follows at once from (66–12) and (66–14) that an ion can describe a circular orbit of radius

$$r = -\frac{mvc}{eH_0\sqrt{1-\beta^2}} \qquad (103\text{–}33)$$

with angular velocity

$$\dot{\phi} = -\frac{eH_0}{mc}\sqrt{1-\beta^2}. \qquad (103\text{–}34)$$

Substituting in (103–12) with $\sin\theta = 1$ we find

$$\alpha_2 = -2r^2\frac{eH_0}{c} = 2\frac{ep_H}{4\pi c}\frac{1}{r} = 2\frac{k}{\xi}. \qquad (103\text{–}35)$$

But, if we eliminate H_0 between (103–32) and (103–33), we find

$$r^2 = \frac{ep_H}{4\pi c}\frac{\sqrt{1-\beta^2}}{mv} = \frac{k}{\sqrt{\dfrac{\alpha_1^2}{c^2} - m^2c^2}} \qquad (103\text{–}36)$$

and
$$\xi = r\sqrt{\frac{\alpha_1^{\,2}}{c^2} - m^2c^2} = \sqrt{k}. \tag{103–37}$$

Hence, from (103–35), $\alpha_2 = 2\sqrt{k}$ and $\epsilon = k/\alpha_2^{\,2} = \frac{1}{4}$. But this is just the case we were discussing. Furthermore, $y = \xi/\alpha_2 = \frac{1}{2}$, agreeing with the value found for the limiting circular orbit in our more general theory.

Since $f(y)$ as plotted in Fig. 113*a* is positive both sides of the double zero $y_3 = y_4 = \frac{1}{2}$, the circular orbit is unstable. This is evident at once from physical considerations, since, if the ion recedes ever so little from the sphere into the weaker portions of the field, the curvature of its path becomes less and it never returns to the circular orbit. On the other hand, if it approaches nearer to the sphere, thereby passing into a region of stronger magnetic intensity, the curvature of its path becomes greater and it departs farther from the circular orbit.

We shall not investigate the open orbit existing when $\epsilon > \frac{1}{4}$ further than we have already done in our preliminary analysis of the trajectory with positive α_2, but shall proceed to the next case.

Case II. The constant α_2 is zero. In this case the ion has zero angular momentum at infinity, and therefore is headed directly for the center of the sphere. As is evident from (103–4), however, it acquires an angular velocity in the negative sense as it approaches the sphere on account of the force exerted by the magnetic field.

We cannot use (103–21) since both y and ϵ become infinite, but must resort to (103–20), which reduces to

$$\frac{d\xi}{d\phi} = -\frac{\xi\sqrt{\xi^4 - k^2}}{k} \tag{103–38}$$

when α_2 is made zero. The integral of this is

$$\xi^2 = \frac{k}{\sin 2(\phi - \beta_2)}, \tag{103–39}$$

which represents the equation of the orbit. Evidently $\xi = \infty$ when $\phi - \beta_2 = 0$ and again when $\phi - \beta_2 = \pi/2$, and has its smallest value $\sqrt{k}$ when $\phi - \beta_2 = \pi/4$. This distance of nearest approach is not so small as that found previously for positive α_2.

Case III. The constant α_2 is negative. It will be convenient in this case to put $y \equiv -\xi/\alpha_2$. Then, in place of (103–21), we have, when ξ increases with ϕ,

$$\frac{dy}{d\phi} = \frac{y\sqrt{y^4 - (y + \epsilon)^2}}{y + \epsilon}. \tag{103–40}$$

The zeros of the function $f(y) \equiv y^4 - (y + \epsilon)^2$ are $y_1 = \frac{1}{2}(-1 - \sqrt{1 - 4\epsilon}), y_2 = \frac{1}{2}(-1 + \sqrt{1 - 4\epsilon}), y_3 = \frac{1}{2}(1 - \sqrt{1 + 4\epsilon})$, $y_4 = \frac{1}{2}(1 + \sqrt{1 + 4\epsilon})$. As ϵ is essentially positive, y_1, y_2 and y_3 are

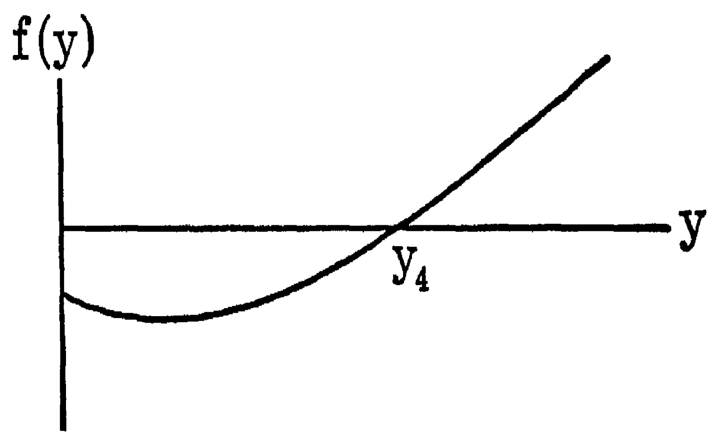

FIG. 115.

negative or complex and therefore of no physical significance. As $f(0) = -\epsilon^2$ and $f(\infty) = \infty$ the curve obtained by plotting $f(y)$ against y is as shown in Fig. 115 for positive values of y. The only trajectories possible are open orbits extending from y_4 to infinity. The distance of nearest approach to the sphere is given by $\xi_4 = \frac{1}{2}(-\alpha_2) + \frac{1}{2}\sqrt{(-\alpha_2)^2 + 4k}$, which has its minimum value $\sqrt{k}$ when $-\alpha_2 = 0$. This is not so small, however, as the nearest distance of approach for positive α_2.

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